On orthoschemes and lattices

Nejc Zadnik

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Abstract

We compile the work on orthoschemes and CAT(0) property that has been done so far, define the concepts used in operating with orthoschemes and provide some examples of concepts that are useful for understanding the subject. We also compile work that has been done towards a 2010 conjecture of Brady and McCammond. This paper is intended as a quick introduction to studying orthoschemes for a newcomer to the field.

1 Introduction

Here we survey the work that has been done on the subject of orthoscheme complexes and CAT(0) property of the lattices related to said complexes. To understand orthoschemes, we first have to understand general order complexes.

Definition 1.1 (Order complex)**.** *The* order complex *of a poset P is a simplicial complex $|P|$ constructed in the following way. We have a vertex v_x in $|P|$ for every $x \in P$, an edge e_{xy} for all $x < y$ and more, we have a k-simplex in $|P|$ with a set of vertices $\{v_{x_0}, v_{x_1}, \ldots, v_{x_k}\}$ for every finite chain $x_0 < x_1 < \cdots < x_k$ in P. When P is bounded, $v_{\mathbf{0}}$ and $v_{\mathbf{1}}$ are the* endpoints *of $|P|$, and the edge $e_{\mathbf{01}}$ connecting them is its* diagonal*.*

Note that the reason we talk about order complexes is because orthoscheme complex puts a metric on the topology of the order complex, meaning we cannot understand orthoschemes without first understanding order complexes.

Definition 1.2 (Orthoscheme complex)**.** *The* orthoscheme complex *of a graded poset P is a metric version of its order complex $|P|$ that assigns the metric of a standard orthoscheme to every top dimensional simplex in $|P|$ (i.e. those corresponding to maximal chains $x_0 < x_1 < \cdots < x_n$) with v_{x_i} corresponding to v_i, resulting in the fact that for all elements $x < y$ in P, the length of the edge connecting v_x and v_y in $|P|$ is $\sqrt{k}$ where k is the rank of P_{xy}. When $|P|$ is turned into a PE complex in this way we say that $|P|$ is endowed with the* orthoscheme metric*. From now on $O(P)$ indicates an orthoscheme complex on P.*

Let's look at an example of a 3-dimensional orthoscheme complex. The distance from $(0,0,0)$ to $(1,0,0)$ is 1, $(0,0,0)$ to $(1,1,0)$ is $\sqrt{2}$, and $(0,0,0)$ to $(1,1,1)$ is $\sqrt{3}$. Notice here that the order of elements in the orthoscheme is important as it is an order complex.

We see that the difference is that in an orthoscheme, we add restrictions that we don't find in an ordinary simplex, as the lines connecting consecutive elements are of length 1 and the simplices representing chains are perpendicular between one another. This is useful because when we have predefined angles and lengths in a structure, a lot more information about properties of our studied space can be extracted. For example, as it can be seen in the definition theorem 1.2, if we have elements in the same chain, their distance can easily be measured (this question is not trivial in a general simplex complex).

Now that we have defined some terms, we can talk about the conjecture that was the reason this paper exists. The main goal of our work was proving the following conjecture from [2]:

Conjecture 1.3 (Curvature of NC_n)**.** *For every n, the orthoscheme complex of the non-crossing partition lattice NC_n is* CAT(0)*.*

Brady and McCammond were interested in this question as it closely related to Artin groups and their non-crossing partitions which is an area that defines some of their work.

We also have some reason to believe that the following conjecture is true, as it is a natural generalization of the above conjecture.

Conjecture 1.4 (Supersolvable lattice is CAT(0))**.** *Let S be a supersolvable lattice with an assigned orthoscheme complex. Then the orthoscheme complex of S is* CAT(0)*.*

2 Definitions and examples

Definition 2.1 (Bounded poset)**.** *A poset is* bounded below *if it has a minimum element* $\mathbf{0}$*,* bounded above *if it has a maximum element* $\mathbf{1}$*, or* bounded *if it has both. Elements* $\mathbf{0}, \mathbf{1}$ *are unique if they exist. A bounded poset is of* rank n *if it's maximal chain is of length* n*.*

The following example will hold some importance in the later parts as cubical complexes are the main thing of interest of this paper and as can be seen from the figure, its shape resembles a cube. I believe it is an easy to understand example that can be useful for initial intuition on the connection between lattices and shapes.

