Extremal set theory as algebraic geometry

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Joint work with ...

Much of the work I will describe is joint with Denys Bulavka and Francesca Gandini.

Denys Bulavka

Francesca Gandini

Throughout, $\mathbb{F}$ will be a field

Convenient to assume characteristic $\neq 2$

Little is lost by assuming $\mathbb{F} = \mathbb{C}$, the complex numbers

The *exterior algebra* over $\mathbb{F}$ is an anti-commutative analogue of a polynomial ring.

$\bigwedge \mathbb{F}^n$ is generated by $e_1, \ldots, e_n$.
If x, y are linear combinations of vars, then

$$x \wedge y = -y \wedge x$$
$$x \wedge x = 0$$

Write $\bigwedge^k \mathbb{F}^n$ for the subspace generated by *k-forms*. Thus,

$\bigwedge^k \mathbb{F}^n$ is an $\binom{n}{k}$-dimensional vector space, generated by $e_S = e_{i_1} \wedge \cdots \wedge e_{i_k}$ for $S = \{i_1 < i_2 < \cdots < i_k\} \subseteq [n]$.

Example: cross products

The cross product from high school physics $v \otimes w$ comprises $\bigwedge \mathbb{R}^3$ together with the "Hodge star" identification $\bigwedge^2 \mathbb{R}^3 \longleftrightarrow \bigwedge^1 \mathbb{R}^3$.

Here, e_S is called an *exterior monomial*.

Exterior algebras: combinatorial set theory

Exterior algebra $\bigwedge \mathbb{F}^n$ is anti-commuting analog to polynom. ring

For $x = \alpha_1 e_1 + \cdots \alpha_n e_n$ in $\bigwedge^1 \mathbb{F}^n$, have $x \wedge x = 0$.

For set $S = \{i_1 < i_2 < \cdots < i_k\} \subseteq [n]$, write $e_S = e_{i_1} \wedge \cdots \wedge e_{i_k}$

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Exterior algebras are useful in areas from differential geometry to algebraic geometry.

In combinatorics, they model set intersections.

Indeed, for $S, T \subseteq [n]$, we have:

$$e_S \wedge e_T = 0 \quad \iff \quad S \cap T \neq \emptyset$$

Example

For $S = \{1, 3, 5\}$, $T = \{3, 4, 5\}$, have

$$(e_1 \wedge e_3 \wedge e_5) \wedge (e_3 \wedge e_4 \wedge e_5) = -e_1 \wedge e_5 \wedge e_3 \wedge e_3 \wedge e_4 \wedge e_5$$
$$= -e_1 \wedge e_5 \wedge 0 \wedge e_4 \wedge e_5 = 0.$$

Exterior algebra $\bigwedge \mathbb{F}^n$ is anti-commuting analog to polynom. ring

$$e_S \wedge e_T = 0 \iff S \cap T \text{ intersect nontrivially}$$

Two Families Theorem (Bollobás, Lovász, 1970s)

Let $A_1, \ldots, A_m$ be r-element subsets of $[n]$
$B_1, \ldots, B_m$ be s-element subsets of $[n]$

such that

$A_i \cap B_i = \emptyset$ for each i, but
$A_i \cap B_j$ is nontrivial for each $i < j$.

Then $m \le \binom{r+s}{r}$.

A beautiful proof

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Then $m \leq \binom{r+s}{r}$. (Note: no n appears!)

Proof. Consider $\bigwedge \mathbb{F}^{r+s}$.
Take n elements $g_1, \dots, g_n$ in general position from $\mathbb{F}^{r+s}$.
Write $g_S = g_{i_1} \wedge \cdots \wedge g_{i_k}$ if $S = \{i_1 < \cdots < i_k\}$.
Thus, $g_A \wedge g_B = 0 \iff A \cap B \neq \emptyset$.
Upper triangular matrix + dual basis argument
$\implies g_{A_1}, \dots, g_{A_m}$ are linearly independent. □

A beautiful proof: now what?

