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(Final project paper)

Particijska lastnost simplicialnih kompleksov

(Partitionability of simplicial complexes)

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Izvleček:
Particijaksa lastnost simplicialnih kompleksov je lastnost, ki se je prvič pojavila v delih Stanley-ja v sedemsetih letih prejšnjega stoletja. Motivirala ga je druga lastnost simplicialnih kompleksov, imenovana lupinavost, čeprav je šibkejša od slednje lastnosti. Med najpomembnejše domneve na tem področju spada t.i. particijska domneva (ang. Partitionability Conjecture), ki jo je leta 1979 postavil Stanley in Garsia. Ta domneva je trdila, da je vsak Cohen-Macaulay kompleks particijski. Po skoraj 40 letih so domnevo ovrgli Duval, Goeckner, Klivans in Martin v članku iz leta 2016. Njihov protiprimer je simplicialni kompleks na samo 16 točkah, ki je Cohen-Macaulayev, vendar ni particijski.
Čeprav se je izkazalo, da je domneva napačna, se je na tem področju postavilo veliko število drugih vprašanj. Ali particijska domneva velja v dimneziji 2? Kaj šteje h-vektor Cohen-Macaulayevega kompleksa?
Obstaja še veliko drugih neodgovorjenih vprašanj o particijski lasnosti. V nalogi je lastnost preučevana na Poincaréjevi homološki sferi. Naše glavno vprašanje je, ali so vse njene triangulacije particijske. Po podatkovni bazi simplicialnih kompleksov Masahira Hachimorija obstaja 16-točkovna triangulacija, za katero je že znano, da je particijska. V našem delu obravnavamo eno od njegovih simetričnih 24-točkovnih triangulacij.

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Abstract:

Partitionability of simplicial complexes is a property that first appeared in the study of shellable and Cohen-Macaulay complexes in the 1970's. It was motivated by another property of simplicial complexes, called shellability, although it is weaker than the latter property. One of the most important conjectures in this field was the so-called Partitionability Conjecture, set by Stanley and Garsia in 1979. This conjecture stated that every Cohen-Macaulay complex is partitionable. It was an open problem for almost 40 years, until until Duval, Goeckner, Klivans and Martin constructed a counterexample in a 2016 paper. Their counterexample is a simplicial complex on only 16 vertices that is Cohen-Macaulay, but not partitionable.
Although the conjecture turned out to be false, it raised a lot of other questions in this area: does the Partitionability Conjecture hold in dimension two? What does the h-vector of a Cohen-Macaulay complex count?
There are many other unanswered questions about partitionability and we will consider some about partitionability of the Poincaré homology sphere. Our main question is whether all its triangulations are partitionable. According to Masahiro Hachimori's database of simplicial complexes, there is a 16-point triangulation which is already known to be partitionable. In this paper, we consider one of its symmetric 24-point triangulations.

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Contents

List of Figures

Appendices

List of Abbreviations

i.e. that is
e.g. for example

1 Introduction

Partitionability of simplicial complexes is a property that first appeared in the study of shellable and Cohen-Macaulay complexes in the 1970's. It was motivated by another property of simplicial complexes, called shellability, although it is weaker than the latter property. One of the most important conjectures in this field was the so-called Partitionability Conjecture, set by Stanley in 1979. This conjecture stated that every Cohen-Macaulay complex is partitionable. It was an open problem for almost 40 years, until until Duval, Goeckner, Klivans and Martin constructed a counterexample in a 2016 paper. Their counterexample is a simplicial complex on only 16 vertices that is Cohen-Macaulay, but not partitionable.
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There are many other unanswered questions about partitionability and we will consider some about partitionability of the Poincaré homology sphere. Our main question is whether all its triangulations are partitionable. According to Masahiro Hachimori's database of simplicial complexes, there is a 16-point triangulation which is already known to be partitionable. In this paper, we consider one of its symmetric 24-point triangulations.

This paper is organized as follows: In the second section we introduce some basic definitions and theorems about simplicial complexes in general and also some basics about homology theory. In the third section, we formally define partitionability, while in the following section we introduce the Poincaré homology sphere. Finally, in the last chapter, we give some results about partitionability of Poincaré homology sphere.

2 Background

In this section we will define the main mathematical object that we will use in this paper, that of simplicial complexes. There are a few ways to define simplicial complexes, but we will begin with a purely combinatorial approach, by defining abstract simplicial complexes. We will then try to find their topological equivalent, i.e. the geometrical realization of abstract simplicial complexes. Throughout the paper, we will see that some of the results and theorems about simplicial complexes sometimes come from combinatorial and algebraic arguments, while others are purely topological.

First, we define an abstract simplicial complex, which is a collection of subsets of a fixed finite set that is closed under inclusion:

Definition 2.1. An ***abstract simplicial complex*** Δ on a vertex set V is a collection of subsets such that whenever $\sigma \in \Delta$ and $\tau \subset \sigma$, then also $\tau \in \Delta$.

Elements of Δ are called ***faces***. More precisely:

- 1-element subsets are 0-dimensional faces
- 2-element subsets are 1-dimensional faces
- 3-element subsets are 2-dimensional faces
- etc.

In general, n-element subsets are called $(n-1)$***-simplices***, or $(n-1)$-dimensional faces. Maximal faces (with respect to inclusion) are called ***facets***. Dimension of a simplicial complex is the maximal dimension among its facets. Further, a complex is ***pure*** if all its facets have the same dimension.

Example 2.2. An easy example of an abstract simplicial complex on vertex set $V = \{1, 2, 3, 4\}$ is

$$\Delta = \{\{1, 2, 3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1\}, \{2\}, \{3\}, \{3, 4\}, \{4\}, \emptyset\}.$$

This complex has one 2-dimensional face, four 1-dimensional faces and four 0-dimensional faces. It has two facets, namely $\{1, 2, 3\}$ and $\{3, 4\}$, and is therefore nonpure.

Now let us see how we can build a topological space C that corresponds to an abstract simplicial complex.
First, consider a simple abstract $(n-1)$-simplex. Place n *linearly independent* points into space $\mathbb{R}^{n-1}$, namely $v_1, v_2, ..., v_n$. Now we can identify this $(n-1)$-simplex with the geometrical $(n-1)$-simplex, i.e. with the set

$$C = \Big\{\gamma_1 v_1 + \gamma_2 v_2 + ... + \gamma_n v_n : \sum_{i=1}^{n} \gamma_i = 1, \gamma_i \geq 0 \text{ for } i = 1, ..., n\Big\}.$$

It is not hard to generalize this construction to any simplicial complex Δ over V:

1. Find the dimension d of Δ, i.e. the dimension of its largest facet
2. Place n points (corresponding to the vertex set V) into space $\mathbb{R}^{n-1}$,
3. Identify each face of Δ with the geometric simplex spanned by the corresponding vertices,
4. We obtain the topological space as the union of all such geometric simplices.

