University of Sarajevo
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Algebraic Characterizations of Distance-regular Graphs

(Master of Science Thesis)

With 49 illustrations

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Abstract

Through this thesis we introduce distance-regular graphs, and present some of their characterizations which depend on information retrieved from their adjacency matrix, principal idempotent matrices, predistance polynomials and spectrum. Let Γ be a finite simple connected graph. In Chapter I we present some basic results from Algebraic graph theory: we prove Perron-Frobenius theorem, we show how to compute the number of walks of a given length between two vertices of Γ, how to compute the total number of (rooted) closed walks of a given length of Γ, we introduce adjaceny matrix $\boldsymbol{A}$ of Γ, principal idempotent matrices $\boldsymbol{E}_i$ of Γ and introduce adjacent (Bose-Mesner) algebra of Γ and Hoffman polynomial of Γ. All of these results are needed in Chapters II and III. In Chapter II we define distance-regular graphs, show some examples of these graph, introduce distance-i matrix $\boldsymbol{A}_i$, $i = 0, 1, ..., D$ (where D is the diameter of graph Γ), introduce predistance polynomials p_i, $i = 0, 1, ..., d$ (d is the number of distinct eigenvalues) of Γ and prove the following sequence of equivalences: Γ is distance-regular $\iff$ Γ is distance-regular around each of its vertices and with the same intersection array $\iff$ distance matrices of Γ satisfy $\boldsymbol{A}_i\boldsymbol{A}_j = \sum_{k=0}^{D} p_{ij}^k \boldsymbol{A}_k$, $(0 \leq i, j \leq D)$ for some constants p_{ij}^k $\iff$ for some constants a_h, b_h, c_h $(0 \leq h \leq D)$, $c_0 = b_D = 0$, distance matrices of Γ satisfy the three-term recurrence $\boldsymbol{A}_h\boldsymbol{A} = b_{h-1}\boldsymbol{A}_{h-1} + a_h\boldsymbol{A}_h + c_{h+1}\boldsymbol{A}_{h+1}, (0 \leq h \leq D)$ $\iff$ $\{I, \boldsymbol{A}, ..., \boldsymbol{A}_D\}$ is a basis of the adjacency algebra $\mathcal{A}(\Gamma)$ $\iff$ $\boldsymbol{A}$ acts by right (or left) multiplication as a linear operator on the vector space span$\{I, \boldsymbol{A}_1, \boldsymbol{A}_2, ..., \boldsymbol{A}_D\}$ $\iff$ for any integer h, $0 \leq h \leq D$, the distance-h matrix $\boldsymbol{A}_h$ is a polynomial of degree h in $\boldsymbol{A}$ $\iff$ Γ is regular, has spectrally maximum diameter $(D = d)$ and the matrix $\boldsymbol{A}_D$ is polynomial in $\boldsymbol{A}$ $\iff$ the number a_{uv}^{ℓ} of walks of length ℓ between two vertices $u, v \in V$ only depends on $h = \partial(u, v)$ $\iff$ for any two vertices $u, v \in V$ at distance h, we have $a_{uv}^h = a_h^h$ and $a_{uv}^{h+1} = a_h^{h+1}$ for any $0 \leq h \leq D-1$, and $a_{uv}^D = a_D^D$ for $h = D$ $\iff$ $\boldsymbol{A}_i\boldsymbol{E}_j = p_{ji}\boldsymbol{E}_j$ (p_{ji} are some constants) $\Leftrightarrow \boldsymbol{A}_i = \sum_{j=0}^{d} p_{ji}\boldsymbol{E}_j \Leftrightarrow$ $\boldsymbol{A}_i = \sum_{j=0}^{d} p_i(\lambda_j)\boldsymbol{E}_j \Leftrightarrow \boldsymbol{A}_i \in \mathcal{A}$, $(i, j = 0, 1, ..., d(= D))$ $\iff$ for every $0 \leq i \leq d$ and for every pair of vertices u, v of Γ, the (u, v)-entry of $\boldsymbol{E}_i$ depends only on the distance between u and v $\iff$ $\boldsymbol{E}_j \circ \boldsymbol{A}_i = q_{ij}\boldsymbol{A}_i$ (q_{ij} are some constants) $\Leftrightarrow \boldsymbol{E}_j = \sum_{i=0}^{D} q_{ij}\boldsymbol{A}_i \Leftrightarrow$ $\boldsymbol{E}_j = \frac{1}{n}\sum_{i=0}^{d} q_i(\lambda_j)\boldsymbol{A}_i$ (where $q_i(\lambda_j) := m_j\frac{p_i(\lambda_j)}{p_i(\lambda_0)}$) $\Leftrightarrow \boldsymbol{E}_j \in \mathcal{D}$ $i, j = 0, 1, ..., d(= D)$ $\iff$ $\boldsymbol{A}^j \circ \boldsymbol{A}_i = a_i^{(j)}\boldsymbol{A}_i$ ($a_i^{(j)}$ are some constants) $\Leftrightarrow \boldsymbol{A}^j = \sum_{i=0}^{d} a_i^{(j)}\boldsymbol{A}_i \Leftrightarrow \boldsymbol{A}^j = \sum_{i=0}^{d}\sum_{l=0}^{d} q_{i\ell}\lambda_l^j\boldsymbol{A}_i \Leftrightarrow$ $\boldsymbol{A}^j \in \mathcal{D}$ $i, j = 0, 1, ..., d$. Finally, in Chapter III, we introduce one interesting family of orthogonal polynomials - the canonical orthogonal system, and prove three more characterizations of distance-regularity which involve the spectrum: Γ is distance-regular $\iff$ the number of vertices at distance k from every vertex $u \in V$ is $|\Gamma_k(u)| = p_k(\lambda_0)$ for $0 \leq k \leq d$ (where $\{p_k\}_{0 \leq k \leq d}$ are predistance polynomials) $\iff$ $q_k(\lambda_0) = \dfrac{n}{\sum_{u \in V} \frac{1}{s_k(u)}}$ for $0 \leq k \leq d$ (where $q_k = p_0 + ... + p_k$, $s_k(u) = |\Gamma_0(u)| + |\Gamma_1(u)| + ... + |\Gamma_k(u)|$ and n is number of vertices) $\iff$ $\dfrac{\sum_{u \in V} n/(n - k_d(u))}{\sum_{u \in V} k_d(u)/(n - k_d(u))} = \sum_{i=0}^{d} \dfrac{\pi_0^2}{m(\lambda_i)\pi_i^2}$ (where $\pi_h = \prod_{\substack{i=0 \\ i \neq h}}^{d} (\lambda_h - \lambda_i)$ and $k_d(u) = |\Gamma_d(u)|$). Largest part of main results on which I would like to bring attention, can be

found in [23], [38], [24] and [9].