Example 1. Set $P = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\} = 2^{\{a,b,c\}}$ ordered by refinement is a poset and not a totally ordered set, as we have reflexivity, antisymmetry and transitivity but for example, the elements $\{a, c\}$ and $\{b\}$ are not comparable as they don't lie in the same chain.

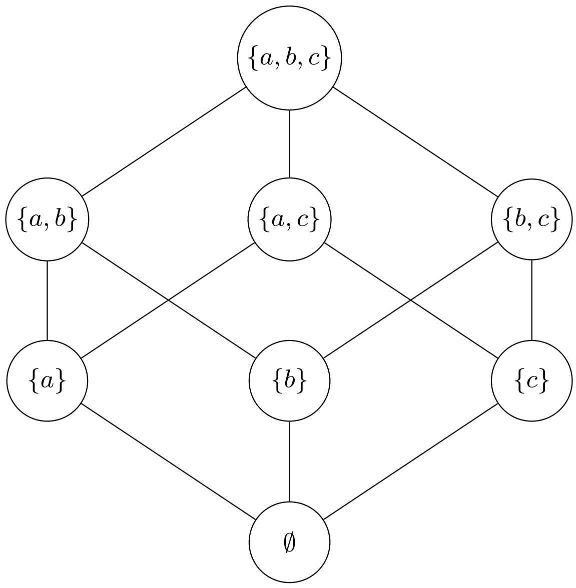

Figure 1: Hasse diagram of P ordered by refinement

Let's choose $\{a, c\}$ and $\{b\}$ in P as before. Their join is $\mathbf{1} = \{a, b, c\}$ and their meet is $\mathbf{0} = \emptyset$ (can be easily read from Figure 1).

Definition 2.2 (Euclidean polytope)**.** Euclidean polytope *is defined as a bounded intersection of a finite collection of closed half-spaces of a Euclidean space. There is also an equivalent definition, where we define it as the convex hull of a finite set of points.*

Definition 2.3 (Faces of Euclidean polytope)**.** *A* proper face *is a nonempty subset in the boundary of a closed half-space in which the entire polytope is contained. Every proper face of a polytope is also a polytope. We also have two* trivial faces*, which are the empty face* $\emptyset$ *and the whole polytope. The* dimension *of a face is defined as the dimension of the smallest affine subspace that contains it entirely.*

We define a 0-dimensional face as a *vertex* and a 1-dimensional face as an *edge*.

Example 2. A filled in unit square $([0, 1] \times [0, 1])$ is an example of a Euclidian polytope in $\mathbb{R}^2$ as it is a convex hull of points $(0, 0), (0, 1), (1, 0), (1, 1)$. If we denote the square as $\square ABCD$, its nontrivial faces are points A, B, C, D, and lines AB, BC, CD, AD.

Definition 2.4 (PE complex)**.** *A piecewise Euclidean complex (later denoted as* PE complex*) is the regular cell complex that we get when we glue together a disjoint union of Euclidean polytopes using isometric identifications of their faces.*

We usually demand that polytopes involved in the construction are embedded into the quotient and that the intersection of any two polytopes also represent a face (intersections are 'nice'). There exists a notion of a complex with *finite shapes*, which means that only finitely many isometry types of polytopes were used in the construction.

Example 3. From a square mentioned in the previous example, we can generate a PE complex generated by faces of dimension 0 - points, 1 - lines, 2 - face (the empty set is of dimension -1), by defining how they are connected.