This definition is also equivalent to saying that a simplical complex on n vertices is a subcomplex of a simplex generated by its n vertices (embedded in $\mathbb{R}^{n-1}$).

Notice that this way we get the following correspondence between the abstract simplicial complex and the topological space:

- 0-dimensional faces correspond to ***points***
- 1-dimensional faces correspond to ***line segments***
- 2-dimensional faces correspond to ***triangles***
- 3-dimensional faces correspond to ***tetrahedra***.

Example 2.3. Consider the same abstract simplical complex as in the previous example: $\Delta = \{\{1,2,3\},\{1,2\},\{2,3\},\{1,3\},\{1\},\{2\},\{3\},\{3,4\},\{4\}\}$. We embed the four points into $\mathbb{R}^3$. The 2-dimensional face (i.e. the largest facet) becomes a triangle in our topological space, the four 1-dimensional faces become line segments, while the four 0-dimensional faces correspond to points in $\mathbb{R}^3$. Of course, we can also embed these objects into space R^2.

Basic counting information is encoded with the f-vector:

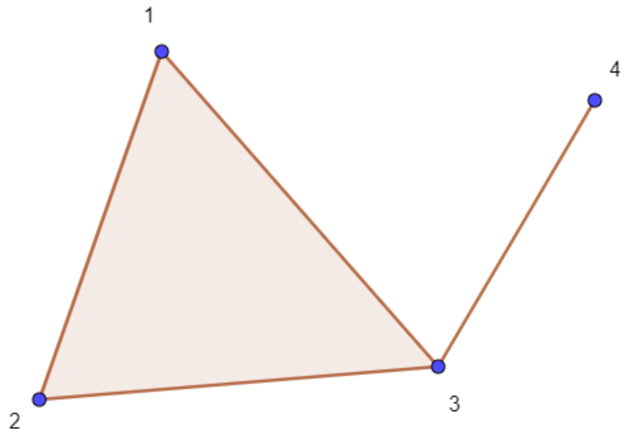

Figure 1: Topological realization of Δ

Definition 2.4. The ***f-vector*** of a simplicial complex Δ of dimension d is a vector $(f_{-1},, f_0, f_1, f_2, ..., f_d)$, where f_i denotes the number of its i-dimensional faces. Additionally, we define $f_{-1} = 0$ if Δ is empty and $f_{-1} = 1$ otherwise, as we consider the empty set to be the (-1)-dimensional face. By using the components of the f-vector as the coefficients of a polynomial, we obtain the ***f-polynomial***:

$$F_\Delta(x) = f_d x^{d+1} + f_{d-1} x^d + ... + f_0 x^1 + f_{-1}.$$

The f-polynomial gives rise to the famous topological invariant called the reduced ***Euler characteristic***:

$$\tilde{\chi} = F_\Delta(-1) = \sum_{i=0}^{d} (-1)^{i-1} f_{i-1}$$

.

Sometimes it is more convenient to study another vector similar to the f-vector, called the h-vector:

Definition 2.5. We define the ***h-vector***, $(h_0, h_1, ..., h_{d+1})$, to be the coefficients of the polynomial

$$H_\Delta = rev_{d+1}(rev_{d+1} F_\Delta(x-1)),$$

where rev is a function that sends $F(x)$ to $F(\frac{1}{x}) \cdot x^{d+1}$.

Now we introduce two operations on simplicial complexes, that may remind us of subtraction and multiplication.

Definition 2.6. A ***relative simplicial complex*** $\Phi = (\Delta, \Gamma)$ is a pair of simplicial complexes Δ and Γ, where the faces of Φ are the faces in Δ that are not in Γ. Here, Δ is the ***absolute*** part of Φ and Γ is the ***relative*** part of Φ.

Note that a relative simplicial complex need not be a simplicial complex, as it might not be closed under inclusion.

Definition 2.7. The ***join*** of two simplicial complexes Δ_1 and Δ_2 with disjoint sets of vertices is a simplicial complex $\Delta_1 * \Delta_2$, whose faces are of the form $\sigma_1 \cup \sigma_2$, where σ_1 is a face of Δ_1 and σ_2 is a face of Δ_2. In other words,

$$\Delta_1 * \Delta_2 := \{\sigma_1 \cup \sigma_2 : \ \sigma_1 \in \Delta_1, \sigma_2 \in \Delta_2\}.$$

Here we show how to compute the f-vector of two simplicial complexes:

Lemma 2.8. *Let Δ_1 and Δ_2 be two simplicial complexes with disjoint vertex sets, and with corresponding f-vectors $(f'_{-1}, f'_0, f'_1, f'_2, ..., f'_{k-1})$ and $(f''_{-1}, f''_0, f''_1, f''_2, ..., f''_{l-1})$. Then the generating function for the f-vector of the join $\Delta = \Delta_1 * \Delta_2$ is given by the product of the f-polynomials of Δ_1 and Δ_2.*

Proof. Let $F_{\Delta_1}(x) = f'_{d-1}x^d + f'_{d-2}x^{d-1} + ... + f'_0x^1 + f'_{-1}$ be the generating function for the f-vector of Δ_1 (i.e. its f-polynomial) and $F_{\Delta_2}(x) = f''_{k-1}x^k + f''_{k-2}x^{k-1} + ... + f''_0x^1 + f''_{-1}$ be the generating function for the f-vector of Δ_2.
Now notice that we obtain the i-dimensional faces as the union of r-dimensional faces of Δ_1 (there are f'_{r+1} of them) and s-dimensional faces of Δ_2 (there are f''_{s+1} of them), where $r + s = i$. Therefore, the number of i-dimensional faces of Δ is

$$f_i = \sum_{r+s=i} f_{r+1} \cdot f_{s+1}.$$

But then notice that then by multiplying $F_{\Delta_1}(x)$ and $F_{\Delta_2}(x)$ we obtain exactly

$$f_{k+l-1}x^{k+l} + f_{k+l-2}x^{k+l-1} + ... + f_0x^1 + f_{-1} = F_\Delta(x),$$

which is the generating function for the f-vector of the join $\Delta = \Delta_1 * \Delta_2$, as desired. □

3 Shellability and Partitionability of Simplicial Complexes

3.1 Definitions and Basic Properties

A pure d-dimensional simplicial complex is said to be ***shellable*** if there is a linear ordering of its facets $\sigma_1, \sigma_2, ..., \sigma_n$ such that $(\cup_{j<i}\sigma_j) \cap \sigma_i$ is a pure $(d-1)$-dimensional simplicial complex, for all $i \geq 2$. In other words, we can obtain the simplicial complex by gluing together facet by facet in a way that the glued part is of dimension one less than our simplicial complex.