Keywords: graph, adjacency matrix, principal idempotent matrices, adjacency algebra, distance matrix, distance $\circ$-algebra, distance-regular graph, distance polynomials, predistance polynomials, spectrum, orthogonal systems

Chapter I

Basic results from Algebraic graph theory

1 Basic definitions from graph theory

We first introduce some basic notation from Algebraic graph theory. Throughout the thesis, $\Gamma = (V, E)$ stands for a (simple and finite) connected graph, with vertex set $V = \{u, v, w, ...\}$ and edge set $E = \{\{u, v\}, \{w, z\}, ...\}$. Two vertices u and v in a graph Γ are called *adjacent* (or *neighbors*) in Γ if $\{u, v\}$ is an edge of Γ. If $e = \{u, v\}$ is an edge of Γ, then e is called *incident* with the vertices u and v.

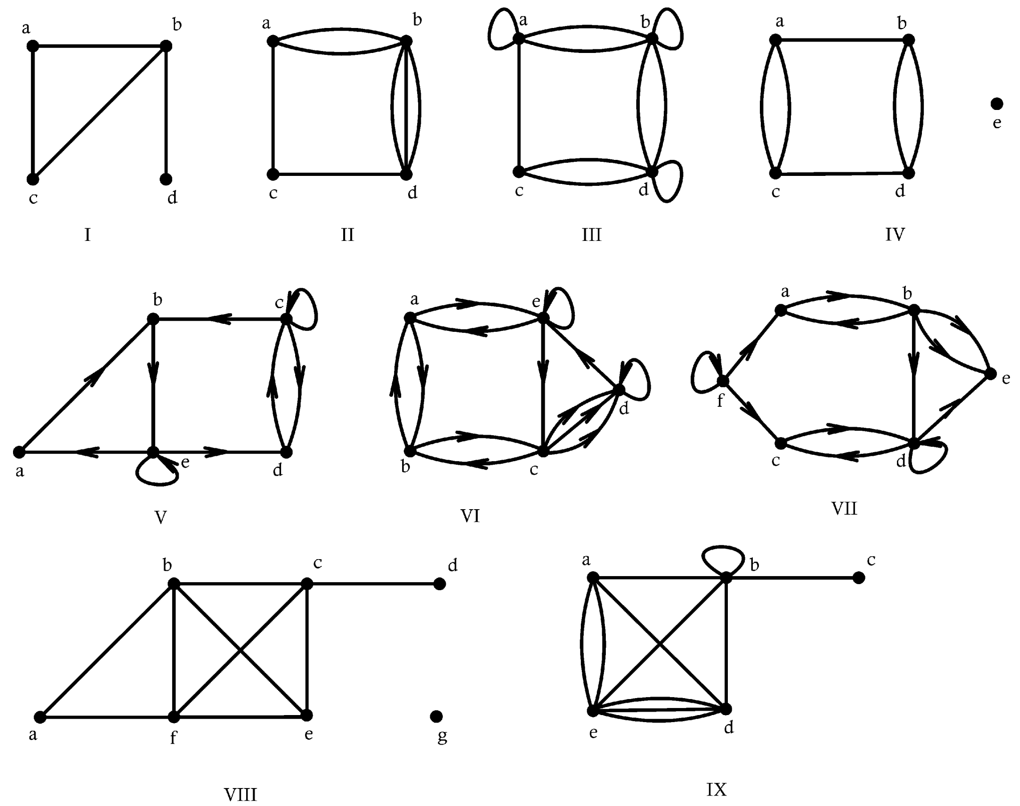


FIGURE 1
Types of graphs (the simple graphs are I and VIII, all others are not simple).

The *degree* (or *valency*) of a vertex $x \in V$ in a graph, denoted by δ_x, is the number of edges that are incident with that vertex x. A graph is *regular* of degree k (or *k-regular*) if every vertex has degree k. Adjacency between vertices u and v ($\{u, v\} \in E$) will be denoted by $u \sim v$.

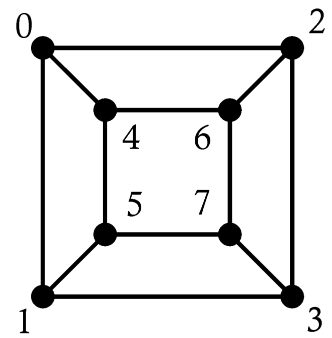
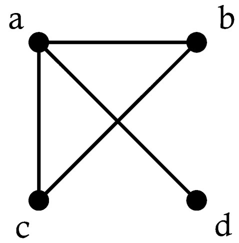


FIGURE 2
The cube ($V = \{0,1,2,3,4,5,6,7\}$,
$E = \{\{0,1\},\{0,2\},\{2,3\},\{1,3\},\{0,4\},\{1,5\},\{2,6\},\{3,7\},\{4,5\},\{4,6\},\{6,7\},\{5,7\}\}$).

Matrix $\boldsymbol{A}$ (or $\boldsymbol{A}(\Gamma)$) stands for *adjacency 01-matrix* of a graph Γ - with rows and columns indexed by the vertices of Γ and $(\boldsymbol{A})_{uv} = 1$ iff $u \sim v$ and equal to 0 otherwise.

A sequence of edges that link up with each other is called a *walk*. The *length* of a walk is the number of edges in the walk. Consecutive edges in a walk must have a vertex in common, so a walk determines a sequence of vertices. In general, a *walk of length n* from the vertex u to the vertex v is a sequence $[x_1, e_1, x_2, e_2, ..., e_n, x_{n+1}]$ of edges and vertices with property $e_i = \{x_i, x_{i+1}\}$ for $i = 1, ..., n$ and $x_1 = u$, $x_{n+1} = v$. The vertices x_2, x_3, ..., x_n are called *internal* vertices. If $[x_1, e_1, x_2, e_2, ..., x_n, e_n, x_{n+1}]$ is a walk from u to v then $[x_{n+1}, e_n, x_n, e_{n-1}, ..., x_2, e_1, x_1]$ is a walk from v to u. We may speak of either of these walks as a *walk between* u and v. If $u = v$, then the walk is said to be *closed*.



$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

FIGURE 3
Simple graph and its adjacency matrix.

For two vertices $u, v \in V$, an *uv-path* (or *path*) is a walk from u to v with all of its edges distinct. A path is called *simple* if all of its vertices are different. A path from u to u is called a *cycle*, if all of its internal vertices are different, and the length of a shortest cycle of a graph is called its *girth*. A simple graph is called *connected* if there is a path between every pair of distinct vertices of the graph.

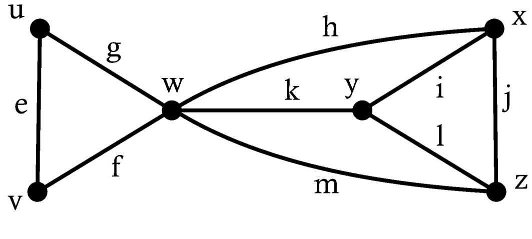


FIGURE 4
Simple connected graph (paths $[u, g, w, k, y, i, x, h, w, f, v]$ and $[u, w, y, z, w, v]$ go from u to v).