Definition 2.5 (Geodesic)**.** *A* geodesic *in a metric space is an isometric embedding of a metric interval.* Local geodesic *is a weaker notions that only requires the image curve to be locally length minimizing (meaning that it is a local isometric embedding).*

Definition 2.6 (Geodesic loops)**.** *A* geodesic loop *is an isometric embedding of a metric circle.* Local geodesic loop *is defined analogously to the above definition for local geodesic.*

Example 4. A path more than halfway along the equator of a 2-sphere is a local geodesic but not a geodesic and a loop that travels around the equator twice is a local geodesic loop but not a geodesic loop. A loop in a PS complex of length less than 2π is called *short* and a PS complex that contains no short local geodesic loops is called *large*.

Remark. A piecewise spherical complex (or PS) is a piecewise Euclidean complex for balls.

Definition 2.7 (Curvature conditions)**.** *Suppose $|K|$ is a PE complex with finite shapes*

- *If the link of every cell in $|K|$ is large, then $|K|$ is said to be* nonpositively curved *or* locally CAT(0)*, moreover if $|K|$ is connected and simply connected, then $|K|$ is* CAT(0)*. Note that such a complex $|K|$ is contractible.*
- *If $|K|$ is a PS complex and the link of every cell in $|K|$ is large, then $|K|$ is* locally CAT(1)*. If $|K|$ itself is large, then $|K|$ is* CAT(1)*.*

Example 5. The following figure (similar to Figure 11.1 in [1]) shows the relation between a standard $\mathbb{R}^2$ space and a space of nonpositive and nonnegative curvature. We can easily see why they are called this way, as $d_2(z,z') < d_1(z,z') < d_3(z,z')$, so $d_2(z,z') - d_1(z,z') < 0$ and $d_3(z,z') - d_1(z,z') > 0$ where the index denotes the triangle in figure from left to right (as we are always talking about Euclidean distance).

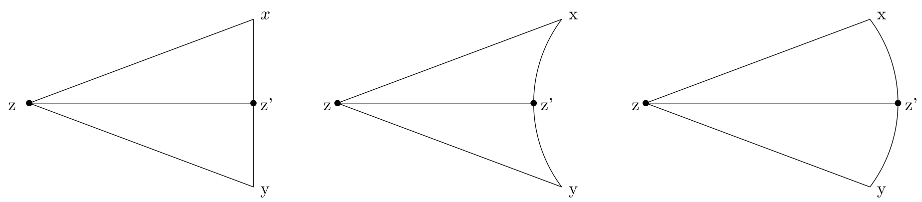

Figure 2: A geodesic triangle in $\mathbb{R}^2$, in CAT(0) and in CAT(1)

Definition 2.8 (Orthoschemes)**.** *We first define a list of $n+1$ in $\mathbb{R}^n$ points $(v_0, v_1, \ldots, v_n)$ in that order. Now we can define a vector $\{u_i\}$ that ranges from v_{i-1} to v_i. We want the vectors $\{u_i\}$ to form an orthogonal basis because this way we can define metric n-simplex called an n-*orthoscheme *which is the convex hull of $\{v_i\}$ (denoted by $\mathsf{Ortho}(v_0, \ldots, v_n)$ or O). In addition, if vectors $\{u_i\}$ form an orthonormal basis, this orthoscheme is called a* standard n-orthoscheme*.*

Notice that every face of an orthoscheme is also an orthoscheme. We take special interest in faces defined by consecutive vertices. We use $O(i,j)$ to denote the face $\mathsf{Ortho}(v_i, \ldots, v_j)$ of $O = \mathsf{Ortho}(v_0, \ldots, v_n)$. *Vertices* of the orthoscheme are points v_i, its *endpoints* are v_0 and v_n, and its *diagonal* is the edge connecting v_0 and v_n.

Example 6. 3-dimensional orthoscheme and a union of 6 3-dimensional orthoschemes glued in a "nice" way.