Now consider the ***face poset*** of a simplicial complex Δ: the partial order on the faces of Δ with respect to inclusion. We say that Δ is ***partitionable*** if the poset can be partitioned into intervals of the form $[R(\sigma), \sigma] = \{\rho \in \Delta \mid R(\sigma) \subseteq \rho \subseteq \sigma, \sigma \text{ is a facet of } \Delta\}$.

Here we give some examples of shellability and partitionability:

Example 3.1. Consider two simplicial complexes, Δ_1 (left side of Figure 2) and Δ_2 (right side of Figure 2). Simplicial complex Δ_1 is clearly shellable, with shelling order of its facets (A, B). The complex is also partitionable, with partitioning $[\emptyset, A], [4, B]$. Complex Δ_2 is not shellable: neither of the orderings of its facets (A, B) and (B, A) is a shelling, as $A \cap B = B \cap A = \{3\}$, which is a 0-dimensional and not 1-dimensional complex. Further, Δ_2 is not even partitionable. Suppose it is, with the following partitioning scheme: $[\sigma_1, A], [\sigma_2, B]$. Clearly, the empty face has to be in one of these two intervals. Because of the symmetry, assume the empty set is in $[\sigma_1, A]$, i.e. that $\sigma_1 = \{\emptyset\}$. But now the second interval $[\sigma_2, B]$ cannot contain both faces $\{3\}$ and $\{4\}$, as σ_2 is either $\{3\}$ or $\{4\}$ (see Figure 4).

Example 3.2. Now consider simplicial complex Δ_3, i.e. the boundary of the tetrahedron on vertices $\{1, 2, 3, 4\}$, with a triangle attached to one of its vertices (see Figure 5). It is not hard to see that Δ_3 is not shellable. Namely, the additional triangle $\tau = \{3, 5, 6\}$ intersects the tetrahedron in a point, i.e. a 0-dimensional face. Therefore, regardless of the order of the facets, the facet τ will intersect its neighboring facets in a 0-dimensional face, instead of 1-dimensional face.

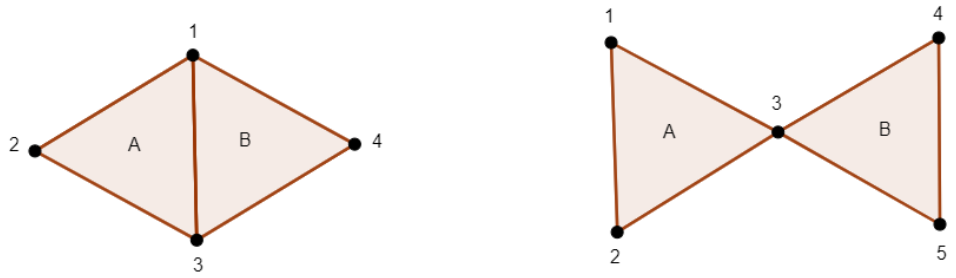

Figure 2: Simplicial complexes Δ_1 and Δ_2

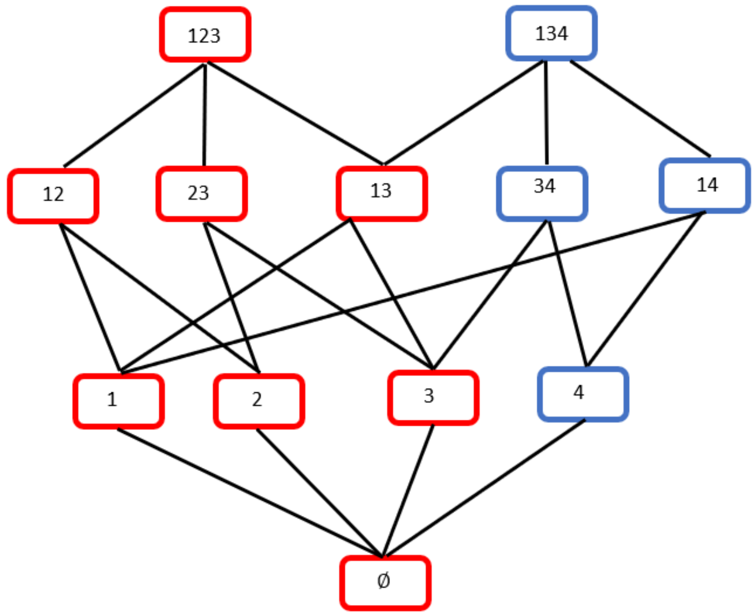

Figure 3: Partitioning scheme for Δ_1

However, Δ_3 is partitionable, with partitioning scheme $[\emptyset, 356], [1, 123], [2, 234], [4, 134], [124, 124]$.

Looking at the example of Δ_1, one might notice that from every shelling we could construct a partitioning scheme. In other words:

Theorem 3.3. *Every shellable simplicial complex is also partitionable.*

Proof. Let $\sigma_1, \sigma_2, ..., \sigma_m$ be a shelling order of simplicial complex Δ and denote by Δ_i the simplicial complex generated by $\sigma_1, ...\sigma_i$, i.e. $\Delta_i = \langle \sigma_1, ...\sigma_i \rangle$. Additionally we define $\Delta_0 = \emptyset$.We will prove that the following intervals form a partitioning scheme of Δ:

$$[\tau_i, \sigma_i], \text{ where } \tau_i = \{x \in \sigma_i : \sigma_i \setminus \{x\} \in \Delta_{i-1}\}.$$