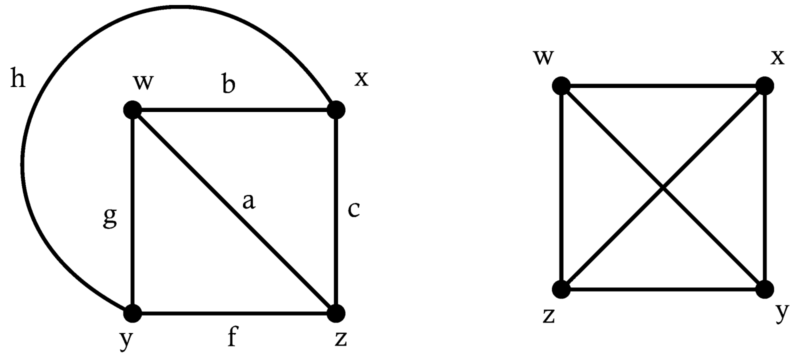


FIGURE 5
Simple graph drawn in two different ways (examples of walks are $[g, a, f, h, b]$, $[c, f, h, c, a, g]$ and $[g, b, c, f, f, a, g]$. Walk $[g, b, c, f, f, a, g]$ has length 7. The vertex sequences for these walks, respectively, are $[y, w, z, y, x, w]$, $[x, z, y, x, z, w, y]$ and $[y, w, x, z, y, z, w, y]$).

The *distance* $\partial(x, y)$ (or $\text{dist}_\Gamma(x, y)$) in Γ of two vertices x, y is the length of a shortest xy-simple path in Γ; if no such simple path exists, we set $\text{dist}(x, y) = \infty$. The *eccentricity* of a vertex u is $\text{ecc}(u) := \max_{v \in V} \text{dist}(u, v)$ and the *diameter* of the graph is $D := \max_{u \in V} \text{ecc}(u)$. The set $\Gamma_k(u)$ denotes the set of vertices at distance k from vertex u. Thus, the degree of vertex u is $\delta_u := |\Gamma_1(u)| \equiv |\Gamma(u)|$.

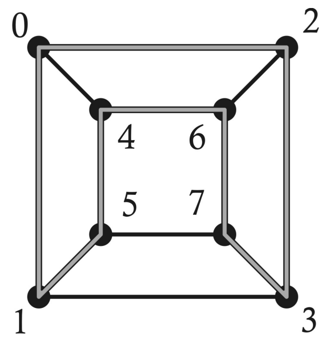
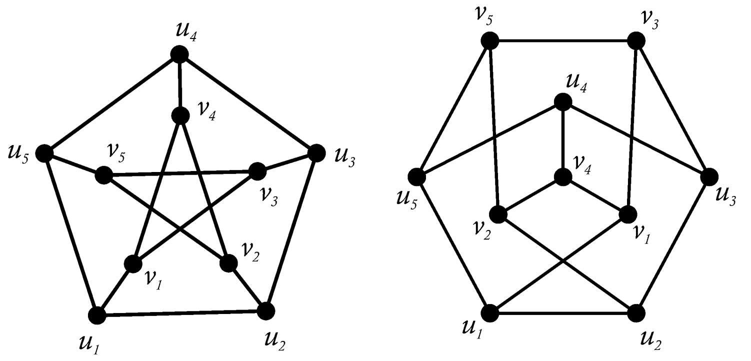


FIGURE 6
A Hamiltonian cycle of the cube (Hamiltonian cycle - cycle that visits every vertex of the graph exactly once, except for the last vertex, which duplicates the first one) where is, for example $\partial(2, 7) = 2$, $\text{ecc}(5) = 3$, $D = 3$, $\delta_4 = 3$.

With $\text{Mat}_{m\times n}(\mathbb{F})$ we will denote the set of all $m \times n$ matrices whose entries are numbers from a field $\mathbb{F}$ (in our case $\mathbb{F}$ is the set of real numbers $\mathbb{R}$ or the set of complex numbers $\mathbb{C}$). For every $B \in \text{Mat}_{n\times n}(\mathbb{F})$ define the *trace* of B by $\text{trace}(B) = \sum_{i=1}^{n} b_{ii} = b_{11} + b_{22} + ... + b_{nn}$. An *eigenvector* of a matrix A is a nonzero $v \in \mathbb{F}^n$ such that $Av = \lambda v$ for some scalar $\lambda \in \mathbb{F}$. An *eigenvalue* of A is a scalar λ such that $Av = \lambda v$ for some nonzero $v \in \mathbb{F}^n$. Any such pair, (λ, v), is called an *eigenpair* for A. We will denote the set of all distinct eigenvalues by $\sigma(A)$. Vector space $\mathcal{E}_\lambda = \ker(A - \lambda I) := \{x \mid (A - \lambda I)x = 0\}$ is called an *eigenspace* for A. For square matrices A, the number $\rho(A) = \max_{\lambda\in\sigma(A)} |\lambda|$ is called the *spectral radius* of A.

The *spectrum* of a graph Γ is the set of numbers which are eigenvalues of $\boldsymbol{A}(\Gamma)$, together with their multiplicities as eigenvalues of $\boldsymbol{A}(\Gamma)$. If the distinct eigenvalues of $\boldsymbol{A}(\Gamma)$ are $\lambda_0 > \lambda_1 > ... > \lambda_{s-1}$ and their multiplicities are $m(\lambda_0)$, $m(\lambda_1)$,...,$m(\lambda_{s-1})$, then we shall write

$$\text{spec}(\Gamma) = \{\lambda_0^{m(\lambda_0)}, \lambda_1^{m(\lambda_1)}, ..., \lambda_{s-1}^{m(\lambda_{s-1})}\}.$$



$$\mathbf{A}=\begin{array}{c} \\ u_1\\u_2\\u_3\\u_4\\u_5\\v_1\\v_2\\v_3\\v_4\\v_5\end{array}\begin{array}{c}\begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 & v_1 & v_2 & v_3 & v_4 & v_5\end{matrix}\\ \begin{bmatrix} 0&1&0&0&1&1&0&0&0&0\\ 1&0&1&0&0&0&1&0&0&0\\ 0&1&0&1&0&0&0&1&0&0\\ 0&0&1&0&1&0&0&0&1&0\\ 1&0&0&1&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&1&1&0\\ 0&1&0&0&0&0&0&0&1&1\\ 0&0&1&0&0&1&0&0&0&1\\ 0&0&0&1&0&1&1&0&0&0\\ 0&0&0&0&1&0&1&1&0&0 \end{bmatrix}\end{array}$$

FIGURE 7
Petersen graph drawn in two ways and its adjacency matrix. It is not hard to compute that trace$(A) = 0$, $\det((A-\lambda I)) = (\lambda-3)(x-1)^5(\lambda+2)^4$, $\sigma(A) = \{3, 1, -2\}$, $\dim(\ker(A-3I)) = 1$, $\dim(\ker(A-I)) = 5$, $\dim(\ker(A+2I)) = 4$, $\rho(A) = 3$, $\text{spec}(\Gamma) = \{3^1, 1^5, -2^4\}$.

