Algebra III - Abstraktna algebra, 22.01.2018.

1. (a) Naj bo dana grupa $G = \mathrm{Mat}_{2\times 2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{R} \right\}$ z operacijo seštevanja po komponentah. Pokaži, da je množica $H = \{A \in G \,|\, \mathrm{sled}(A) = 0\}$ podgrupa grupe G.

(b) Naj bo H prava podgrupa grupe G (to je, $H \leq G$ in $H \neq G$). Pokaži, da je $\langle G \backslash H \rangle = G$.

Re.

(a) $B = \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix} \Rightarrow \mathrm{trag}(B) = 0 \Rightarrow H \neq \emptyset$.

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, $A, B \in H \Rightarrow a + d = 0,\ x + w = 0 \Rightarrow a + x + d + w = 0 \Rightarrow A + B \in H$.

$-A = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$.

(b) $G = \{e, a_1, ..., a_k, a_{k+1}, ...\}$, $H = \{e, a_1, ..., a_k\} \Rightarrow G \backslash H = \{a_{k+1}, a_{k+2}, ...\}$ $(G \backslash H \subseteq G)$. Če $\langle G \backslash H \rangle \neq G$ potem $\exists b \in G$ t.d. $b \notin \langle G \backslash H \rangle$.

1° $b \in H$. $\exists c \notin H$ t.d. $cH \neq H \Rightarrow cH \subseteq G \backslash H$ $(cH \cap H = \emptyset) \Rightarrow \exists d \in G \backslash H$, $cb = d \Rightarrow$ $b = c^{-1}d \in G \backslash H \Rightarrow b \notin H$, protislovje.

2° $b \notin H \Rightarrow b \in G \backslash H$, protislovje.

2. Naj bo

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 7 & 5 & 1 & 3 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 7 & 3 & 5 & 1 & 2 & 6 & 4 \end{pmatrix}.$$

Izračunaj π^{-1} in π^{2018}.

Re.

$|\pi| = 15$, $\pi^{2018} = (178)(24356)$, $\pi^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 5 & 2 & 6 & 4 & 3 & 8 & 1 \end{pmatrix}$.

3. (a) Naj bo $G = (\mathrm{Mat}_{2\times 2}(\mathbb{Z}), +)$ in $H = \{A \in G \,|\, \mathrm{sled}(A) = 0\}$. Pokaži, da je H podgrupa edinka v G in da je $G/H \cong \mathbb{Z}$.

(b) Konstruiraj Cayley-evo tabelo za $\mathrm{Aut}(\mathbb{Z}_8)$.

Re.

(a) $\phi : G \to \mathbb{Z}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow a + d$, ϕ je homomorfizem, $\phi(G) = \mathbb{Z}$, $\ker(\phi) = H$.

(b) $f_1(1) = 1$, $f_3(1) = 3$, $f_5(1) = 5$, $f_7(1) = 7$.

$\circ$	f_1	f_2	f_3	f_4
f_1	f_1	f_3	f_5	f_7
f_3	f_3	f_1	f_7	f_5
f_4	f_5	f_7	f_1	f_3
f_5	f_7	f_5	f_3	f_1

4. Center grupe G je definiran na naslednji način: $Z(G) = \{a \in G \,|\, ax = xa \text{ za } \forall x \in G\}$. Pokaži, da je

$$Z(\mathrm{GL}_2(\mathbb{R})) = \mathbb{R} \cdot \mathrm{id}.$$

Re.

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow X \in \mathrm{GL}_2(\mathbb{R})$.

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$AX = XA \Rightarrow a = d,\ b = c$.

$Y = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow Y \in \mathrm{GL}_2(\mathbb{R})$.

$AY = YA \Rightarrow b = 0$.

Več na `http://osebje.famnit.upr.si/~penjic/`