Proof idea: associate set with elements of $\bigwedge^r \mathbb{F}^{r+s}$, $\bigwedge^s \mathbb{F}^{r+s}$,
use dual pairing to get linear independence
and an upper bound by dimension

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Question: What else can you use this idea for?

Theorem (Erdős, Ko, Rado, 1938–1961)

Let $r \leq n/2$. If $A_1, \ldots, A_m$ are r-element subsets of $[n]$ such that every pair A_i, A_j intersects nontrivially, then $m \leq \binom{n-1}{r-1}$.

EKR Theorem has a large number of known proofs.
Is there a slick one using exterior algebra and duality?

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Is there a slick one using exterior algebra and duality?

I don't know.

Proofs with dual pairings are known. I like the "polynomial method" proof of Füredi, Hwang, and Weichsel.

Exterior extension of EKR

Although I don't know a proof as pretty as the Lovász proof of the Two Families Theorem, there is a nice extension of EKR.

Say that a subset L of $\bigwedge \mathbb{F}^n$ is *self-annihilating* if $L \wedge L = \{w_1 \wedge w_2 : w_1, w_2 \in L\} = 0$.

Theorem (Scott and Wilmer, 2021)

Let $r \leq n/2$. If L is a self-annihilating subspace of $\bigwedge^r \mathbb{F}^n$, then $\dim L \leq \binom{n-1}{r-1}$.

The EKR theorem follows by letting the monomials $e_{A_1}, \ldots, e_{A_m}$ form a basis of L.

Scott and Wilmer were also interested in the Two Families Theorem, but proved their self-annihilating theorem from EKR.

Exterior extension of EKR: another proof?

Scott-Wilmer Theorem:
When $r \leq n/2$, self-annihilating subspace of $\bigwedge^r \mathbb{F}^n$ has dim$\leq \binom{n-1}{r-1}$

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kill a fly with an elephant gun

To take excessive, overcomplicated, or extravagant means or force to accomplish something relatively minor or simple.

Farlex Dictionary of Idioms

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Farlex Dictionary of Idioms

So I want to tell you how to use some big or little tools from easy* algebraic geometry to prove Scott-Wilmer (and hence EKR).

Algebraic geometry background: Zariski topology

The *Zariski topology* on affine $\mathbb{F}^n$ or projective $\mathrm{P}\mathbb{F}^n$ has as closed sets the zero loci of sets of polynomials or homogenous polynomials.

The Zariski topology is T_1, but not Hausdorff.
Few of the definitions from a first topology course are useful.

Most definitions from a first topology course have useful analogs.

"Standard" topology	Zariski topology
$\mathbb{R}^n$, S^n	affine $\mathbb{F}^n$, projective $\mathrm{P}\mathbb{F}^n$
metric space	quasi-projective space ?
limits of sequences	limits of (polynomial) curves
closed subspace	variety
compact	complete
Hausdorff	separated

Algebraic geometry background: Zariski topology dictionary

Zariski topology on $\mathbb{F}^n$ or $\mathrm{P}\mathbb{F}^n$ has as closed sets the zero loci of sets of (homogenous) polynomials.

The translation from standard topological notions to Zariski is often made by writing both in terms of products.

Example: Hausdorff and separable

A topological space X is Hausdorff iff the diagonal subspace in $X \times X$ is closed.

A variety* X is *separated* if the diagonal subspace in $X \times X$ is closed.

Topological product vs product of affine/projective spaces

Metric spaces are Hausdorff; quasi-projective varieties are separated

Setting up for algebraic geometry proof of Scott-Wilmer .

Zariski topology on $\mathbb{F}^n$ or $\mathrm{P}\mathbb{F}^n$ has as closed sets the zero loci of sets of (homogenous) polynomials.

The *Grassmannian* $Gr_k V$ is a projective variety (closed subset of projective space) whose points correspond to k-dimensional subspaces of vector space V.

Thus, the Grassmannian is a geometric object representing the family of fixed-dimensional subspaces.

Algebraic geometry is useful for Scott-Wilmer and EKR because:

Lemma

The family of self-annihilating subspaces L of $\bigwedge^r \mathbb{F}^n$ is a Zariski closed subset of $Gr(\bigwedge^r \mathbb{F}^n)$.