Let $\sigma(A)$ be the set of all (different) eigenvalues for some matrix A, and let $\lambda \in \sigma(A)$. The *algebraic multiplicity* of λ is the number of times it is repeated as a root of the characteristic polynomial (recall that polynomial $p(\lambda) = \det(A-\lambda I)$ is called the *characteristic polynomial* for A). In other words, $\text{alg mult}_A(\lambda_i) = a_i$ if and only if $(x-\lambda_1)^{a_1}...(x-\lambda_s)^{a_s} = 0$ is the characteristic equation for A. When $\text{alg mult}_A(\lambda) = 1$, λ is called a *simple eigenvalue*. The *geometric multiplicity* of λ is $\dim\ker(A-\lambda I)$. In other words, $\text{geo mult}_A(\lambda)$ is the maximal number of linearly independent eigenvectors associated with λ.

Matrix $A \in \text{Mat}_{n\times n}(\mathbb{F})$ is said to be a *reducible matrix* when there exists a permutation matrix P (a permutation matrix is a square 0-1 matrix that has exactly one entry 1 in each row and each column and 0s elsewhere) such that $P^\top AP = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$, where X and Z are both square. Otherwise A is said to be an *irreducible matrix*. $P^\top AP$ is called a *symmetric permutation* of A - the effect of $P^\top AP$ is to interchange rows in the same way as columns are interchanged.

In the rest of this chapter we recall some basic results from algebraic graph theory, that we will need later:

(a.1) Since Γ is connected, $\boldsymbol{A}$ is an irreducible nonnegative matrix. Then, by the Perron-Frobenius theorem, the maximum eigenvalue λ_0 is simple, positive (in fact, it coincides with the spectral radius of $\boldsymbol{A}$), and has a positive eigenvector $\boldsymbol{v}$, say, which is useful to normalize in such a way that $\min_{u\in V} \boldsymbol{v}_u = 1$. Moreover, Γ is regular if and only if $\boldsymbol{v} = \boldsymbol{j}$, the all-1 vector (then $\lambda_0 = \delta$, the degree of Γ).

(a.2) The number of walks of length $l \geq 0$ between vertices u and v is $a^l_{uv} := (\boldsymbol{A}^l)_{uv}$.

(a.3) If $\Gamma = (V, E)$ has spectrum $\text{spec}(\Gamma) = \{\lambda_0^{m(\lambda_0)}, \lambda_1^{m(\lambda_1)}, ..., \lambda_d^{m(\lambda_d)}\}$ then the total number of (rooted) closed walks of length $l \geq 0$ is $\text{trace}(\boldsymbol{A}^l) = \sum_{i=0}^d m(\lambda_i)\lambda_i^l$.

(a.4) If Γ has $d+1$ distinct eigenvalues, then $\{I, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^d\}$ is a basis of the adjacency or Bose-Mesner algebra $\mathcal{A}(\Gamma)$ of matrices which are polynomials in $\boldsymbol{A}$. Moreover, if Γ has diameter D,

$\dim\mathcal{A}(\Gamma) = d + 1 \geq D + 1$,

because $\{I, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^D\}$ is a linearly independent set of $\mathcal{A}(\Gamma)$. Hence, the diameter is always less than the number of distinct eigenvalues: $D \leq d$.

(a.5) A graph $\Gamma = (V, E)$ with eigenvalues $\lambda_0 > \lambda_1 > ... > \lambda_d$ is a regular graph if and only if there exists a polynomial $H \in \mathbb{R}_d[x]$ such that $H(\boldsymbol{A}) = \boldsymbol{J}$, the all-1 matrix. This polynomial is unique and it is called the Hoffman polynomial. It has zeros at the eigenvalues λ_i, $i \neq 0$, and $H(\lambda_0) = n := |V|$. Thus,

$$H = \frac{n}{\pi_0} \prod_{i=1}^{d} (x - \lambda_i),$$

where $\pi_0 := \prod_{i=1}^{d} (\lambda_0 - \lambda_i)$.

2 Perron-Frobenius theorem

(2.01) Lemma

Let $\langle \cdot, \cdot \rangle$ be the standard inner product for $\mathbb{R}^n$ ($\langle x, y \rangle = x^\top y$), and let A be a real symmetric $n \times n$ matrix. If $\mathcal{U}$ is an A-invariant subspace of $\mathbb{R}^n$, then $\mathcal{U}^\perp$ is also A-invariant.

Proof: Recall that for a subspace $\mathcal{U}$ is said to be A-invariant if $Au \in \mathcal{U}$ for all $u \in \mathcal{U}$. We want to prove that $Av \in \mathcal{U}^\perp$ for all $v \in \mathcal{U}^\perp$.

Since A is real symmetric matrix, for any two vectors u and v, we have

$$\langle v, Au \rangle = v^T(Au) = (v^T A)u = (Av)^T u = \langle Av, u \rangle. \tag{1}$$

If $u \in \mathcal{U}$, then $Au \in \mathcal{U}$; hence if $v \in \mathcal{U}^\perp$ then $\langle v, Au \rangle = 0$. Consequently, by equation (1), $\langle Av, u \rangle = 0$ whenever $u \in \mathcal{U}$ and $v \in \mathcal{U}^\perp$. This implies that $Av \in \mathcal{U}^\perp$ whenever $v \in \mathcal{U}^\perp$, and therefore $\mathcal{U}^\perp$ is A-invariant. □

(2.02) Lemma

Consider arbitrary rectangular matrix P of order $m \times n$ in which columns are linearly independent. The column space of P is A-invariant if and only if there is a matrix D such that $AP = PD$.

Proof: Denote by $\mathcal{M}$ the column space of P, i.e. $\mathcal{M} = \text{span}\{P_{*1}, P_{*2}, ..., P_{*n}\}$ where P_{*i} is ith column of matrix P. Because columns of P are linearly independent we have $\dim(\mathcal{M}) = n$.

($\Rightarrow$) Assume that $\mathcal{M}$ is A-invariant. That means $AP_{*1} \in \mathcal{M}$, $AP_{*2} \in \mathcal{M}$,..., $AP_{*n} \in \mathcal{M}$. Since $\mathcal{M} = \text{span}\{P_{*1}, P_{*2}, ..., P_{*n}\}$ and vectors $P_{*1}, P_{*2}, ..., P_{*n}$ are linearly independent they form a basis for vector space $\mathcal{M}$. Now, there are unique coefficients $d_{ij} \in \mathbb{F}$ such that

$$\begin{aligned} AP_{*1} &= d_{11}P_{*1} + d_{21}P_{*2} + ... + d_{n1}P_{*n} \\ AP_{*2} &= d_{12}P_{*1} + d_{22}P_{*2} + ... + d_{n2}P_{*n} \\ &\;... \\ AP_{*n} &= d_{1n}P_{*1} + d_{2n}P_{*2} + ... + d_{nn}P_{*n} \end{aligned}$$

which gives

$$A \begin{bmatrix} P_{*1} & P_{*2} & ... & P_{*n} \end{bmatrix} = \begin{bmatrix} P_{*1} & P_{*2} & ... & P_{*n} \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} & ... & d_{1n} \\ d_{21} & d_{22} & ... & d_{2n} \\ \vdots & \vdots & ... & \vdots \\ d_{n1} & d_{n2} & ... & d_{nn} \end{bmatrix}$$

or, simply, $AP = PD$.