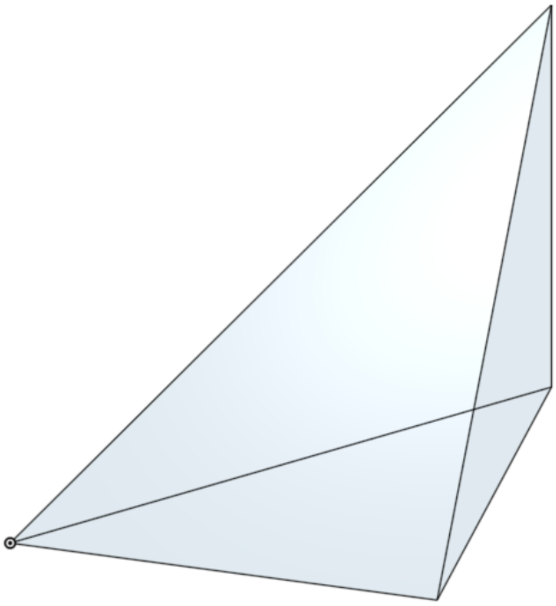

Figure 3: 3-dimensional orthoscheme

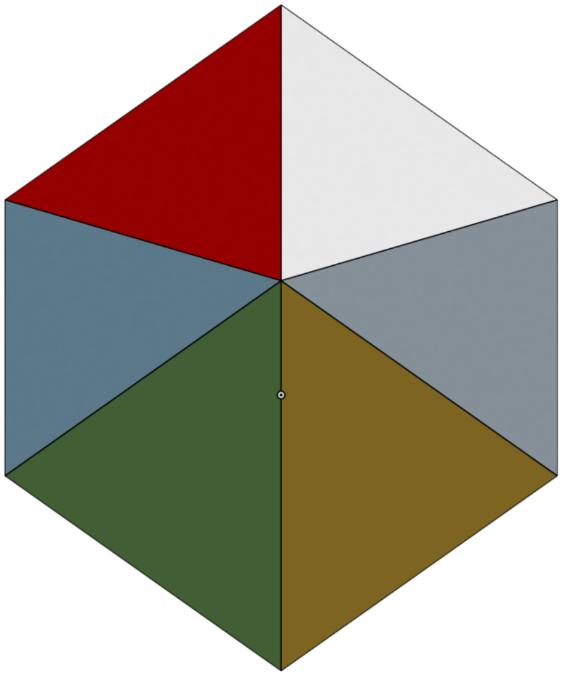

Figure 4: View from above of 6 orthoschemes forming a cube

The following figure will show an exploded view of the 6 orthoschemes

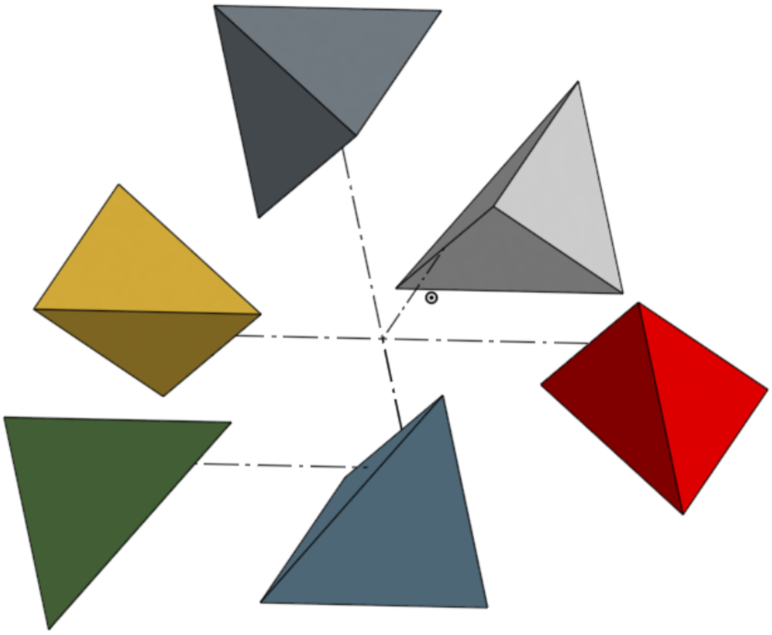

Figure 5: Exploded view of 3

Definition 2.9 (Non-crossing partition)**.** *Partition of a set* $[n] = \{1, \ldots, n\}$ *is a* noncrossing *partition if, when you place the elements* $1, \ldots, n$ *as vertices of regular* n*-gon in consecutive order, convex hulls of its blocks are pairwise disjoint.*

Example 7. The following example can be found in [4].

