# 8 Homework 8 (Vector norms & Inner-product spaces)

**1.** Show that Euclidean norm satisfy the following properties

(i) $\|\boldsymbol{x}\| \geq 0 \quad \text{and} \quad \|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = 0.$

(ii) $\|\alpha\boldsymbol{x}\| = |\alpha|\|\boldsymbol{x}\| \quad \text{for all scalars } \alpha.$

(iii) $\|\boldsymbol{x}+\boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|.$

**2.** For a general inner-product space $\mathcal{V}$, explain why each of the following statements must be true.

(a) If $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 0$ for all $\boldsymbol{x} \in \mathcal{V}$, then $\boldsymbol{y} = \boldsymbol{0}$.

(b) If $\langle \alpha\boldsymbol{x}, \boldsymbol{y}\rangle = \overline{\alpha}\langle \boldsymbol{x}, \boldsymbol{y}\rangle$ for all $x, y \in \mathcal{V}$ and for all scalars $\alpha$.

(c) $\langle \boldsymbol{x}+\boldsymbol{y}, \boldsymbol{z}\rangle = \langle \boldsymbol{x}, \boldsymbol{z}\rangle + \langle \boldsymbol{y}, \boldsymbol{z}\rangle$ for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{V}$.

**3.** Let $\mathcal{V}$ be an inner-product space with an inner product $\langle \boldsymbol{x}, \boldsymbol{y}\rangle$. Explain why the function defined by $\| \star \| = \sqrt{\langle \star, \star\rangle}$ satisfies the first two norm properties from Problem 1. That is

(i) $\|\boldsymbol{x}\| \geq 0 \quad \text{and} \quad \|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = 0.$

(ii) $\|\alpha\boldsymbol{x}\| = |\alpha|\|\boldsymbol{x}\| \quad \text{for all scalars } \alpha.$

**4.** For a real inner-product space with $\| \star \|^2 = \langle \star, \star\rangle$, derive the inequality

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle \leq \frac{\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2}{2}.$$

**5.** Let $\mathrm{Mat}_{m\times n}(\mathbb{R})$ denote given vector space, the set of all $m \times n$ matrices. Show that function defined with

$$\langle A, B\rangle = \mathrm{trag}(A^\top B)$$

is an inner product on a vector space $\mathrm{Mat}_{m\times n}(\mathbb{R})$.

**6.** Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ denote a given matrix. In space $\mathbb{R}^n$ lets define product with

$$\langle x, y\rangle = (Ax)^\top Ay$$

Discuss and carefully explain for what kind of matrices $A$

(a) given product will be inner product on $\mathrm{Mat}_{n\times n}(\mathbb{R})$;

(b) given product will yield that equality $\langle x, y\rangle = x^\top y$ hold.

**7. (challenge) Equivalent Norms.** Vector norms are basic tools for defining and analyzing limiting behavior in vector spaces $\mathcal{V}$. A sequence $\{\boldsymbol{x}_k\} \subset \mathcal{V}$ is said to converge to $x$ (write $\boldsymbol{x}_k \to \boldsymbol{x}$) if $\|\boldsymbol{x}_k - \boldsymbol{x}\| \to 0$. This depends on the choice of the norm, so, ostensibly, we might have $\boldsymbol{x}_k \to \boldsymbol{x}$ with one norm but not with another. Fortunately, this is impossible in finite-dimensional spaces because all norms are equivalent in the following sense.

**Problem:** For each pair of norms, $\| \star \|_a$, $\| \star \|_b$ on a $n$-dimensional space $\mathcal{V}$, exhibit positive constants $\alpha$ and $\beta$ (depending only on the norms) such that

$$\alpha \leq \frac{\|\boldsymbol{x}\|_a}{\|\boldsymbol{x}\|_b} \leq \beta \qquad \text{for all nonzero vectors in } \mathcal{V}.$$