($\Leftarrow$) Assume that there is a matrix D such that $AP = PD$. Because $\mathcal{M} = \text{span}\{P_{*1}, P_{*2}, ..., P_{*n}\}$ and vectors $P_{*1}, P_{*2}, ..., P_{*n}$ are linearly independent they form basis for vector space $\mathcal{M}$. First note that AP_{*i} is the i-th column of AP. Since $AP = PD$, this is equal to the i-th column of PD. But i-th column of PD is $d_{1i}P_{*1} + d_{2i}P_{*2} + ... + d_{n,i}P_{*n}$. Therefore, AP_{*i} is a linear combination of $P_{*1}, ..., P_{*n}$.

Now, pick arbitrary $x \in \mathcal{M}$. We know that there unique scalars $c_1, c_2, ..., c_n \in \mathbb{F}$ such that $x = c_1P_{*1} + c_2P_{*2} + ... + c_nP_{*n}$. Now we have

$$Ax = A(c_1P_{*1} + c_2P_{*2} + ... + c_nP_{*n}) = c_1AP_{*1} + c_2AP_{*2} + ... + c_nAP_{*n}.$$

Every AP_{*i} is linear combination of $P_{*1}, P_{*2}, ..., P_{*n}$, therefore $Ax \in \mathcal{M}$. □

(2.03) Lemma

Let A be a real symmetric matrix. If u and v are eigenvectors of A with different eigenvalues, then u and v are orthogonal.

Proof: Suppose that $Au = \mu u$ and $Av = \eta v$. As A is symmetric, equation (1) implies that $\mu\langle v, u\rangle = \langle v, Au\rangle = \langle Av, u\rangle = \eta\langle v, u\rangle$. As $\mu \neq \eta$, we must have $\langle v, u\rangle = 0$. □

(2.04) Lemma

The eigenvalues of a real symmetric matrix A are real numbers.

Proof: Let u be an eigenvector of A with eigenvalue λ. Then by taking the complex conjugate of the equation $Au = \lambda u$ we get $\overline{Au} = \overline{\lambda u}$, which is equivalent with $A\overline{u} = \overline{\lambda}\overline{u}$ and so $\overline{u}$ is also an eigenvector of A. Now since eigenvector are not zero we have $\langle u, \overline{u}\rangle > 0$. Vectors u and $\overline{u}$ are eigenvectors of A, and if they have different corresponding eigenvalues λ and $\overline{\lambda}$, than by Lemma 2.03 $\langle u, \overline{u}\rangle = 0$, a contradiction. We can conclude $\lambda = \overline{\lambda}$ and the lemma is proved. □

(2.05) Lemma

Let A be an $n \times n$ real symmetric matrix. If $\mathcal{U}$ is a nonzero A-invariant subspace of $\mathbb{R}^n$, then $\mathcal{U}$ contains a real eigenvector of A.

Proof: We know from Lemma 2.04 that the eigenvalues of a real symmetric matrix A are real numbers. Pick one real eigenvalue, θ say. Notice that we can find at last one real eigenvector for θ (we know from definition of eigenvalue that there is some nonzero eigenvector v, and if this vector have entry(s) which are complex we can consider equations $Av = \theta v$ and $A\overline{v} = \theta\overline{v}$ (this is true) from which $A(v + \overline{v}) = \theta(v + \overline{v})$). Hence a real symmetric matrix A has at least one real eigenvector (any vector in the kernel of $(A - \theta I)$, to be precise).

Let R be a matrix whose columns form an orthonormal basis for $\mathcal{U}$. Then, because $\mathcal{U}$ is A-invariant, $AR = RB$ for some square matrix B (Lemma 2.02). Since $R^TR = I$ we have

$$R^TAR = R^TRB = B,$$

which implies that B is symmetric, as well as real. Since every symmetric matrix has at least one eigenvalue, we may choose a real eigenvector u of B with eigenvalue λ. Then $ARu = RBu = \lambda Ru$. Now, since $u \neq \mathbf{0}$ and the columns of R are linearly independent we have $Ru \neq \mathbf{0}$. Notice that if $v = [v_1, ..., v_n]^T$ then $Av = v_1A_{*1} + ... + v_nA_{*n}$ where A_{*i} are ith column of matrix A. Therefore, Ru is an eigenvector of A contained in $\mathcal{U}$. □

(2.06) Lemma

Let A be a real symmetric $n \times n$ matrix. Then $\mathbb{R}^n$ has an orthonormal basis consisting of eigenvectors of A.

Proof: Let $\{u_1, ..., u_m\}$ be an orthonormal (and hence linearly independent) set of $m < n$ eigenvectors of A, and let $\mathcal{M}$ be the subspace that they span. Since A has at least one eigenvector we have $m \geq 1$. The subspace $\mathcal{M}$ is A-invariant, and hence $\mathcal{M}^{\perp}$ is A-invariant (Lemma 2.01), and so $\mathcal{M}^{\perp}$ contains a (normalized) eigenvector u_{m+1} (Lemma 2.05). Then $\{u_1, ..., u_m, u_{m+1}\}$ is an orthonormal set of $m+1$ eigenvectors of A. Therefore, a simple induction argument shows that a set consisting of one normalized eigenvector can be extended to an orthonormal basis consisting of eigenvectors of A. □

(2.07) Proposition

Suppose that A is an $n \times n$ matrix, with entries in $\mathbb{R}$. Suppose further that A has eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}$, not necessarily distinct, with corresponding eigenvectors $v_1, ..., v_n \in \mathbb{R}^n$ and that $v_1, ..., v_n$ are linearly independent. Then

$$P^{-1}AP = D$$

where $P = \begin{bmatrix} v_1 & v_2 & ... & v_n \end{bmatrix}$ *and* $D = \operatorname{diag}(\lambda_1, \lambda_2, ..., \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ \vdots & \vdots & ... & \vdots \\ 0 & 0 & ... & \lambda_n \end{bmatrix}$.

Proof: Since v_1, ... v_n are linearly independent, they form a basis for $\mathbb{R}^n$, so that every $u \in \mathbb{R}^n$ can be written uniquely in the form

$$u = \alpha_1 v_1 + ... + \alpha_n v_n, \text{ where } \alpha_1, ..., \alpha_n \in \mathbb{R}, \tag{2}$$

and

$$Au = A(\alpha_1 v_1 + ... + \alpha_n v_n) = \alpha_1 A v_1 + ... + \alpha_n A v_n = \lambda_1 \alpha_1 v_1 + ... + \lambda_n \alpha_n v_n \tag{3}$$

Writing $c = (\alpha_1, \alpha_2, ..., \alpha_n)^{\top}$ we see that Equation (2) and (3) can be rewritten as

$$u = Pc \quad \text{and} \quad Au = P \begin{bmatrix} \lambda_1 \alpha_1 \\ \vdots \\ \lambda_n \alpha_n \end{bmatrix} = PDc$$

respectively, so that

$$APc = PDc.$$

Note that $c \in \mathbb{R}^n$ is arbitrary. This implies that $(AP - PD)c = \mathbf{0}$ for every $c \in \mathbb{R}^n$. Hence we must have $AP = PD$. Since the columns of P are linearly independent, it follows that P is invertible. Hence $P^{-1}AP = D$ as required. □

Suppose that A is an $n \times n$ matrix, with entries in $\mathbb{R}$. We say that A is *diagonalizable* if there exists an invertible matrix P, with entries in $\mathbb{R}$, such that $P^{-1}AP$ is a diagonal matrix, with entries in $\mathbb{R}$.

