9 Homework 9 (Orthogonal Vectors & Gram-Schmidt Procedure)

1. With respect to the inner product for matrices given by $\langle A, B\rangle = \text{trace}(A^\top B)$, verify that the set

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \right\}$$

is an orthonormal basis for $\text{Mat}_{2\times 2}(\mathbb{R})$, and then compute the Fourier expansion of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ with respect to $\mathcal{B}$.

2. Using the trace inner product described with $\langle A, B\rangle = \text{trace}(A^\top B)$, determine the angle between the following pairs of matrices.

(a) $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

(b) $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -2 \\ 2 & 0 \end{pmatrix}$.

3. If $\{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_n\}$ is an orthonormal basis for an inner-product space $\mathcal{V}$, explain why

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \sum_i \langle \boldsymbol{x}, \boldsymbol{u}_i\rangle\langle \boldsymbol{u}_i, \boldsymbol{y}\rangle$$

holds for every $x, y \in \mathcal{V}$.

4. Let $\mathcal{L}$ denote vector subspace of space $\text{Mat}_{n\times n}(\mathbb{R})$ defined with

$$\mathcal{L} = \left\{ A \in \text{Mat}_{n\times n}(\mathbb{R}) \,|\, AX - XA = 0, X = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right\}.$$

Considering standard inner product for matrices $\langle A, B\rangle = \text{trace}(A^\top B)$ find orthonormal basis for $\mathcal{L}$.

5. Let $\mathcal{V} = \text{Mat}_{n\times n}(\mathbb{R})$ be given inner product space with $\langle A, B\rangle = \text{trace}(A^\top B)$ and let $\mathcal{L}$ denote subspace of $\mathcal{V}$ given with

$$\mathcal{L} = \left\{ \begin{pmatrix} 0 & 3 \\ 3 & -3 \end{pmatrix}, \begin{pmatrix} -2 & -2 \\ 6 & -2 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \right\}$$

Find orthonormal basis for $\mathcal{L}$.

6. Let $\mathcal{P}_3$ denote inner product space of all polynomials with degree ≤ 3, with inner product

$$\langle p, q\rangle = \frac{1}{4}\sum_{i=0}^{3} p(\lambda_i)q(\lambda_i)$$

where $\lambda_0 = 3$, $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = -3$. Use Gram-Schmidt procedure on set of polynomials $\{-1, x, -x^2, x^3\}$ and find orthonormal basis $\{p_1(x), p_2(x), p_3(x), p_4(x)\}$ for $\mathcal{P}_3$ which have additional property that

$$\|p_i\|^2 = p_i(\lambda_0) \qquad \text{for } i = 1, 2, 3, 4.$$

7. Consider real inner product space $\mathcal{P}_2$, where for two given polynoimals

$$p = p(x) = p_0 + p_1 x + p_2 x^2 \qquad \text{and} \qquad q = q(x) = q_0 + q_1 x + q_2 x^2$$

inner product is defined on the following way

$$\langle p, q\rangle = p_0 q_0 + p_1 q_1 + p_2 q_2.$$

Check are the following polynomials linearly independent in $\mathcal{P}_2$

$$u_1 = 3 + 4x + 5x^2, \qquad u_2 = 9 + 12x + 5x^2, \qquad u_3 = 1 - 7x + 25x^2$$

and use them to find orthonormal basis for $\mathcal{P}_2$.

8. (challenge) Linear Correlation. Suppose that an experiment is conducted, and the resulting observations are recorded in two data vectors

$$\boldsymbol{x}_1 = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \boldsymbol{y}_1 = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \qquad \text{and let} \quad \boldsymbol{e} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Problem. Determine to what extent the y_i's are linearly related to the x_i's. That is, measure how close $\boldsymbol{y}$ is to being a linear combination $\beta_0 \boldsymbol{e} + \beta_1 \boldsymbol{x}$.

9. (challenge) Fourier Series. Let $\mathcal{V}$ be the inner-product space of real-valued functions that are integrable on the interval $(-\pi, \pi)$ and where the inner product and norm are given by

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)g(t)\,\mathrm{d}t \quad \text{and} \quad \|f\| = \left(\int_{-\pi}^{\pi} f^2(t)\,\mathrm{d}t \right)^{1/2}$$

(a) Verity that the set of trigonometric functions

$$\mathcal{B}' = \{1, \cos t, \cos 2t, ..., \sin t, \sin 2t, \sin 3t, ...\}$$

is a set of mutually orthogonal vectors, and normalize each vector.

(b) Given and arbitrary $f \in \mathcal{V}$, construct its Fourier expansion.

(c) The infinite series $F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$, that you will get in part (b), where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt\,\mathrm{d}t \qquad \text{and} \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt\,\mathrm{d}t$$

is called the **Fourier series** expansion for $f(t)$. Find the Fourier series expansion for **square wave function** f defined by

$$f(t) = \begin{cases} -1 & \text{when } -\pi < t < 0, \\ 1 & \text{when } 0 < t < \pi. \end{cases}$$