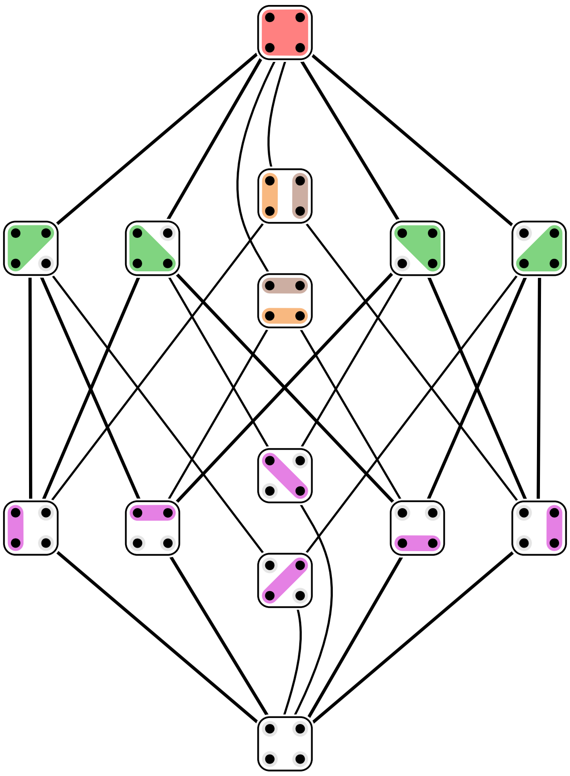

Figure 6: 14 non-crossing partitions of a 4 element set ordered by refinement in a Hasse diagram (NC_4)

Here we can see all the non-crossing partitions and the order between them and as we can see, the partition $\{\{1, 3\}, \{2, 4\}\}$ is missing as that is a crossing partition. It's the only crossing partition of this 4 element set. In the following figure, we can easily see, why that is the crossing element.

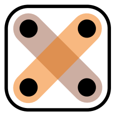

Figure 7: Crossing element $\{\{1,3\},\{2,4\}\}$

Example 8. The following diagram shows NC_4 without the top or bottom element. Since the orthoscheme complex of a Boolean lattice is a cube, a full-height Boolean sublattice of a lattice L gives a full-dimensional cube sitting inside the orthoscheme complex of L. Every colour represents a different cube (5 in total). This diagram may be of some use, when looking at the way different faces interact.

