5 Homework 5 (Linear transformations)

1. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a given linear operator with coordinate matrix $T = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$ with respect to canonical basis $\{\vec{i}, \vec{j}\}$. Let $\vec{a} = \vec{i} + \vec{j}$ and $\vec{b} = \vec{i} - 2\vec{j}$.

(a) Find $T(\vec{a})$, $T(\vec{b})$.

(b) For which $\alpha \in \mathbb{R}$ are vectors $T(\vec{a})$, $T(\vec{a} + \alpha\vec{b})$ collinear?

2. Let $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ be coordinate matrix of T in canonical basis $\mathcal{S}$ (with another words $[T]_{\mathcal{S}} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ where $\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$). coordinate matrix of T with respect to the basis $\mathcal{S}' = \left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} \right\}$ (with another words find $[T]_{\mathcal{S}'}$).

3. Let φ denote a linear transformation $\varphi : \mathbb{R}^4 \longrightarrow \mathbb{R}$ such taht $\varphi \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 1$, $\varphi \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} = 1$, $\varphi \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = 0$, $\varphi \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = 0$. Find $\varphi \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$.

4. Let $T : \mathcal{P}_2 \longrightarrow \mathrm{Mat}_{2\times 2}(\mathbb{R})$ be a given linear transformation defined by:

$$T(p) = \begin{pmatrix} p(0) & p(-1) \\ p(1) & p(2) \end{pmatrix}.$$

Find coordinate matrix of $T \in \mathcal{L}(\mathcal{S}, \mathcal{S}')$ with respect to the pair $(\mathcal{S}, \mathcal{S}')$, where $\mathcal{S}$ and $\mathcal{S}'$ are standard basis for $\mathcal{P}_2$ and $\mathrm{Mat}_{2\times 2}(\mathbb{R})$, respectively. Find also one basis for both image $\mathrm{im}(T)$ and null space $\ker(T)$. Discus and give an answer, is there some polynomial $q \in \mathcal{P}_2$ such that $T(q) = \begin{pmatrix} 2 & 1 \\ 5 & 4 \end{pmatrix}$? ($\mathcal{P}_2$ is space of all polynomials of degree ≤ 2).

5. Let $T : \mathrm{Mat}_{2\times 2}(\mathbb{R}) \longrightarrow \mathrm{Mat}_{2\times 2}(\mathbb{R})$ be a given operator defined with

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a - b & -a + b + 2c \\ a - c - d & -a + 2c + d \end{bmatrix}.$$

(a) Find some basis for $\ker(T)$ and $\mathrm{im}(T)$.

(b) Find coordinate matrix of T with respect to the standard basis of $\mathrm{Mat}_{2\times 2}(\mathbb{R})$.

6. Let $T : \mathcal{P}_3 \to \mathcal{P}_3$ denote a given linear operator on space $\mathcal{P}_3$ of all polynomials with real coefficients or degree at most 3, where for any $p(x) \in \mathcal{P}_3$, T is defined on the following way

$$T(p(x)) = xp'(x)$$

that is product of x with derivative $p'(x)$ of polynomial $p(x)$. Show that T is linear operator. Find coordinate matrix representation A for T with respect to $\mathcal{B} = \{1, x, x^2, x^3\}$ and find $A[q(x)]_{\mathcal{B}}$ where $q(x) = q_0 + q_1 x + q_2 x^2 + q_3 x^3$.

7. Let $T : \mathcal{P}_2 \longrightarrow \mathrm{Mat}_{2\times 2}(\mathbb{R})$ be a given linear transformation given by

$$T(a + bt + ct^2) = \begin{pmatrix} a - 2b & b + c \\ -2a - 4c & -2a + 4b \end{pmatrix}$$

Find coordinate matrix representation of T with respect to pair or standard basis (that is find $[T]_{\mathcal{S}\mathcal{S}'}$ where $\mathcal{S}$ and $\mathcal{S}$' are standard basis for $\mathcal{P}_2$ and $\text{Mat}_{2\times 2}(\mathbb{R})$ respectively). Find some basis for rang and null space of T (that is find basis for $\text{im}(T)$ and $\ker(T)$) ($\mathcal{P}_2$ is space of all real polynomials of degree ≤ 2).