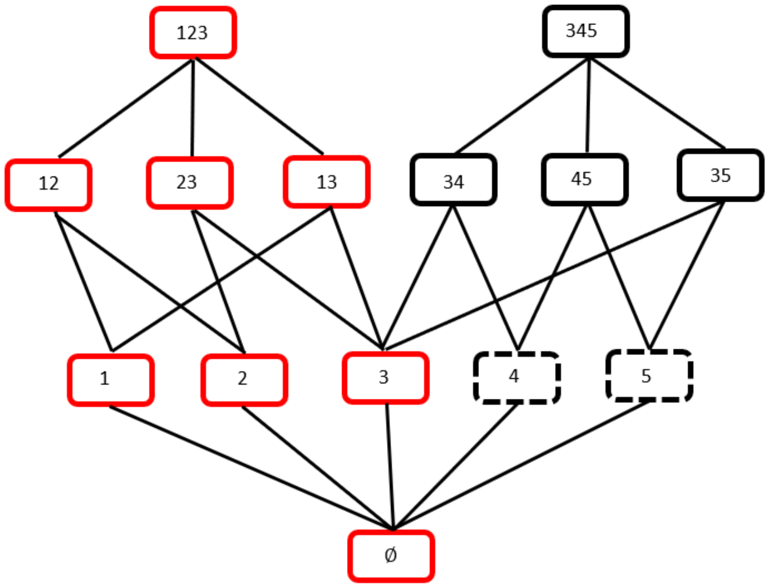

Figure 4: Partitioning fails for Δ_2

In other words, we want to prove that any "new" face γ of σ_i that is not contained in Δ_{i-1}, has to contain τ_i, i.e. γ contained in the interval $[\tau_i, \sigma_i]$.
($\Rightarrow$) First, let γ a face of σ_i that does not contain τ_i, i.e. $\tau_i \subsetneqq \gamma$. That means that there is some vertex x in τ_i that is not in γ. But then $\gamma \subseteq \sigma_i \setminus \{x\}$, so it is a face of Δ_{i-1}.
($\Leftarrow$) Now suppose that γ contains τ_i, i.e. $\tau_i \subseteq \gamma$. By definition of shellability, $\langle\sigma_i\rangle \cap \Delta_{i-1}$ is of codimension 1, its each face γ is in $\sigma_i \setminus \{x\}$, for some vertex x. But then γ is not contained in Δ_{i-1}, as desired. □

Example 3.4. Consider again the simplicial complex Δ_1 with facets $\{1, 2, 3\}$ and $\{1, 3, 4\}$ (see the left side of Figure 2). The first interval of the partitioning scheme is $[\tau_1, \sigma_1]$ will be $[\emptyset, A]$. For the second interval, notice that vertex 4 is the only vertex in B for which $B \setminus \{4\} \in \Delta_1 (= A)$ holds. Therefore, $\tau_2 = \{4\}$, and the second interval in the scheme is $[\{4\}, B]$.

Example 3.5. Consider three tetrahedra glued together over the common simplex $\{1, 2, 3\}$: $\sigma_1 = \{1, 2, 3, 4\}$, $\sigma_2 = \{1, 2, 3, 5\}$ and $\sigma_3 = \{1, 2, 3, 6\}$, shown on Figure 6. This simplicial complex is clearly shellable with shelling order $\sigma_1, \sigma_2, \sigma_3$ (all the facets all intersect at a simplicial complex $\{1, 2, 3\}$ of codimension 1).
Here, $\tau_2 = \{5\}$ and $\tau_3 = \{6\}$. So, the partitioning scheme is $[\emptyset, \sigma_1]$, $[5, \sigma_2]$ and $[6, \sigma_3]$.

A natural question to ask whether there are some necessary and sufficient conditions for shellability. Indeed, it turns out in the one dimension case, we have the following result.

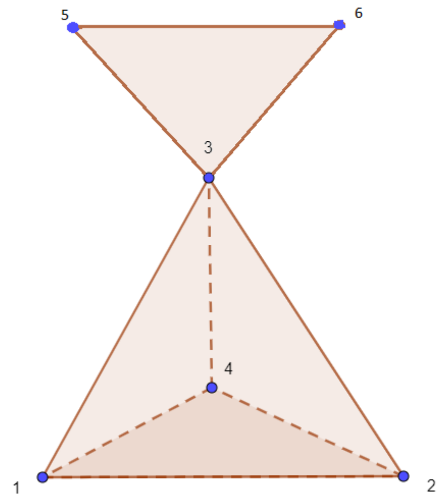

Figure 5: Simplicial complex Δ_3

Lemma 3.6. *A pure 1-dimensional simplicial complex is shellable if and only if it is connected.*

Proof. First, suppose that a disconnected pure 1-dimensional simplicial complex (i.e. a graph) with shelling $e_1, e_2, ..., e_n$. Let C be the connected component containing e_1 and let e_i be the first facet, i.e. edge that is not in C. But then $(\cup_{j<i} e_j) \cap e_i$ is the empty set, which is clearly not 1-dimensional, contradiction.
On the other hand, suppose we have a connected 1-dimensional simplicial complex Δ, i.e. a connected graph. Start with any edge e_1. At each step add an edge e_i such that $e_1, e_2, ..., e_{i-1}$ and e_i are in the same connected component of Δ. So, $(\cup_{j<i} e_j) \cap \{e_j\}$ is either a single vertex $\{v\}$ or two vertices $\{v, u\}$, both pure 0-dimensional complex, as desired. Because Δ is connected, we are guaranteed that we will eventually order all edges (i.e. facets) of Δ, therefore Δ is shellable. □

We now give one result about pure 2-dimensional simplicial complexes:

Theorem 3.7. *Every simplicial complex that is homeomorphic to a 2-ball is shellable.*

Unfortunately, it turns out that in dimension 2 and higher dimensions there is no easy characterization of shellability. Therefore, we will focus our attention on a weaker

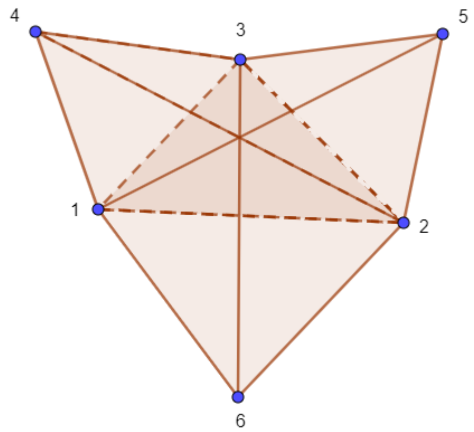

Figure 6: Three glued tetrahedra

property, namely partitionability. Here we give a result describing the partition in terms of the h-vector.