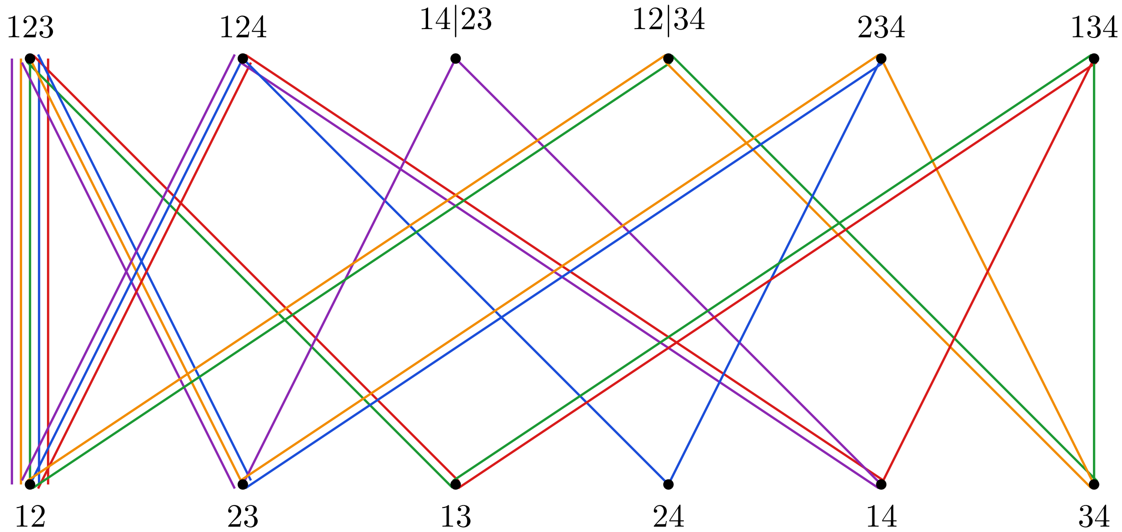

Figure 8: Representation of NC_4 with top and bottom element removed

3 Main problem

Main goal of our work was to make progress towards proving conjecture 1.3 from [2] and conjecture 1.4 but we were unfortunately not very successful. Some work on this problem has been done by other authors. It is best we begin with a word on modular and supersolvable lattices.

Definition 3.1 (Modular lattice)**.** *A* modular lattice *L is a ranked lattice that satisfies*

$$\rho(x) + \rho(y) = \rho(x \vee y) + \rho(x \wedge y) \quad \forall x, y \in L$$

where ρ is the rank function.

Example 9. The following figure shows a simple modular lattice.

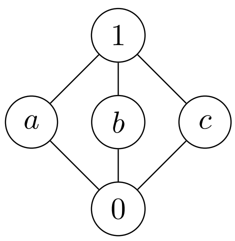

Figure 9: Graph of a simple modular lattice on elements $a, b, c, \mathbf{0}, \mathbf{1}$

Definition 3.2 (Supersolvable lattice)**.** *A lattice L is a* supersolvable lattice *if it has a chief chain m and for any chain c, the smallest sublattice in L that still contains all of the elements of m and c is distributive.*

Example 10. Example 1 is an example of a supersolvable lattice.

The reason we are interested in modular lattices is that there are some similarities between them and supersolvable lattices, so there exists a possibility that the proofs could be extended to supersolvable and maybe even non-crossing partition lattices. It is useful to remember that every modular lattice is supersolvable, so theorems that apply to supersolvable lattices also apply to modular lattices.

Theorem 3.3 (Modular lattice is CAT(0))**.** *Let M be a modular lattice of finite rank. Then $O(M)$ is* CAT(0).

As the proof of the theorem Theorem 3.3 can be found in the paper [3] (Section 7.4), where the theorem is stated, we believe it is of more value if we do a sketch of the proof and explain the steps.

Sketch of proof. We will be using Euclidean metric (and denoting the distance function as $d(x, y)$) and we pick $x, y \in O(M)$ and by a theorem of Dedekind-Birkhoff (theorem 7.12 in [3]) there also exists a 'distributive' lattice $\mathcal{D}$ s.t. $x, y \in O(\mathcal{D})$. Here, 'distributive' is a generalization of Boolean, and we henceforth assume that D is a Boolean sublattice, so that $|\mathcal{D}|$ is a hypercube.
The main theorem that the proof relies on is stated in [1] (Proposition 11.4) and it states that for a space to be CAT(0), it suffices to show that for any $z \in O(M), t \in [0, 1]$ and $p \in \gamma(x, y)$ (geodesic from x to y) with $d(x, p) = td(x, y)$ it holds that

$$d^2(z, p) \leq (1 - t)d^2(z, x) + td^2(z, y) - t(1 - t)d^2(x, y).$$

This inequality can be easily understood looking at fig. 2 as a triangle inequality for curved triangles. The next step is done in the following way; a maximal chain C in $\mathcal{D}$ with $p \in |C|$. Here comes the main step of the proof. We define z'. We think of $O(\mathcal{D})$ as a hypercube with $\gamma(x, y) \in O(\mathcal{D})$ and of z' as a projection of z onto the simplex corresponding to C (thought of as sitting inside $\mathcal{D}$). So x, y, p, z' can be viewed as points in that hypercube and by that, we get the equality

$$d^2(z', p) = (1 - t)d^2(z', x) + td^2(z', y) - t(1 - t)d^2(x, y),$$

stemming from the fact that we are using Euclidean distance. From this we can deduce the rest. The proof in full detail is stated in the paper. $\square$