Setting up for algebraic geometry proof of Scott-Wilmer ..

Lemma: Self-annihilating subspaces L of $\bigwedge^r \mathbb{F}^n$ form a Zariski closed subset of $Gr(\bigwedge^r \mathbb{F}^n)$.

Sketch of Lemma. Given two elements w and u in $\bigwedge^k \mathbb{F}^n$, written in the e_S basis, their product is expressed coordinatewise with homogenous polynomials.

Example

$$
\begin{aligned}
&(\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3) \wedge (\beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3) \\
= &(\alpha_1\beta_2 - \alpha_2\beta_1)\, e_{12} + (\alpha_1\beta_3 - \alpha_3\beta_1)\, e_{13} + (\alpha_2\beta_3 - \alpha_3\beta_2)\, e_{23}
\end{aligned}
$$

Thus, the set $\{w \times u \,:\, w \wedge u = 0\}$ is Zariski closed in $\left(\bigwedge^k \mathbb{F}^n\right)^2$.

The desired now follows by yoga well-known to algebraic geometers, and possibly partly written down somewhere.

Lemma: Self-annihilating subspaces L of $\bigwedge^r \mathbb{F}^n$ form a Zariski closed subset of $Gr(\bigwedge^r \mathbb{F}^n)$.

We now get out the elephant gun:

Theorem (Borel Fixed-Point, 1956)

If X is a projective variety, and G is a connected, solvable, linear algebraic group acting nicely on X, then X has a point that is fixed by G.

Here we take :
X to be the variety of k-dimensional self-annihilating subspaces,
G to be the group of upper triangular matrices. (for $\mathbb{F} = \mathbb{C}$)

A fixed-point by the upper triangular matrices is generated by monomials supported by a *shifted* family of sets.

Erdős-Ko-Rado for a shifted family is an easy induction!

Borel Fixed-Point Theorem: Borel group acting nicely on projective variety has a fixed point.

I did not make this technique up.
Draisma, Kraft, and Kuttler earlier used BFPT for extensions of:

Theorem (Gerstenhaber, 1958; others)

Let L be a vector space consisting of $n \times n$ nilpotent matrices over $\mathbb{F}$. Then $\dim L \leq \binom{n}{2}$, with equality only if L is conjugate to the space of strictly upper triangular matrices.

The algebraic groups perspective gives a connection between
- Erdős-Ko-Rado problems and
- Gerstenhaber problems.

Borel fixed versus shifted

Let's see the connection between Borel fixed and shifted.

A family of sets $\mathcal{A}$ is *shifted* if for every $i < j$, whenever $A \in \mathcal{A}$ contains j but not i, then also $A \setminus j \cup i$ is in $\mathcal{A}$.

Example: triangular matrix vs shifted

Consider linear map f sending $e_1 \mapsto e_1$ and $e_3 \mapsto e_1 + e_3$.
If $e_{23} = e_2 \wedge e_3$ is in Borel fixed subspace L, then also
$f(e_{23}) = -e_1 \wedge e_2 + e_2 \wedge e_3$ is in L, and hence
e_{12} is in L.

Shifted families have long been a tool in extremal combinatorics.

Shifted family of sets: for every $i < j$,
$A \in \mathcal{A}$ contains j but not $i \implies A \setminus j \cup i \in \mathcal{A}$

The proof of Erdős, Ko, and Rado of EKR goes as follows:
Given an initial intersecting family of sets $\mathcal{A}$,
iteratively transform it into a shifted family of the same size.
Again, EKR for shifted is an easy induction.

A proof of the Borel Fixed Point Theorem goes as follows:
Given an initial point on projective variety X,
iteratively move it along closed curves to a fixed point.

Is there a deeper relation?

Proof of Scott-Wilmer, mark II .