Theorem 3.8. *Let Δ be a pure partitionable d-dimensional simplicial complex and let $(h_0, h_1, ..., h_{d+1})$ be its h-vector. Then the component h_i counts the number of intervals of height $d+1-i$ in the partitioning scheme.*

Proof. Consider the f-polynomial of Δ:
In the partitioning scheme, the interval of kind $[\emptyset, \text{facet}]$ contributes a total of 2^d faces of Δ. As the interval $[\emptyset, \text{facet}]$ is isomorphic to the Boolean lattice B_d, we see that the interval contributes 1 face of dimension 0, $\binom{d+1}{1}$ faces of dimension 1, $\binom{d+1}{2}$ faces of dimension 2, etc. In other words, this interval contributes $r_0(1+x)^{d+1}$ to the f-polynomial (where $r_0 = 1$).
The interval of kind $[\text{vertex}, \text{facet}]$ is isomorphic to the Boolean lattice B_{d-1}. It contributes 1 face of dimension 1, $\binom{d}{1}$ faces of dimension 2, etc. If r_1 is the number of such intervals in the partitioning scheme, the corresponding generating function is $r_1 x(1+x)^d$.
In general, the interval of kind $[(i-1)\text{-dimensional face}, \text{facet}]$ (i.e. interval of height $d-i+1$) is isomorphic to Boolean lattice B_{d-i}. Each interval corresponds to the generating function $x^i(1+x)^{d-i+1}$. If there are r_{i+1} such intervals, the corresponding generating function is $r_i x^i(1+x)^{d-i+1}$.
Therefore, when we sum all these generating functions, we obtain the generating function for the total number of faces in Δ, i.e. the f-polynomial of Δ:

$$F_\Delta(x) = r_0(1+x)^{d+1} + r_1 x(1+x)^d + ... + r_i \cdot x^i(1+x)^{d+1-i} + ... + r_{d+1}x^{d+1}.$$

Now,

$$H_\Delta = rev_{d+1}((rev_{d+1}F_\Delta)(x-1)).$$

Notice that $rev_{d+1}(r_i \cdot x^i(1+x)^{d+1-i}) = r_i(1+x)^{d+1-i}$, so

$$(rev_{d+1}F_\Delta)(x-1) = r_0x^{d+1} + r_1x^d + ... + r_ix^{d+1-i} + ... + r_{d+1}.$$

Further,

$$H_\Delta(x) = rev_{d+1}(r_0x^{d+1} + r_1x^d + ... + r_ix^{d+1-i} + ... + r_{d+1}),$$

i.e.

$$H_\Delta(x) = r_0 + r_1x^1 + ... + r_ix^i + ... + r_{d+1}x^{d+1},$$

meaning that $h_i = r_i$, for all $i = 0, 1, ..., d$.

□

From this theorem, we get one trivial consequence:

Theorem 3.9. *The h-vector of partitionable simplicial complexes is always non-negative.*

For more details about partitionability and shellability see [2], [14] and [10].

3.2 Partitionability Conjecture

In algebraic topology, it is common to associate an algebraic structure to the topological objects that we want to examine. Of course, we would like to do this in a such a way that we can use our knowledge of algebra to conclude something about object of interest.

In this case, we associate a ring to every simplicial complex as follows:

Given a simplicial complex Δ on n vertices, we define the quotient ring $A_\Delta = \mathbf{R}[x_1, ..., x_n]/I$, where I is the ideal generated by all monomials $x_{a_1}x_{a_2}...x_{a_k}$ s.t. the set $\{a_1, ..., a_k\}$ does not form a face of Δ.

It can be shown that the h-vector of Δ is precisely the Hibert function of A_Δ, where the Hilbert fuction is a definition frequently used in commutative algebra. A natural question to ask is which simplicial complexes correspond to some special class of rings. In our case, we will be looking at the simplicial complexes corresponding to Cohen-Macaulay rings. There simplicial complexes are called ***Cohen-Macaulay*** and have a number of interesting properties.

Theorem 3.10. *The h-vector of a Cohen-Macaulay simplicial complexes is always non-negative.*

The theorem is proven by Macaulay and can be found in Section 2 of [11]. The methods he used can be described as a multiset version of the Kruskal-Katona Theorem, a result about sufficient and necessary for a vector to be an f-vector of a simplicial complex. Macaulay also used the relation between h-vectors and Hilbert series. Finally, mathematicians including Stanley observed the connection with simplicial complexes, producing a result about Cohen-Macaulay simplicial complexes.

When looking at the last two results, i.e. Consequence 3.9 and Theorem 3.10, one may wonder if there is some connection between partitionable and Cohen-Macaulay complexes. Does one condition imply the other?
Another reason for this belief is that we know that shellable complexes are partitionable, and one can also show that shellable complexes are Cohen-Macaulay. Further, Björner found an example of a partitionable complex that is not Cohen-Macaulay (this is actually the simplicial complex from Example 3.2). Therefore one direction is not possible. The other direction remained open for quite some time. Under such considerations, Stanley made the following conjecture in 1979:

Conjecture 1. (Partitionability Conjecture) Every Cohen-Macaulay complex is partitionable.

This was a strong conjecture. If it were true, it would connect the algebraic property of Cohen-Macaulayness with the combinatorial property of partitionability.The conjecture was open for nearly 40 years, until it was disproven by Duval, Goeckner, Klivans and Martin in [3] (see also [4]). They constructed a relatively small counterexample – they found a simplicial complex on only 16 vertices that is Cohen-Macaulay, but not partitionable.

Proof sketch. We overview a slight improvement to the main argument from [3].
First, we start by finding a non-partitionable relative simplicial complex that is Cohen-Macaulay, denoted by $Q = (X, A)$. The exact complex can be found in [3]; it is a relative simplicial complex on 10 vertices and with 14 facets, with f-vector $(1, 7, 11, 5, 0)$. Now we want to somehow obtain a non-relative simplicial complex out of Q, while preserving Cohen-Macaulayness. One way to do this is by gluing some number of copies of Q along their mutual subcomplex A. This way we obtain a non-relative simplicial complex S. It can be proven that the complex S is indeed Cohen-Macaulay.

It remains to see if it is possible to obtain a non-partitionable complex S this way. It turns out it is enough to glue 19 copies of Q. We know that the f-vector of S is $(1, 64, 391, 594, 266)$ and then we easily compute the h-vector as well: $(1, 60, 205, 0, 0)$. Suppose that S is partitionable. Now, by Theorem 3.8, every facet of S has to match either to the empty set, a vertex or an edge. Then, every copy of Q must match at least one facet to face of A. Otherwise, there would exist a copy of Q that matches all of its facets to faces of that same copy, implying that Q is partitionable, a contradiction. Additionally, one facet of S has to be matched to the empty set, i.e. there is an interval in the partitioning scheme that contains at least one vertex of A. Therefore, there are at least 19+1=20 faces of A of dimension 0, 1 and 2 in our partitioning scheme, which is a contradiction, as A contains $1 + 7 + 11 = 19$ such faces.

The authors from [3], using a more complicated argument, showed that it suffices to glue only three copies of Q to obtain a non-partitionable Cohen-Macaulay simplicial complex.

□

Some questions still remain open:

- Is every 2-dimensional Cohen–Macaulay simplicial complex partitionable?
- Is every Cohen–Macaulay manifold partitionable?

This thesis was motivated by a special case of the second question: we ask whether some triangulations of a specific Cohen-Macaulay 3-manifold, called the Poincaré homology sphere, are partitionable.