Other known results on the non-crossing partition lattice are less general. The following was shown in [5].

Theorem 3.4 (NC_6 is CAT(0))**.** *The non-crossing partition complex NC_6 is* CAT*(0).*

A big problem we have stumbled upon in orthoschemes is finding a geodesic between elements that do not lie in the same chain. Remember that for the same chain, the distance is just $\sqrt{k}$, where k is the distance between those two elements in the chain. In the following lemma, we actually put an upper limit to the distance for a single orthoscheme and a combination of orthoschemes, glued in such a way that they produce a bounded lattice (unique **0** and **1** element).

Lemma 3.5. *Let $O(L)$ be an orthoscheme complex generated from a lattice L, where L is a graded bounded lattice. Then $\forall x, y \in O(L) : d(x, y) \leq \sqrt{n}$ where n is the grade of the orthoscheme complex.*

Proof. Let's choose an n-dimensional orthoscheme complex $O_n(L)$, where L is bounded. As it is generated by a bounded lattice, it has a bottom element **0** and top element **1** and $d(\mathbf{0}, \mathbf{1}) = \sqrt{n}$ by definition. Note that the diagonal exists in every orthoscheme simplex of height n. Now let's choose a point c on the geodesic from **0** to **1** that is equidistant from both points, so the distance $d(\mathbf{0}, c) = d(c, \mathbf{1}) = \frac{\sqrt{n}}{2}$ and create an n-dimensional closed ball $\overline{B}(c, \frac{\sqrt{n}}{2})$.
We have $c = (\frac{1}{2}, \ldots, \frac{1}{2})$ and arbitrary $x = (x_1, \ldots, x_n)$, both in $[0, 1]^n$. We can write the distance $d(x, c) = \sqrt{\sum_{i=1}^{n}(x_i - \frac{1}{2})^2} \implies d^2(x, c) = \sum_{i=1}^{n}(x_i - \frac{1}{2})^2$. We also know that $|x_i - \frac{1}{2}| \leq \frac{1}{2}$ as $x_i \in [0, 1]$. This means $||x - c||_\infty \leq \frac{1}{2}$. Now we can use the fact that

$$||x||_\infty \leq ||x||_2 \leq \sqrt{n}||x||_\infty$$

and get $d^2(x, c) = ||x - c||_2^2 \leq n||x - c||_\infty^2 \leq \frac{n}{4} \implies d(x, c) \leq \frac{\sqrt{n}}{2}$.
By the way, glueings are allowed, they should also match up in this geodesic, so every orthoscheme in this complex should lie in the ball, if at least one lies in it. If we now choose arbitrary $x, y \in O(L)$, we know that $d(x, c) + d(c, y) \leq \frac{\sqrt{n}}{2} + \frac{\sqrt{n}}{2} = \sqrt{n}$ as $x, y \in O(L) \subset \overline{B}(c, \frac{\sqrt{n}}{2})$. $\square$

The path constructed in the above lemma is not in general a geodesic, that is, there may be a shorter path between x and y. We don't know a general way to recognize geodesics in the orthoscheme metric.

4 Finishing words

Unfortunately we haven't been able to prove anything of real significance towards solving this problem, but we have compiled the work we have done in hopes that it will help somebody who is interested in this problem, to easily pick up this problem and take a shot at it themselves.

I also owe great thanks to my mentor Russ Woodroofe for all the help and insightful remarks about the problem and on the structure of the paper.

References

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