(2.08) Proposition

Suppose that A is an $n \times n$ matrix, with entries in $\mathbb{R}$. Suppose further that A is diagonalizable. Then A has n linearly independent eigenvectors in $\mathbb{R}^n$.

Proof: Suppose that A is diagonalizable. Then there exists an invertible matrix P, with entries in $\mathbb{R}$, such that $D = P^{-1}AP$ is a diagonal matrix, with entries in $\mathbb{R}$. Denote by v_1, ... v_n the columns of P. From $AP = PD$ (where $D = \operatorname{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$) it is not hard to show that

$$Av_1 = \lambda_1 v_1, \; ..., \; A_n v_n = \lambda_n v_n.$$

It follows that A has eigenvalues $\lambda_1, ..., \lambda_n \in \mathbb{R}$, with corresponding eigenvectors $v_1, ..., v_n \in \mathbb{R}^n$. Since P is invertible and $v_1, ..., v_n$ are the columns of P, it follows that the eigenvectors $v_1, ..., v_n$ are linearly independent. □

(2.09) Proposition

Let M be a $n \times n$ real symmetric matrix. Then there exist an orthogonal matrix $P = \begin{bmatrix} v_1 & v_2 & ... & v_n \end{bmatrix}$ such that

$$M = PDP^\top$$

where D is diagonal matrix whose diagonal entries are the eigenvalues of M, namely $D = \operatorname{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$, not necessarily distinct, with corresponding eigenvectors $v_1, v_2, ..., v_n \in \mathbb{R}$.

Proof: Since M is a real symmetric $n \times n$ matrix then $\mathbb{R}^n$ has an orthonormal basis consisting of eigenvectors of M (Lemma 2.06). Denote these eigenvectors by $v_1, v_2, .., v_n$, and set them like columns of matrix P ($P = \begin{bmatrix} v_1 & v_2 & ... & v_n \end{bmatrix}$). Notice that

$$P^\top P = \begin{bmatrix} — v_1 — \\ — v_2 — \\ \vdots \\ — v_n — \end{bmatrix} \begin{bmatrix} | & | & & | \\ v_1 & v_2 & ... & v_n \\ | & | & & | \end{bmatrix} = I_{n \times n}$$

That is

$$P^{-1} = P^\top. \tag{4}$$

Next, since $\{v_1, v_2, .., v_n\}$ is linearly independent set of eigenvectors, we have $P^{-1}MP = D$ (Proposition 2.07), where D is diagonal matrix whose diagonal entries are the eigenvalues of M, namely $D = \operatorname{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$, not necessarily distinct, which correspond to eigenvectors $v_1, ..., v_n \in \mathbb{R}^n$. With another words $M = PDP^{-1}$. By Equation (4) the result follows. □

(2.10) Theorem (Rayleigh's quotient)

Let $\langle \cdot, \cdot \rangle$ be the standard inner product for $\mathbb{R}^n$ ($\langle x, y \rangle = x^\top y$), and let M be a real symmetric matrix with largest eigenvalue λ_0. Then

$$\frac{\langle y, My \rangle}{\langle y, y \rangle} \leq \lambda_0, \quad \forall y \in \mathbb{R}^n \backslash \{0\}$$

with equality if and only if y is an eigenvector of M with eigenvalue λ_0.

Proof: Since M is a symmetric matrix it can be written as $M = PDP^\top$ where P is some orthogonal matrix having the eigenvectors of M as columns and D is diagonal matrix whose diagonal entries are the eigenvalues of M, not necessarily distinct, that correspond to columns of P (Lemma 2.09). Then for arbitrary $y \in \mathbb{R}^n \backslash \{0\}$

$$\langle y, My \rangle = y^\top My = y^\top PDP^\top y = \underbrace{(y^\top P)}_{1 \times n} \underbrace{D}_{n \times n} \underbrace{(P^\top y)}_{n \times 1} \leq (y^\top P)\lambda_0 I(P^\top y) = \lambda_0 y^\top y = \lambda_0 \langle y, y \rangle.$$

The result for first part (inequality) follows.

For second part we want to show that equality hold if and only if y is an eigenvector of M with eigenvalue λ_0. We have

$$\frac{\langle y, My \rangle}{\langle y, y \rangle} = \lambda_0 \;\Leftrightarrow\; \langle y, PDP^\top y \rangle = \langle y, \lambda_0 y \rangle \;\Leftrightarrow\; \langle P^\top y, DP^\top y \rangle = \langle P^\top y, \lambda_0 I(P^\top y) \rangle$$

$$\Leftrightarrow \ \langle P^\top y, (D - \lambda_0 I)P^\top y\rangle = 0 \ \Leftrightarrow \ (P^\top y)_i = 0 \ \forall i : \lambda_i \neq \lambda_0$$

so y is orthogonal to all columns of P that are eigenvectors of the eigenvalues λ_i ($\lambda_i \neq \lambda_0$). But then y must be in the eigenspace of the eigenvalue λ_0. The result for second part (equality) follows. □

(2.11) Proposition (independent eigenvectors)

Let $\{\lambda_1, \lambda_2, ..., \lambda_k\}$ *be a set of distinct eigenvalues for* A.

(i) If $\{(\lambda_1, x_1), (\lambda_2, x_2), ..., (\lambda_k, x_k)\}$ *is a set of eigenpairs for* A, *then* $\mathcal{S} = \{x_1, x_2, ..., x_k\}$ *is a linearly independent set.*

(ii) If $\mathcal{B}_i$ *is a basis for* $\ker(A - \lambda_i I)$, *then* $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup ... \cup \mathcal{B}_k$ *is a linearly independent set.*

Proof: (i) Suppose $\mathcal{S}$ is a dependent set. If the vectors in $\mathcal{S}$ are arranged so that $\mathcal{M} = \{x_1, x_2, ..., x_r\}$ is a maximal linearly independent subset, then

$$x_{r+1} = \sum_{i=1}^{r} \alpha_i x_i,$$

and multiplication on the left by $A - \lambda_{r+1} I$ produces

$$(A - \lambda_{r+1} I)x_{r+1} = \sum_{i=1}^{r} \alpha_i (A - \lambda_{r+1} I)x_i,$$

$$0 = \sum_{i=1}^{r} \alpha_i (\lambda_i - \lambda_{r+1}) x_i.$$

Because $\mathcal{M}$ is linearly independent, $\alpha_i(\lambda_i - \lambda_{r+1}) = 0$ for each i. Consequently, $\alpha_i = 0$ for each i (because the eigenvalues are distinct), and hence $x_{r+1} = 0$. But this is impossible because eigenvectors are nonzero. Therefore, the supposition that $\mathcal{S}$ is a dependent set must be false.