8. Let $A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a given linear operator defined on the following way

$$A(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x+y \\ 0 \\ y-z \end{bmatrix}$$

(a) Find $\text{im}(T)$, $\ker(T)$ and their basis.
(b) If $\mathcal{L} = \{(x, y, z)^\top \in \mathbb{R}^3 \,|\, x = y\}$ and

$$\mathcal{M} = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \,|\, A(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) \in \mathcal{L}\}$$

explain what is $\mathcal{M}$ and find its basis.

9. Let $T : V^3(0) \longrightarrow V^3(0)$ be a given map defined with

$$T(a\vec{i} + b\vec{j} + c\vec{k}) = (a - 2b + c)\vec{i} + 3a\vec{j} - (2a - 4c)\vec{k}.$$

Show that T is a linear operator and find coordinate matrix representation of T with respect to $\mathcal{B} = \{\vec{i} - \vec{j}, 2\vec{i} + \vec{j}, \vec{i} + \vec{k}\}$ (with another words find $[T]_\mathcal{B}$).

10. In space of all real arrays $\mathbb{R}^\mathbb{N}$ ($\mathbb{R}^\mathbb{N} = \{(a_1, a_2, a_3, ..., a_n, a_{n+1}, ...) \,|\, a_n \in \mathbb{R}, n \in \mathbb{N}\}$) it is given a set

$$\mathcal{L} = \{(a_n)_{n\in\mathbb{N}} \in \mathbb{R}^\mathbb{N} \,|\, a_{n+2} - 2a_n = 0, n \in \mathbb{N}\}.$$

Show that the map $T : \mathcal{L} \to \mathcal{L}$ which $(a_n)_{n\in\mathbb{N}}$ map to $(a_{n+2})_{n\in\mathbb{N}}$ is a linear operator. Find the coordinate matrix of T with respect to basis $\mathcal{B} = \{(1, 0, 2, 0, 4, 0, 8, ...), (0, 1, 0, 2, 0, 4, 0, 8, ...)\}$.

11. Let $T : \mathcal{P}_3 \to \text{Mat}_{2\times 2}(\mathbb{R})$ be a given linear transformation defined with

$$T(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = \begin{bmatrix} a_0 + a_3 - a_2 & a_0 + 2a_1 - a_2 \\ a_3 & a_0 - a_2 \end{bmatrix}$$

Find the coordinate matrix of T with respect to the pair of standard basis, and find $\ker(T)$, $\text{im}(T)$, $\dim \text{im}(T)$ and $\dim \ker(T)$. ($\mathcal{P}_3$ is a space of all polynomials of degree ≤ 3).

12. Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be a given linear operator defined with

$$T(a + bt + ct^2) = a + b + c + (a + 3b)t + (a - b + 2c)t^2.$$

Find the coordinate matrix of $T \in \mathcal{L}(\mathcal{P}_2, \mathcal{P}_2)$ with respect to $\mathcal{B} = \{1 - t, t - t^2, 1 + t^2\}$. Moreover, find a basis for $\ker(T)$ and $\text{im}(T)$.

13. **(IMC 2011.)** For some $A \in \text{Mat}_{3\times 3}(\mathbb{R})$ let λ_1, λ_2 and λ_3 denote three different real numbers such that

$$p(x) = \det(A - xI) = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3).$$

For $i = 1, 2, 3$ denote by V_i spaces $V_i := \ker(A - \lambda_i I)$. Show that

(a) $\dim(V_i) = 1$.

(b) $\mathbb{R}^3 = V_1 + V_2 + V_3$.

(c) $\text{trace}(A) = \lambda_1 + \lambda_2 + \lambda_3$.

(d) Discuss and carefully explain is it possible that for A we have

$$A^2 + A^\top = I \quad \text{and} \quad \text{trace}(A) = 0.$$