Univerza na Primorskem
Fakulteta za matematiko, naravoslovje in informacijske tehnologije
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NAME: ____________________ STUDENT NUMBER: ____________________

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Algebra II - Linear algebra, exam

1. Suppose $A \in \mathrm{Mat}_{3\times 4}(\mathbb{R})$, and $Ax = 0$ has exactly 2 special solutions:

$$x_1 = (1,1,1,0)^\top \qquad \text{and} \qquad x_2 = (-2,-1,0,1)^\top.$$

Remembering that A is 3 by 4, find its row reduced echelon form R. Find the dimensions of all four fundamental subspaces $\ker(A)$, $\mathrm{im}(A)$, $\ker(A^\top)$, $\mathrm{im}(A^\top)$.

You have enough information to find bases for one or more of these subspaces - find those bases.

2. The map $T : \mathcal{P}_2 \to \mathcal{P}_2$, is defined with

$$(T(p))(x) = \frac{p(x+2) - p(x)}{2}.$$

Show that T is linear operator. Find $\ker(T)$, $\mathrm{im}(T)$, $\dim\ker(T)$ and $\dim\mathrm{im}(T)$. Find coordinate matrix of T with respect to basis $\mathcal{B} = \{1, x+1, x^2\}$. ($\mathcal{P}_2$ is the set of all real polynomials of degree ≤ 2).

3. (a) Find a linear combination w ($w \neq \mathbf{0}$) of the linearly independent vectors v and u that is perpendicular to u.

(b) For the 2-column matrix $A = [u, v]$, find Q (orthonormal columns) and R (2 by 2 upper triangular) so that $A = QR$.

(c) In terms of Q only, using $A = QR$, find the projection matrix P onto the plane spanned by u and v.

4. Let T be a linear operator on a finite-dimensional vector space $\mathcal{V}$. If λ is an eigenvalue of T having (algebraic) multiplicity m, show that then

$$1 \leq \dim(E_\lambda) \leq m$$

where $E_\lambda = \{x \in \mathcal{V} \,|\, T(x) = \lambda x\}$. (Recall: The algebraic multiplicity of λ is the number of times it is repeated as a root of the characteristic polynomial. In other words, $\mathrm{alg\,mult}_A(\lambda_i) = a_i$ if and only if $(x-\lambda_1)^{a_1} \ldots (x-\lambda_s)^{a_s} = 0$ is the characteristic equation for A.)

Instructions: Please, write your solutions only with ink or ballpoint pen in blue or black colour. You must return this sheet of paper together with your solutions. You can use calculator. All pages with your solutions must be marked in the following way: "page-number/number-of-pages".