The *shifting operation* $\text{shift}_{i \leftarrow j}$ replaces a family $\mathcal{A}$ with

$$\text{shift}_{i \leftarrow j} \mathcal{A} = \{A \in \mathcal{A} \,:\, j \notin A \text{ or } i \in \mathcal{A} \text{ or } A \setminus j \cup i \in \mathcal{A}\} \cup \{A \setminus j \cup i \,:\, j \in A \text{ and } i \notin A\}.$$

An algebraic geometry analog comes from the map $N_{i \leftarrow j}(t)$ sending

$e_j \mapsto te_j + e_i$, and fixing all other e_k.

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An algebraic geometry analog comes from the map $N_{i \leftarrow j}(t)$ sending

$$e_j \mapsto te_j + e_i, \text{ and fixing all other } e_k.$$

If L is a space of k-forms generated by monomials supported by $\mathcal{A}$, then $\lim_{t \to 0} N_{i \leftarrow j}(t)L$ is gen'ed by monomials supported by $\text{shift}_{i \leftarrow j} \mathcal{A}$.

(Observed by Knutson, Murai, probably others.)

Example: shifting with limits

Consider again $e_{23} = e_2 \wedge e_3$ and $N_{1 \leftarrow 3}(t)$.
Then

$$N_{1 \leftarrow 3}(t)e_{23} = -e_1 \wedge e_3 + t \cdot e_2 \wedge e_3 \\ \to -e_1 \wedge e_3 \text{ as } t \to 0.$$

Map $N_{i \leftarrow j}(t)$ sends $e_j \mapsto te_j + e_i$, and fixes all other e_k.
Limit of $N_{i \leftarrow j}(t)$ realizes shifting of Erdős, Ko, Rado.

A second algebraic geometry-based proof of Scott-Wilmer is as follows:

iteratively apply $\lim_{t \to 0} N_{i \leftarrow j}(t)$ to L,
reducing to Borel fixed,
and apply EKR for shifted families of sets.

Thus, this proof unpacks the Borel fixed point theorem and applies its proof technique directly. (See Lie-Kolchin!)

Message: extremal set theorists sometimes are really doing algebraic geometry.

Map $N_{i \leftarrow j}(t)$ sends $e_j \mapsto te_j + e_i$, and fixes all other e_k.
Limit of $N_{i \leftarrow j}(t)$ realizes shifting of Erdős, Ko, Rado.

Limits here work similar to limits in calculus: if you have an indeterminate form, then "cancel zeros".

Example: limits

Let $L = \text{span}\{e_{13}, e_{23}\}$, and consider $N_{1 \leftarrow 3}(t)$.
Then $N_{1 \leftarrow 3}(t)L = \text{span}\{e_{13}, -e_{13} + te_{23}\} = \text{span}\{e_{13}, te_{23}\} = L$.
Thus, $\lim_{t \to 0} N_{1 \leftarrow 3}(t)L = L$.

Details are surprisingly hard to find in an accessible form:
see Newstead, *Introduction to moduli problems and orbit spaces*.

Now on to the good part

Recall again the theorem of Gerstenhaber:

Theorem (Gerstenhaber, 1958; others)

Let L be a vector space consisting of $n \times n$ nilpotent matrices over $\mathbb{F}$. Then $\dim L \leq \binom{n}{2}$, with equality only if L is conjugate to the space of strictly upper triangular matrices.

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(The so-called *strict EKR Theorem*.)

Strict EKR: $\binom{n-1}{r-1}$ attainable only if
all sets have common intersection, or $r = n/2$.

More is true:

Theorem (Hilton and Milner, 1967)

Let $r \leq n/2$. If $A_1, \ldots, A_m$ are r-element subsets of $[n]$ such that every pair A_i, A_j intersects nontrivially, but $\bigcap A_i = \emptyset$, then $m \leq \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$.

Thus, if $r < n/2$ and $\bigcap A_i = \emptyset$, then the number of sets is a lot smaller than the best possible.

That is, if you are close to best numerically, then you are close to the standard construction structurally.

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Hilton-Milner is a *stability* result.

Strict and stability results ...

Strict EKR: $\binom{n-1}{r-1}$ attainable only if the obvious construction
Stability: if not the obvious construction, then small

Scott and Wilmer conjectured a strict version of their theorem.
I asked whether a stability form of their theorem holds.

