10 Homework 10 (Complementary Subspaces)

1. Let $\mathcal{M}$ be subspace of $\mathbb{R}^4$ defined with

$$\mathcal{M} = \{(z_1, z_2, z_3, z_4)^\top \in \mathbb{R}^4 \,|\, z_1 + 2z_2 + z_3 = 0,\ 2z_1 + z_2 - z_3 = 0, z_1 + 5z_2 + 4z_3 = 0\}.$$

Find $\mathcal{N}$ such that $\mathcal{M}$ and $\mathcal{N}$ are complementary subspaces of a space $\mathbb{R}^4$.

2. In vector space $\mathbb{R}^5$, let $\mathcal{M}$ be subspace spanned by $(0, 0, 1, 0, 0)^\top$ and $(0, 1, 0, 1, 0)^\top$ and let

$$\mathcal{L} = \{(x_1, x_2, x_3, x_4, x_5)^\top \in \mathbb{R}^5 \,|\, x_1 - x_2 + x_3 = 0,\ 2x_1 - 2x_2 + x_3 + x_4 = 0\}.$$

(a) Find a basis and dimensions for $\mathcal{M}$ and $\mathcal{L}$.

(b) Find a dimension of subspace $\mathcal{M} \cap \mathcal{L}$.

(c) Find a basis for complementary subspace $\mathcal{K}$ of the space $\mathcal{L}$ (i.e. find a basis for subspace $\mathcal{K}$ where $\mathcal{L}$ and $\mathcal{K}$ are complementary subspaces of a space $\mathbb{R}^5$).

3. Let $Q : \mathcal{P}_3 \to \mathcal{P}_3$ ($\mathcal{P}_3$ is a vector space of all polynomials of degree ≤ 3) denote a given linear operator defined with

$$Q(p) = \text{all polynomials of degree 2 which graph pass through the points } (-1; p(-1)),\ (0; p(0)) \text{ and } (1; p(1)).$$

(a) Find coordinate matrix of Q with the respect to the standard basis.

(b) Find complementary subspace $\mathcal{N}$ of the space $\mathcal{M} = \ker(Q)$ in $\mathcal{P}_3$.

4. Let

$$\mathcal{L} = \{(x_1, x_2, x_3, x_4)^\top \in \mathbb{R}^4 \,|\, -x_1 + x_2 + x_3 + x_4 = 0,\ x_1 - x_2 + x_3 + x_4 = 0,$$
$$x_1 + x_2 - x_3 + x_4 = 0,\ x_1 + x_2 + x_3 - x_4 = 0\}$$

denote a given set. Show that $\mathcal{L}$ is a subspace of $\mathbb{R}^4$, find a basis, dimension and find complementary subspace of $\mathcal{L}$ in $\mathbb{R}^4$.

5. In a vector space $\mathcal{P}_4$ of all real polynomials of degree ≤ 4 it is given a set

$$\mathcal{M} = \{p \in \mathcal{P}_4 \,|\, p'(0) = p(1),\ p''(0) = 2p(-1)\}.$$

Show that $\mathcal{M}$ is a vector subspace of $\mathcal{P}_4$, find a basis and dimension, and find complementary subspace of $\mathcal{M}$ in $\mathcal{P}_4$.

6. (challenge) Angle between Complementary Subspaces. The angle between nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^n$ was defined to be the number $0 \leq \theta \leq \pi/2$ such that $\cos\theta = \frac{\boldsymbol{v}^\top \boldsymbol{u}}{\|\boldsymbol{v}\|_2 \|\boldsymbol{u}\|_2}$. It's natural to try to extend this idea to somehow make sense of angles between subspaces of $\mathbb{R}^n$. Here we introduce angle between a pair of complementary subspaces.

When $\mathbb{R}^n = \mathcal{M} \oplus \mathcal{N}$, the angle (also known as the minimal angle) between $\mathcal{M}$ and $\mathcal{N}$ is defined to be the number $0 \leq \theta \leq \pi/2$ that satisfies

$$\cos\theta = \max\left\{\frac{\boldsymbol{v}^\top \boldsymbol{u}}{\|\boldsymbol{v}\|_2 \|\boldsymbol{u}\|_2} : \boldsymbol{u} \in \mathcal{M}, \boldsymbol{v} \in \mathcal{N}\right\} = \max\left\{\boldsymbol{v}^\top \boldsymbol{u} : \boldsymbol{u} \in \mathcal{M}, \boldsymbol{v} \in \mathcal{N}, \|\boldsymbol{v}\|_2 = 1, \|\boldsymbol{u}\|_2 = 1\right\}.$$

While this is a good definition, it's not easy to use - especially if one wants to compute the numerical value of $\cos\theta$. Can you explain what would be easiest way to compute numerical value of $\cos\theta$? Justify your answer!