4 The Poincaré conjecture and Poincaré homology sphere

4.1 History of the Poincaré conjecture

One of the most important questions in topology is whether two spaces are the same, i.e. whether we can obtain the second space by "stretching" the first one. We call those "stretchings" homeomorphisms, and if they exist, we say that the two spaces are homeomorphic.
In some cases, this is relatively easy to see: consider for example a line and a plane. They are not the same, because if we take a point out of both spaces, the first one becomes disconnected, while the other one is still connected. But what happens if we consider for example the Klein bottle and the torus? You may try to stretch the Klein bottle to get the torus, or use some other trick, but you will not be successful. But how can we prove that no matter how much you stretch one space, you will never be able to get the other one?

For this reason, mathematicians at the beginning of 20th century tried to find some invariants of topological spaces, which could tell them if two spaces are homeomorphic or not. Namely, a topological invariant is a property of topological space that is preserved under homeomorphism. That is, find such property, i.e. function, that if two spaces have different values of that function, they are not homeomorphic. One direction would be to look at the combinatorial properties of the spaces. This branch of topology is the so-called algebraic topology, and one of the founders of this field is the French mathematician Henri Poincaré. Around 1900, Poincaré developed a tool called homology, which associates an abelian groups to a space and in some way counts the number of n-dimensional holes in that space. He conjectured that homology could directly tell us whether a 3-manifold (a space that locally looks like space $\mathbb{R}^3$) is a 3-sphere.
However, 4 years later, he constructed a counterexample, the so-called ***Poincaré homology sphere***. This space had the same homology as the 3-sphere, but is was not homeomorphic to it. He then defined another property of a topological called ***simply***

connectedness: a topological space is said to be be ***simply connected*** if it is path-connected and if any path p from A to B can be continuously transformed into any other path q from A to B, while keeping both endpoints A and B fixed. In the same paper as the counterexample, Poincaré formulated the famous ***Poincaré conjecture***: he asked whether any closed simply connected 3-manifold had to be a 3-sphere. This is also the form of the conjecture that was best known until it was proved by Perelman (see [7–9]):

Theorem 4.1 (Perelman's Theorem, conjectured by Poincaré)**.** *Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.*

This problem turned to be a very important question in topology, as many other important results were obtained along the way of solving the conjecture. Moreover, it motivated the creation of entirely new fields of mathematics, like differential geometry among others.The conjecture even became one of the seven Millennium Prize Problems in 2000. It remained open for almost 100 years, until the Russian mathematician Grigori Perelman showed the conjecture to be correct in 2003. To this day, it remains the only one of the seven Millennium Prize Problems which have been solved.

4.2 Homology

As previously said, homology theory is a branch of algebraic topology that allows us to distinguish topological spaces (in our case simplicial complexes) by counting the number of their holes. In this section we will describe the main idea of this concept.

As a first example, consider a 2-sphere and a 3-ball. Both of them are path-connected (i.e. they have a single connected component) and none of them have 1-dimensional holes. However, a 2-sphere has a 2-dimensional hole in the middle, while the 3-ball does not. Therefore, we can conclude that a 2-sphere is different (i.e. not homeomorphic) to a 3-ball.

However, it is not always easy to tell what a "hole" in a space actually means, especially in higher dimensions. For this reason, mathematicians formalized the notion of holes by associating a sequence of abelian groups to each topological space in the following way:
The i-th component of that sequence, denoted by $H_i(X)$, is a abelian group that will describe the number of i-dimensional holes in space X. It is computed by a complicated algebraic procedure. The element $H_i(X)$ is the i-th ***homology group***. The

rank of this group tells us about the number of essentially different holes (or loops) in X.

Here are some examples:

Example 4.2. As mentioned, the 3-sphere S^3 has one connected component, no 1 and 2-dimensional holes and one 3-dimensional hole. This corresponds to the following homology groups:

$$\tilde{H}_i(S^3) = \begin{cases} \mathbb{Z}, & \text{for } i = 0, 3 \\ \{0\}, & \text{otherwise.} \end{cases}$$

Here $\mathbb{Z}$ is an abelian group generated by one element, while $\{0\}$ is the trivial group.

Example 4.3. Now consider a torus T. As the sphere S^2, it has one connected component and one 2-dimensional hole. But, it has two essentially different 1-dimensional holes, i.e. circles, colored with red and blue on the Figure 7. Therefore, the first homology group has to be generated by two generators:

$$\tilde{H}_i(T) = \begin{cases} \mathbb{Z}, & \text{for } i = 0, 2 \\ \mathbb{Z} \times \mathbb{Z}, & \text{for } k = 1, \\ \{0\}, & \text{otherwise.} \end{cases}$$

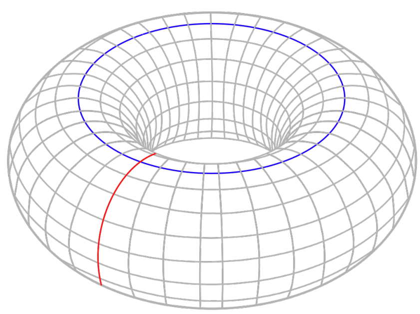

Figure 7: Torus T

As mentioned in the section 4.1, we should emphasize that two spaces with same homology groups do not need to be homeomorphic. This is exactly what Poincaré showed by constructing the Poincaré homology sphere: he found an object with same homology groups as S^3 (see Example 4.2), but he showed that it is not homeomorphic to S^3. In the next section we will talk more about the construction of Poincaré homology sphere, as well as some of its properties.

4.3 Poincaré homology sphere

There are multiple constructions of the Poincaré homology sphere (denoted P), with different approaches. Several of those constructions can be found in [6]. Here we will consider the one using a dodecahedron:

Construction: Start with a solid dodecahedron. Then identify (glue) the opposite faces of the dodecahedron by twisting one of the faces in a pair by clockwise rotation of $\frac{\pi}{5}$.

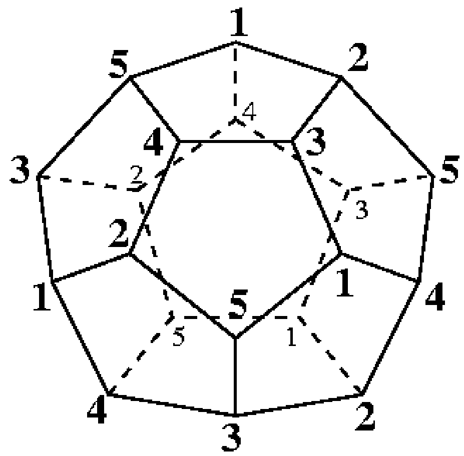

Figure 8: Poincaré homology sphere (due to Hachimori from [5])

It is not hard to see that P is a manifold. Firstly, a dodecahedron is clearly a 3-manifold, as it is homeomorphic to a 3-ball. Then, by gluing its faces, we in fact glue two half-spaces, therefore locally we "see" the whole space $\mathbb{R}^3$.