Setting:
If L is a self-annihilating subspace of $\bigwedge^r \mathbb{F}^n$ ($r < n/2$) and
L is not annihilated by any 1-form in $\bigwedge^1 \mathbb{F}^n$,
then does a strict/stable version of Scott-Wilmer hold?

Cautionary example

Consider $n = 6$ and $r = 3$. Then

$$\begin{matrix} e_{123} + e_{456} & e_{124} + e_{356} & e_{125} + e_{346} & e_{126} + e_{345} & e_{134} + e_{235} \\ e_{135} + e_{246} & e_{136} + e_{245} & e_{145} + e_{236} & e_{146} + e_{235} & e_{156} + e_{234} \end{matrix}$$

generate a self-annihilating subspace of dimension $10 = \binom{6-1}{3-1}$,
but fail to be annihilated by any 1-form $\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$.

A Hilton-Milner type theorem for exterior algebras

Hilton-Milner: if $A_1, \ldots, A_m$ pairwise intersect but $\bigcap A_i = \emptyset$,
then $m \leq \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$.

In the work with Bulavka and Gandini, we solve this problem:

Theorem (Bulavka, Gandini, and me, 2025+)

Let $r \leq n/2$. If L is a self-annihilating subspace of $\bigwedge^r \mathbb{F}^n$, and L is not annihilated by any 1-form, then $\dim L \leq \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$.

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The proof is more involved than either the proof(s) of Scott-Wilmer or of Hilton-Milner.

The difficulty is "not annihilated by any 1-form" is not Zariski closed
so moving along curves can destroy the condition,
while elements like $e_{123} + e_{456}$ cause trouble for HM techniques.
In the last few minutes, I want to illustrate some of the ideas.

Hilton-Milner: if $A_1, \ldots, A_m$ pairwise intersect but $\bigcap A_i = \emptyset$,
then $m \leq \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$.

The main idea of the proof of the Hilton-Milner theorem is to reduce to a shifted system with the same properties.

Sketch: Shift, shift, shift.
If $\text{shift}_{i \leftarrow j}$ would give a common intersection, then:
don't do that, and instead
relabel i, j as $1, 2$, and shift everything else, then
fix up with $1, 2$ at the end.

Difficulty in exterior algebra setting:
$\lim N_{i \leftarrow j}(t)L$ may be divisible by a 1-form ℓ other than e_i.
Instead, change basis to $f_1 = \ell$, $f_2 = e_i - e_j$, and continue.
Controlling failure to be generated by monomials is important.

Shifting terminates

In the combinatorial set theory setting, it is obvious and straightforward to prove that shifting operations eventually terminate.

In the exterior algebra setting, especially where we want to avoid certain i, j, it is still obvious, but not as straightforward to prove.

Indeed:
Performing $\text{shift}_{i \leftarrow j}$ over $i, j \in I \subseteq [n]$, we can get to a stable system in at most $\binom{|I|}{2}$ operations.
Performing $\lim N_{i \leftarrow j}$ over $i, j \in I \subseteq [n]$, we can get to a stable system in at most $|I| - 1 + \binom{|I|}{2}$ operations.

The extra operations make the subspace L "monomial enough" that subsequent operations are easy to control.

Questions

Question: What other theorems and/or conjectures from extremal set theory have exterior algebra analogs?

Question: Can limits of curves in varieties help solve problems in extremal combinatorics?

An unexpected benefit: a new proof of HM

As an unexpected benefit of this project, and another with Bulavka, we found a new short proof of the Hilton-Milner theorem.

The standard proof of Füredi and Frankl is short and elementary, but involves a difficult (to me!) step.

We found a new proof that follows the same sketch as EKR, and which uses only standard ideas. One can get a common proof of EKR, HM, and strict HM.

(Wu, Li, Feng, Liu, and Yu arXived a paper with a similar idea a few days after ours.)


Thank you! Hvala!
Teşekkürler!

Let Δ be a simplicial complex...

One more thing.

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