(ii) Assume that basis $\mathcal{B}_t$ for $\ker(A - \lambda_t I)$ is of the form

$$\{v_{t1}, v_{t2}, ..., v_{tr_t}\}, \quad 1 \leq t \leq k.$$

Because $\ker(A - \lambda_t I)$ is a vector space that mean

$$\sum_{i=1}^{r_t} c_i v_{ti} \in \ \ker(A - \lambda_t I) \ \text{ for arbitrary } c_i \in \mathbb{F}.$$

Now, consider equation

$$\sum_{i=1}^{k} \sum_{j=1}^{r_i} \alpha_{ij} v_{ij} = \mathbf{0} \text{ for unknown } \alpha_{ij} \in \mathbb{F}.$$

We can rewrite this equation in the following form:

$$\underbrace{\sum_{j=1}^{r_1} \alpha_{1j} v_{1j}}_{\in \ker(A - \lambda_1 I)} + \underbrace{\sum_{j=1}^{r_2} \alpha_{2j} v_{2j}}_{\in \ker(A - \lambda_2 I)} + ... + \underbrace{\sum_{j=1}^{r_k} \alpha_{kj} v_{kj}}_{\in \ker(A - \lambda_k I)} = \mathbf{0},$$

and this is possible if and only if

$$\sum_{j=1}^{r_i} \alpha_{ij} v_{ij} = \mathbf{0}, \quad i = 1, 2, ..., k.$$

By assumption the v_{*j}'s from $\{v_{i1}, v_{i2}, ..., v_{ir_i}\}$, $1 \leq i \leq k$, are linearly independent, and hence $\alpha_{ij} = 0 \ \forall i, j$. Therefore, $\mathcal{B}$ is linearly independent. □

Recall: Let $\sigma(A)$ be a set of all (distinct) eigenvalues for some matrix A, and let $\lambda \in \sigma(A)$. The algebraic multiplicity of λ is the number of times it is repeated as a root of the characteristic polynomial. In other words, $\operatorname{alg\,mult}_A(\lambda_i) = a_i$ if and only if $(x-\lambda_1)^{a_1}...(x-\lambda_s)^{a_s} = 0$ is the characteristic equation for A. The geometric multiplicity of λ is $\dim\ker(A-\lambda I)$. In other words, $\operatorname{geo\,mult}_A(\lambda)$ is the maximal number of linearly independent eigenvectors associated with λ.

(2.12) Theorem (diagonalizability and multiplicities)

A matrix $A \in \operatorname{Mat}_{n\times n}(\mathbb{C})$, is diagonalizable if and only if

$$\operatorname{geo\,mult}_A(\lambda) = \operatorname{alg\,mult}_A(\lambda)$$

for each $\lambda \in \sigma(A)$.

Proof: $(\Leftarrow)$ Suppose $\operatorname{geo\,mult}_A(\lambda_i) = \operatorname{alg\,mult}_A(\lambda_i) = a_i$ for each eigenvalue λ_i. If there are k distinct eigenvalues, and if $\mathcal{B}_i$ is a basis for $\ker(A-\lambda_i I)$, then $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup ... \cup \mathcal{B}_k$ contains $\sum_{i=1}^k a_i = n$ vectors. We just proved in Proposition 2.11(ii) that $\mathcal{B}$ is a linearly independent set, so $\mathcal{B}$ represents a complete set of linearly independent eigenvectors of A, and we know this insures that A must be diagonalizable.

$(\Rightarrow)$ Conversely, if A is diagonalizable, and if λ is an eigenvalue for A with $\operatorname{alg\,mult}_A(\lambda) = a$, then there is a nonsingular matrix P such that

$$P^{-1}AP = D = \begin{pmatrix} \lambda I_{a\times a} & 0 \\ 0 & B \end{pmatrix}$$

where $\lambda \notin \sigma(B)$. Consequently,

$$\operatorname{rank}(A-\lambda I) = \operatorname{rank} P(D-\lambda I)P^{-1} = \operatorname{rank}\left(P\begin{bmatrix} \lambda I_{a\times a} - \lambda I & 0 \\ 0 & B-\lambda I \end{bmatrix} P^{-1}\right) =$$

$$= \operatorname{rank}(B-\lambda I) = n - a \quad\Rightarrow\quad a = n - \operatorname{rank}(A-\lambda I),$$

and thus $\operatorname{geo\,mult}_A(\lambda) = \dim\ker(A-\lambda I) = n - \operatorname{rank}(A-\lambda I) = a = \operatorname{alg\,mult}_A(\lambda)$. □

Recall: Matrix $A \in \operatorname{Mat}_{n\times n}(\mathbb{F})$ is said to be a reducible matrix when there exists a permutation matrix P (a permutation matrix is a square 0-1 matrix that has exactly one entry 1 in each row and each column and 0s elsewhere) such that $P^\top AP = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$, where X and Z are both square. Otherwise A is said to be an irreducible matrix. $P^\top AP$ is called a symmetric permutation of A - the effect of $P^\top AP$ is to interchange rows in the same way as columns are interchanged.

(2.13) Theorem (Perron-Frobenius)

Let M be a nonnegative irreducible symmetric matrix. Then the largest eigenvalue λ_0 has algebraic multiplicity 1 and has an eigenvector whose entries are all positive. For all other eigenvalues we have $|\lambda_i| \leq \lambda_0$.

Proof: Suppose x is an eigenvector of M for the eigenvalue λ_0, i.e. $Mx = \lambda_0 x$. Let $y = |x|$ (entry-wise i.e. $y = (|x_1|, ..., |x_n|)$). Since M is nonnegative matrix we have $\langle y, My\rangle \geq \langle x, Mx\rangle$ and this imply

$$\frac{\langle y, My\rangle}{\langle y, y\rangle} = \frac{y^\top My}{y^\top y} = \frac{y^\top My}{x^\top x} \geq \frac{x^\top Mx}{x^\top x} = \frac{\langle x, Mx\rangle}{\langle x, x\rangle} \overset{Thm. 2.10}{=} \lambda_0,$$

that is $\frac{\langle y, My \rangle}{\langle y, y \rangle} \geq \lambda_0$. By Theorem 2.10 $\dfrac{\langle z, Mz \rangle}{\langle z, z \rangle} \leq \lambda_0, \ \forall z \in \mathbb{R}^n \backslash \{0\}$. We may conclude

$$\frac{\langle y, My \rangle}{\langle y, y \rangle} = \lambda_0$$

so y must be a non-negative eigenvector for the eigenvalue λ_0 (see Teorem 2.10).