The Poincaré homology sphere has a number of interesting properties. First of all, it is a counterexample to the Poincaré 1900 conjecture. Secondly, there are several known relatively small triangulations of the Poincaré homology sphere, one with 16, and the other with 24 vertices, among others. The 16-point triangulation is the smallest one known. We will concentrate on the 24-point triangulation, which is simmetrical. According to Masahiro Hachimori's database of simplicial complexes (see [5]), the first is known to be partitionable. We are interested in whether the 24-point triangulation is partitionable or not.

5 Results

5.1 24-vertex triangulation of P

In this section we will consider the 24-point triangulation of P. We follow the presentation of this triangulation from the paper [1] of Björner and Lutz . The idea is that we start with a dodecahedron with faces identified as explained in the previous chapter. However, such dodecahedron is not simplicial for two reasons: first, it does not consist of tetrahedra (i.e. 3-simplicies). The second problem are the "face identifications". Therefore, if we subdivide this body enough, this fixes our problem. Here is the subdivision used in Björner's and Lutz's paper:
First, we inscribe the dual icosahedron in the dodecahedron. The vertices of the dodecahedron are numbered 1, 2, 3, 4 and 5, the vertices of the icosahedron by 6, 7, 8, 9, 10 and 11, while the corresponding top vertices of the cones over pentagons are numbered 61, 62, 71, 72,..., 111, 112. The middle point of the dodecahedron is 24.

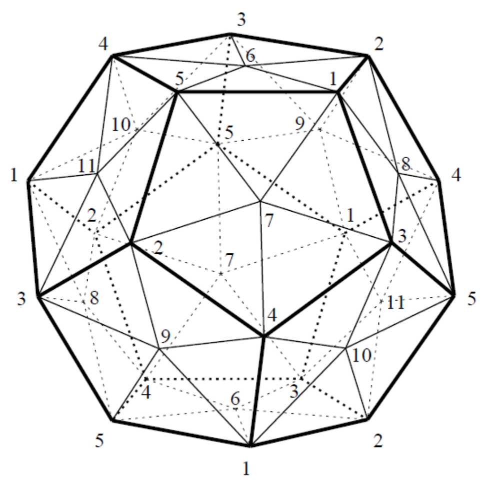

Figure 9: 24-point triangulation of Poincaré homology sphere (diagram due to Björner and Lutz from [1])

We determined the facets this triangulation as follows:
Firstly, we know that the Poincaré sphere is invariant under the alternating group A_5, a group of 60 elements. Therefore, we only had to find the orbits of the facets under A_5. To obtain all these orbits of Poincaré homology sphere, we used the computer algebra system GAP (see [12]). We found a representative of each orbit, and then applied the

elements of the group A_5 to obtain the rest of the elements of that specific orbit. For the complete list of facets of this triangulation see Appendix A.

5.2 Smaller triangulations of P

The 24-point triangulation is not the smallest known triangulation of Poincaré homology sphere. The smallest one known has 16 vertices and was first described by Björner and Lutz in [1].
The triangulation is obtained using a computer program BISTELLAR, which, given a known triangulation of a manifold, returns a smaller triangulation, by randomly applying a sequence of moves, called the ***bistellar moves***.
It is still not known whether this is the smallest triangulation possible, although it is shown in [13] that the minimal triangulation contains at least 11 vertices.

5.3 Ideas for partitioning scheme of Poincaré homology sphere

In this section we present some techniques that we tried for partitioning the 24-point triangulation of the Poincaré homology sphere.

First, we notice that a typical small triangulation of a 3-ball should be shellable. Then our main goal is to find subcomplexes that are homeomorphic to the 3-ball that will cover as many facets of the Poincaré sphere as possible. Then we could add the remaining faces to this structure. We found a few such complexes:

- Consider the triangulated icosahedron in the middle. This is a simplicial complex generated by all the facets that contain the middle vertex 24. This complex has 20 facets.

- Consider the cone over the face of the dodecahedron, i.e. the cone over the pentagon. This is also clearly a 3-ball. As there are six such pentagons, each of them covering 10 facets of the Poincaré sphere, in total they cover $10 \cdot 6 = 60$ facets.

Once we have a subcomplex of the Poincaré sphere that is surely partitionable, we have to deal with the partitioning scheme for the rest of the faces. That is, we partition the relative simplicial complex that is left.

6 Conclusion

In this paper we give an overview about partitionability of simplicial complexes. We gave some some results and tools about partitionability. Later in paper, we were particularly interested in partitioning the Poincaré homology sphere, more specifically a 24-point triangulation of this homology sphere. We give some ideas about this partitioning. However, some question still remain open: is every triangulation of the Poincaré sphere partitionable? Moreover, is every Cohen-Macaulay manifold partitionable?

7 Povzetek naloge v slovenskem jeziku

Particijaksa lastnost simplicialnih kompleksov je lastnost, ki se je prvič pojavila v delih Stanley-ja v sedemsetih letih prejšnjega stoletja. Motivirala ga je druga lastnost simplicialnih kompleksov, imenovana lupinavost, čeprav je šibkejša od slednje lastnosti. Med najpomembnejše domneve na tem področju spada t.i. particijska domneva (ang. Partitionability Conjecture), ki jo je leta 1979 postavil Stanley. Ta domneva je trdila, da je vsak Cohen-Macaulay kompleks particijski. Po skoraj 40 letih so domnevo ovrgli Duval, Goeckner, Klivans in Martin v članku iz leta 2016. Njihov protiprimer je simplicialni kompleks na samo 16 točkah, ki je Cohen-Macaulayev, vendar ni particijski. Čeprav se je izkazalo, da je domneva napačna, se je na tem področju postavilo veliko število drugih vprašanj. Ali particijska domneva velja v dimneziji 2? Kaj šteje h-vektor Cohen-Macaulayevega kompleksa?
Obstaja še veliko drugih neodgovorjenih vprašanj o particijski lasnosti. V nalogi je lastnost preučevana na Poincaréjevi homološki sferi. Naše glavno vprašanje je, ali so vse njene triangulacije particijske. Po podatkovni bazi simplicialnih kompleksov Masahira Hachimorija obstaja 16-točkovna triangulacija, za katero je že znano, da je particijska. V našem delu obravnavamo eno od njegovih simetričnih 24-točkovnih triangulacij.

