Univerza na Primorskem
Fakulteta za matematiko, naravoslovje in informacijske tehnologije
Koper, 17.04.2018.

NAME: ____________________ STUDENT NUMBER: ____________________

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Algebra II - Linear algebra, 1st midterm

1. Let $\mathcal{V}$ and $\mathcal{W}$ be finite-dimensional vector spaces and $T : \mathcal{V} \to \mathcal{W}$ be a given linear transformation. Let $\mathcal{V}_0$ be a subspace of $\mathcal{V}$. If T is bijection show that $T(\mathcal{V}_0)$ is a subspace of $\mathcal{W}$ and that $\dim(\mathcal{V}_0) = \dim(T(\mathcal{V}_0))$.

2. (a) Find values of a parameter $a \in \mathbb{R}$ such that the set

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1-a \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2-a \\ \vdots \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ (n-1)-a \end{pmatrix} \right\}$$

is a basis for space $\mathbb{R}^n$. For such a write $(1, 2, 3, ..., n)^\top$ as linear combination of vectors from $\mathcal{B}$. (50%)

(b) Let A be a fixed $n \times n$ matrix, and define $T : \mathrm{Mat}_{n\times n}(\mathbb{R}) \to \mathrm{Mat}_{n\times n}(\mathbb{R})$ by $T(B) = AB$. When does $\mathrm{im}(T) = \mathrm{Mat}_{n\times n}(\mathbb{R})$? (50%)

3. Let $\mathcal{S} = \left\{ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ denote standard basis for $\mathbb{R}^2$ and let $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$ be coordinate matrix of $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ with respect to $\mathcal{B} = \{x_1, x_2\}$, where $x_1 = 2e_1 + e_2$, $x_2 = -e_1 - e_2$.

(a) Find coordinate matrix of T with respect to $\mathcal{S}$.

(b) Find all vectors $v \in \mathbb{R}^2$ such that $v^\top v = 1$ and $T(v) = 2v$.

4. (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined with $T(a, b, c) = (a + b, b + c, 0)$. Show that xy-plane$= \{(x, y, 0) \,|\, x, y \in \mathbb{R}\}$ and the x-axis $= \{(x, 0, 0) \,|\, x \in \mathbb{R}\}$ are T-invariant subspaces of $\mathbb{R}^3$. (30%)

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined with $T(a, b, c) = (-b + c, a + c, 3c)$. Define space $\mathcal{W}$ on the following way $\mathcal{W} := \mathrm{span}\{e_1, T(e_1), T^2(e_1), ...\}$ where $e_1 = (1, 0, 0)$. Explain if $\mathcal{W}$ is T-invariant subspaces of $\mathbb{R}^3$. Find basis and dimension of $\mathcal{W}$. (70%)

Instructions: Please, write your solutions only with ink or ballpoint pen in blue or black colour. You must return this sheet of paper together with your solutions. You can use calculator. All pages with your solutions must be marked in the following way: "page-number/number-of-pages".