Univerza na Primorskem
Fakulteta za matematiko, naravoslovje in informacijske tehnologije
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NAME: ____________________ STUDENT NUMBER: ____________________

SURNAME: ____________________ SIGNATURE: ____________________

Algebra II - Linear algebra, exam

1. Let $\mathcal{W}_1, ..., \mathcal{W}_n$ be subspaces of a vector space $\mathcal{V}$. For each $1 \le i \le n$, let $Z_i = \sum_{j \neq i} \mathcal{W}_j$. Prove that the sum $\sum_{i=1}^n \mathcal{W}_i$ is direct if and only if $\mathcal{W}_i \cap Z_i = \{\mathbf{0}\}$ for each i.

2. Let $\mathcal{S} = \left\{ e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ denote standard basis for $\mathbb{R}^3$ and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined on the following way

$$T(ae_1 + be_2 + ce_3) = (a - 2b + c)e_1 + 3a \cdot e_2 - (2a - 4c)e_3$$

(a) Show that T is a linear operator.

(b) Find coordinate matrix of T with respect to basis $\{e_1 - e_2, 2e_1 + e_2, e_1 + e_3\}$.

3. An operator $T \in \mathcal{L}(\mathcal{V})$ is called positive definite if T is self-adjoint and $\langle Tv, v \rangle > 0$ for all $v \in \mathcal{V}$. Prove a self-adjoint operator T on a finite dimensional vector space $\mathcal{V}$ is positive definite if and only if all of its eigenvalues are positive.

4. (a) Find the eigenvalues of

$$C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad C^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

(b) Those are both permutation matrices. What are their inverses C^{-1} and $(C^2)^{-1}$?

(c) Find the determinants of C and $C + I$ and $C + 2I$.

Instructions: Please, write your solutions only with ink or ballpoint pen in blue or black colour. You must return this sheet of paper together with your solutions. You can use calculator. All pages with your solutions must be marked in the following way: "page-number/number-of-pages".