V drugem poglavju predstavimo nekaj osnovnih definicij simplicialnih kompleksov. Začeli bomo z abstraktnimi simplicialnimi kompleksi, ki so povsem kombinatorični objekti. Nato predstavimo, kako lahko iz abstraktnih kompleksov sestavimo simplicialni kompleks kot topološki prostor. Potem definiramo spoj dveh simplicialnih kompleksov in tudi f in h -vektorjev. Definiramo tudi relativen simplicialni kompleks, operacijo ki izgleda kot odštevanje.

V tretjem poglavju govorimo o lupinavosti in particijski lastnosti simplicialnih kompleksov, ki sta tesno povezani. Pokazali smo, da lupinavost implicira particijsko lastnost. Predstavimo nekaj primerov. Nato predstavimo nekaj zadostnih in potrebnih pogojev za lupinavost. Nadalje podamo nekaj rezultatov in trditev, ki jih uporabimo pozneje v nalogi. Med njimi je en o pomenu h -vektorja za čiste particijske simplicialne komplekse. Nato predstavimo particijsko domnevo in protiprimer, ki jo je ovrgel. Poglavje zaključimo z navedbo nekaj odprtih vprašanj s tega področja topologije.

Četrto poglavje je posvečeno znameniti Poincaréjevi domnevi, idejah za njen dokaz in njeni prejšnji formulaciji. Podamo kratek uvod o veji topologije, ki jo je Poincaré razvil v svojem življenju, o homologiji. Homologija je topološka invarianta, ki nam pomaga razlikovati različne topološke prostore (v našem primeru simplicialne komplekse). Ohlapno rečeno, to stori tako, da šteje število lukenj vsake dimenzije znotraj danega topološkega prostora. Potem predstavimo Poincaréjevo homološko sfero, enega zanimivih topoloških objektov. Poincarejeva homološka sfera je objekt, ki ima isto homologijo (ali natančneje, ima iste homološke grupe) kot 3-sfera. Kljub temu ni homeomorfna 3-sferi. Predstavimo konstrukcijo in nekatere njene lastnosti.

V zadnjem poglavju predstavimo nekatere triangulacije Poincaréjeve homološke sfere. Naš poudarek je predvsem na 24-točkovni triangulaciji. Podamo celoten seznam glavnih simpleksov te triangulacije in na kratko pojasnimo postopek, ki smo ga uporabili za pridobitev liste. Nadalje podamo nekaj idej za dokaz particijske lastnosti te triangulacije.

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Appendices

A Facets of the 24-point triangulation of the Poincaré homology sphere

List of all 130 facets of the triangulation:

(1, 2, 6, 61)	(1, 2, 6, 62)	(1, 2, 8, 81)
(1, 2, 8, 82)	(1, 2, 10, 101)	(1, 2, 10, 102)
(1, 2, 61, 101)	(1, 2, 62, 81)	(1, 2, 82, 102)
(1, 3, 7, 71)	(1, 3, 7, 72)	(1, 3, 8, 81)
(1, 3, 8, 82)	(1, 3, 11, 111)	(1, 3, 11, 112)
(1, 3, 71, 81)	(1, 3, 72, 111)	(1, 3, 82, 112)
(1, 4, 9, 91)	(1, 4, 9, 92)	(1, 4, 10, 101)
(1, 4, 10, 102)	(1, 4, 11, 111)	(1, 4, 11, 112)
(1, 4, 91, 101)	(1, 4, 92, 111)	(1, 4, 102, 112)
(1, 5, 6, 61)	(1, 5, 6, 62)	(1, 5, 7, 71)
(1, 5, 7, 72)	(1, 5, 9, 91)	(1, 5, 9, 92)
(1, 5, 61, 91)	(1, 5, 62, 71)	(1, 5, 72, 92)
(1, 61, 91, 101)	(1, 62, 71, 81)	(1, 72, 92, 111)
(1, 82, 102, 112)	(2, 3, 6, 61)	(2, 3, 6, 62)
(2, 3, 9, 91)	(2, 3, 9, 92)	(2, 3, 11, 111)
(2, 3, 11, 112)	(2, 3, 61, 111)	(2, 3, 62, 92)
(2, 3, 91, 112)	(2, 4, 7, 71)	(2, 4, 7, 72)
(2, 4, 8, 81)	(2, 4, 8, 82)	(2, 4, 9, 91)
(2, 4, 9, 92)	(2, 4, 71, 91)	(2, 4, 72, 82)
(2, 4, 81, 92)	(2, 5, 7, 71)	(2, 5, 7, 72)
(2, 5, 10, 101)	(2, 5, 10, 102)	(2, 5, 11, 111)
(2, 5, 11, 112)	(2, 5, 71, 112)	(2, 5, 72, 102)

(2, 5, 101, 111)	(2, 61, 101, 111)	(2, 62, 81, 92)
(2, 71, 91, 112)	(2, 72, 82, 102)	(3, 4, 6, 61)
(3, 4, 6, 62)	(3, 4, 7, 71)	(3, 4, 7, 72)
(3, 4, 10, 101)	(3, 4, 10, 102)	(3, 4, 61, 72)
(3, 4, 62, 102)	(3, 4, 71, 101)	(3, 5, 8, 81)
(3, 5, 8, 82)	(3, 5, 9, 91)	(3, 5, 9, 92)
(3, 5, 10, 101)	(3, 5, 10, 102)	(3, 5, 81, 101)
(3, 5, 82, 91)	(3, 5, 92, 102)	(3, 61, 72, 111)
(3, 62, 92, 102)	(3, 71, 81, 101)	(3, 82, 91, 112)
(4, 5, 6, 61)	(4, 5, 6, 62)	(4, 5, 8, 81)
(4, 5, 8, 82)	(4, 5, 11, 111)	(4, 5, 11, 112)
(4, 5, 61, 82)	(4, 5, 62, 112)	(4, 5, 81, 111)
(4, 61, 72, 82)	(4, 62, 102, 112)	(4, 71, 91, 101)
(4, 81, 92, 111)	(5, 61, 82, 91)	(5, 62, 71, 112)
(5, 72, 92, 102)	(5, 81, 101, 111)	(17, 61, 72, 82)
(17, 61, 72, 111)	(17, 61, 82, 91)	(17, 61, 91, 101)
(17, 61, 101, 111)	(17, 62, 71, 81)	(17, 62, 71, 112)
(17, 62, 81, 92)	(17, 62, 92, 102)	(17, 62, 102, 112)
(17, 71, 81, 101)	(17, 71, 91, 101)	(17, 71, 91, 112)
(17, 72, 82, 102)	(17, 72, 92, 102)	(17, 72, 92, 111)
(17, 81, 92, 111)	(17, 81, 101, 111)	(17, 82, 91, 112)
(17, 82, 102, 112)		