Math 131
Upper bounds – January 21, 2009

1. Working with inequalities

Three useful rules for working with inequalities:

1. $|a \cdot b| = |a| \cdot |b|$
2. $|a + b| \leq |a| + |b|$
3. $|a - b| \leq |a| + |b|$

Rule (3) may seem surprising. Let us see how to prove it. We calculate

$$|a - b| = |a + (-b)| \leq |a| + |-b| = |a| + |b|.$$

2. Bounds

Definition 1. An *upper bound* for a function f is a number U so that: for all x, we have $f(x) \leq U$.
A *lower bound* for a function f is a number L so that: for all x, we have that $f(x) \geq L$.
A *bound in absolute value*, which is what we will usually refer to as just a *bound*, is a number M so that $|f(x)| \leq M$ for all x.

Notice that if M is a bound in absolute value for f, then $-M$ and M are lower and upper bounds for f, and conversely that if L and U are lower and upper bounds, then $\max(|L|, |U|)$ is a bound for f in absolute value.

We'll usually be interested in bounds in absolute value, since they are convenient and quick to work with.

Definition 2. We say f has an *upper bound* U on the interval $[a, b]$ if: for all x on $[a, b]$, we have $f(x) \leq U$. Similarly for lower bounds and bounds in absolute values.

Example 3. *Some bounds*

1. $|\sin x| \leq 1$ for all x. Thus, 1 is a bound (in absolute value) for $\sin x$. So is 2, and 3, and 3.17, but not 0.98.
2. $|x^3| \leq 27$ on the interval $[-3, 1]$.
3. $|e^x| \leq e^2$ on the interval $[-5, 2]$. We could also say that $|e^x| \leq 9$ on the interval $[-5, 2]$, since $9 > e^2$.

2.1. **Geometric interpretation.** Geometrically, an upper bound is a horizontal line that the graph of the function does not go above. Similarly, a lower bound is a horizontal line that the graph does not go below. A bound in absolute value 'traps' the graph of the function in a band between the horizontal lines $y = -M$ and $y = M$.

3. Example

We find a bound for $12\sin x^2 - x\cos x$ on the interval $[-3, 2]$. Using the rules from Part 1, we break it apart:

$$\begin{aligned} |12\sin x^2 - x\cos x| &\leq |12\sin x^2| + |x\cos x| \\ &= 12|\sin x^2| + |x| \cdot |\cos x|. \end{aligned}$$

Since $|x| \leq 3$ on the interval $[-3, 2]$, and since $|\sin *| \leq 1$ and $|\cos *| \leq 1$, we get that

$$|12\sin x^2 - x\cos x| \leq 12 \cdot 1 + 3 \cdot 1 = 15 \text{ (on the interval } [-3, 2]).$$

Note: this is <u>not</u> the least possible bound. However, for many applications it is good enough.

4. Why is this useful?

For us, the main applications will be with the Sandwich Theorem, and the definition of limit.

Bounds for functions have applications throughout mathematics, and you will make more use of them in Calc II, as well as any other future mathematics classes you take.

5. Limits of products with bounded functions

Fact. $\lim_{x\to a} g(x) = 0$ *if and only if* $\lim_{x\to a} |g(x)| = 0$.

Theorem. *If* $|f(x)| \leq M$, *and* $\lim_{x\to a} g(x) = 0$, *then*

$$\lim_{x\to a} f(x) \cdot g(x) = 0,$$

even if $\lim_{x\to a} f(x)$ *doesn't exist.*

Proof. We apply the normal Sandwich Theorem.

$$-M \cdot |g(x)| \leq f(x) \cdot g(x) \leq M \cdot |g(x)|,$$

and since $|g(x)| \to 0$, the function $f(x) \cdot g(x)$ is trapped to go to zero as well. $\square$

Exercise. We learned in class that $\lim_{x\to 0} \sin \frac{1}{x}$ does not exist. Show that $\lim_{x\to 0} x \sin \frac{1}{x} = 0$, nonetheless.