4 Homework 4 (Basis and dimension)

1. Let $\mathcal{V}$ denote vector space of all matrices of form 2×2 over the field of real numbers. Let $\mathcal{W}_1$ be the set of all matrices of form $\begin{pmatrix} x & -x \\ y & z \end{pmatrix}$ and let $\mathcal{W}_2$ be the set of all matrices of the form $\begin{pmatrix} a & b \\ -a & c \end{pmatrix}$. Find a basis and the dimensions of the four subspaces $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_1+\mathcal{W}_2$ and $\mathcal{W}_1\cap\mathcal{W}_2$.

2. Let

$$\mathcal{V}=\left\{\begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \,|\, z_1-2\overline{z_2}+z_3=0, z_1+\overline{z_2+z_3}+z_4=0\right\}$$

be a given subspace of a vector space $\mathrm{Mat}_{2\times 2}(\mathbb{C})$. Find a basis and the dimension of $\mathcal{V}$.

3. Let $\mathcal{M}$ and $\mathcal{L}$ denote subspaces of vector space $\mathbb{R}^5$, where $\mathcal{M}$ is spanned by vectors $(0,0,1,0,0)^\top$ and $(0,1,0,1,0)^\top$ and $\mathcal{L}$ is

$$\mathcal{L}=\{(x_1,x_2,x_3,x_4,x_5)^\top \in \mathbb{R}^5 \,|\, x_1-x_2+x_3=0,\ 2x_1-2x_2+x_3+x_4=0\}$$

(a) Find a basis and the dimension of $\mathcal{M}$ and $\mathcal{L}$. (b) Find a basis and the dimension of $\mathcal{M}\cap\mathcal{L}$ and $\mathcal{M}+\mathcal{L}$.

4. Let $\mathcal{V}$ be vector space $\mathbb{R}^3$ spanned by vectors x_1, x_2, x_3 (x_1, x_2 and x_3 are linearly independent)

$$\mathcal{V}=\mathrm{span}\left\{x_1=\begin{pmatrix}1\\-1\\3\end{pmatrix}, x_2=\begin{pmatrix}0\\1\\-1\end{pmatrix}, x_3=\begin{pmatrix}0\\3\\-2\end{pmatrix}\right\}$$

Recall that this means that $\forall v\in\mathcal{V}\ \exists$ unique $\alpha_1,\alpha_2,\alpha_3\in\mathbb{R}$ s.t. $v=\alpha_1x_1+\alpha_2x_2+\alpha_3x_3$. Let $\mathcal{V}^*$ denote the set of all linear mapping from $\mathcal{V}$ to $\mathbb{R}$ that is

$$\mathcal{V}^*=\mathcal{L}(\mathcal{V},\mathbb{R})=\{T:\mathcal{V}\to\mathbb{R} \mid T \text{ is linear}\}.$$

Now for every $j\in\{1,2,3\}$ lets defined $T_j\in\mathcal{V}^*$ on the following way

$$T_j(a_1x_1+a_2x_2+a_3x_3)=a_j$$

(a) Show that $\mathcal{B}^*=\{T_1,T_2,T_3\}$ is a basis for $\mathcal{V}^*$.

(b) Compute T_1, T_2 and T_3.

Remark: Solutions for (a) and (b) are independent between themselves. Space $\mathcal{V}^*$ is called dual space of $\mathcal{V}$, and a basis $\mathcal{B}^*$ is called dual basis of $\mathcal{B}$.

5. Show that $\mathcal{V}=\{A\in\mathrm{Mat}_{2\times 2}(\mathbb{R}) \,|\, \mathrm{trace}(A)=0\}$ is subspace of a vector space $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ (where $\mathrm{trace}(A)=$ sum of diagonal entries of A). Find a basis and the dimension. Basis that you get extend to full basis of $\mathrm{Mat}_{2\times 2}(\mathbb{R})$.

6. In space of all real sequences $\mathbb{R}^{\mathbb{N}}$ ($\mathbb{R}^{\mathbb{N}}=\{(a_1,a_2,a_3,...,a_n,a_{n+1},...) \,|\, a_n\in\mathbb{R}, n\in\mathbb{N}\}$) let $\mathcal{L}$ be a given set

$$\mathcal{L}=\{(a_n)_{n\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}} \,|\, a_{n+2}-2a_n=0, n\in\mathbb{N}\}.$$

Show that $\mathcal{L}$ is subspace of $\mathbb{R}^{\mathbb{N}}$ and find its basis and the dimension.

<u>Lemma</u> If $A\in\mathrm{Mat}_{m\times n}(\mathbb{R})$ and $B\in\mathrm{Mat}_{n\times p}$ then

$$\mathrm{rank}(AB)\le\mathrm{rank}(B)-\dim\Big(\ker(A)\cap\mathrm{rank}(B)\Big) \qquad \text{and}$$

$$\mathrm{rank}(AB)\le\min\{\mathrm{rank}(A),\ \mathrm{rank}(B)\}.$$

7. **(IMC 2012.)** Let $n\ge 3$ be a fixed positive integer, and let $A\in\mathrm{Mat}_{n\times n}(\mathbb{R})$ denote a matrix that has zeros along the main diagonal and strictly positive real numbers off the main diagonal. Determine the smallest possible number r such that

$$\dim(\mathrm{im}(A))=r.$$

For r that you get, give an example of matrix.