Now we want to show that all entries in y are strictly positive, and this we will show by contradiction: we will assume that there exist i such that $y_i = 0$ and we will see that this assumption imply that $y_j = 0$ for all $j = 1, 2, ..., n$.

Assume that $y_i = 0$ for some $i \in \{1, 2, ..., n\}$. Then

$$0 = \lambda_0 y_i = (\lambda_0 y)_i = (My)_i = \sum_{j=1}^{n} \underbrace{(M)_{ij}}_{\geq 0} \underbrace{y_j}_{\geq 0} .$$

Since all elements in sum are nonnegative, then for each $j = 1, 2, ..., n$, $(M)_{ij}$ or y_j must be equal to 0. If $(M)_{ij} = 0$ for all $j = 1, 2, ..., n$, then, since M is symmetric, we would have that M is reducible matrix, a contradiction. So there must bi some j such that $(M)_{ij} \neq 0$. Now consider different case. If there exist one and just one j such that $(M)_{ij} = 0$, and that j is i i.e. if $(M)_{ii} = 0$, and $(M)_{ij} > 0$ for all $j \neq i$, $j = 1, 2, ..., n$ then we would obtain that all entries in y are equal to 0, a contradiction. So, there must be some $j \in \{1, 2, ..., n\}$ such that $j \neq i$ and $(M)_{ij} \neq 0$. For this j we must have that $y_j = 0$. Repeating this process over and over for every such y_j (and on similar way using irreducibility and fact that y is eigenvalue) we get that $y = 0$, which is a contradiction.

Assumption that there exist some $i \in \{1, 2, ..., n\}$ such that $y_i = 0$ lead us in contradiction, so it is not true. Therefore, entries of eigenvector y are all strictly positive, which also implies that

$$\text{any eigenvector } x \text{ for the eigenvalue } \lambda_0 \text{ cannot have entries that are } 0. \tag{5}$$

Next we want to show that $\operatorname{alg\,mult}_A(\lambda_0) = 1$. First consider geometric multiplicity. Suppose there are two linearly independent eigenvectors $x_1, x_2 \in \ker(A - \lambda_0 I)$ for the eigenvalue λ_0. Then vector $z = \alpha x_1 + \beta x_2$ is also eigenvector for the eigenvalue λ_0, for every $\alpha, \beta \in \mathbb{R}$. This means that for some choice of α and β we can find one entry $z_i = 0$, a contradiction with (5). So, eigenvalue λ_0 must have geometric multiplicity 1. Since for diagonalizable matrices algebraic multiplicity is equal to geometric multiplicity for every eigenvalue λ (Theorem 2.12), and every symmetric matrix is diagonalizable (by Lemma 2.06 and Proposition 2.07) we may conclude that $\operatorname{alg\,mult}_A(\lambda_0) = 1$.

It is only left to shown that for all other eigenvalues λ_i of M we must have $|\lambda_i| \leq \lambda_0$. Assume that there exist some eigenvalue λ_i such that $|\lambda_i| > \lambda_0$. Let y be eigenvector that correspond to λ_i. Notice that

$$My = \lambda_i y \iff y^\top My = \lambda_i y^\top y \iff \frac{\langle y, My \rangle}{\langle y, y \rangle} = \lambda_i.$$

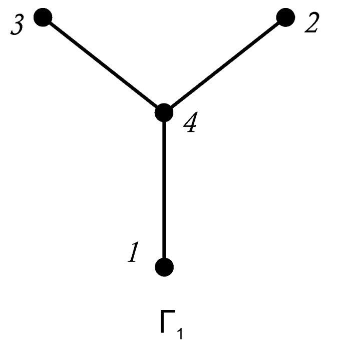
If we denote by z vector $z = |y|$, since $|\langle y, My \rangle| = |y^\top My| = |y|^\top M|y| = z^\top Mz$ (matrix M is nonnegative) we have

$$\frac{|\langle y, My \rangle|}{|\langle y, y \rangle|} = |\lambda_i| \iff \frac{\langle z, Mz \rangle}{\langle z, z \rangle} = |\lambda_i| \ (> \lambda_0).$$

So we have find vector $z \in \mathbb{R}^n \backslash \{0\}$ such that $\frac{\langle z, Mz \rangle}{\langle z, z \rangle} > \lambda_0$, a contradiction (with Rayleigh's quotient (Theorem 2.10)). The result follows. □

(2.14) Example

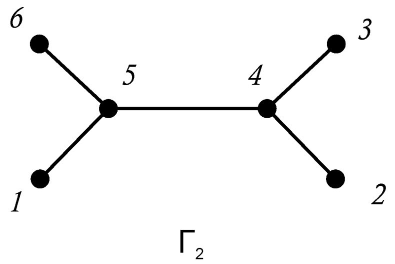
a) Consider matrix $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$. Characteristic polynomial of A is $\mathrm{char}(\lambda) = \lambda^2(\lambda - \sqrt{3})(\lambda + \sqrt{3})$. It follow that maximal eigenvalue $\lambda_0 = \sqrt{3}$ is simple, positive and coincides with spectral radius of A. Eigenvector for eigenvalue λ_0 is $\boldsymbol{v} = (1, 1, 1, \sqrt{3})^\top$, so it is positive.



$$\mathbf{A} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

FIGURE 8
Simple graph Γ_1 and its adjacency matrix.

b) Consider matrix $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$. Characteristic polynomial of A is $\mathrm{char}(\lambda) = \lambda^2(\lambda - 1)(\lambda - 2)(\lambda + 1)(\lambda + 2)$. It follow that maximal eigenvalue $\lambda_0 = 2$ is simple, positive and coincides with spectral radius of A. Eigenvector for eigenvalue λ_0 is $\boldsymbol{v} = (1, 1, 1, 2, 2, 1)^\top$, so it is positive.



$$\mathbf{A} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

FIGURE 9
Simple graph Γ_2 and its adjacency matrix. ◊

(2.15) Proposition

Let Γ be a regular graph of degree k. Then:
(i) k is an eigenvalue of Γ.
(ii) If Γ is connected, then the multiplicity of k is one.
(iii) For any eigenvalue λ of Γ, we have $|\lambda| \leq k$.

Proof: Recall that, degree of v is the number of edges of which v is an endpoint, and that graph is regular of degree k (or k-valent) if each of its vertices has degree k.

(i) Let $\boldsymbol{j} = [1\ 1\ ...\ 1]^\top$; then if A is the adjacency matrix of Γ we have

$$Au = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} + a_{12} + ... + a_{1n} \\ \vdots \\ a_{n1} + a_{n2} + ... + a_{nn} \end{bmatrix} \overset{\forall \text{ vertex has degree k}}{=\!=} \begin{bmatrix} k \\ \vdots \\ k \end{bmatrix} = k\boldsymbol{j},$$

so that k is an eigenvalue of Γ.

(ii) Let $x = [x_1\ x_2\ ...,\ x_k]^\top$ denote any non-zero vector for which $Ax = kx$ (that is let x be arbitrary eigenvector that correspond to eigenvalue k) and suppose that x_j is an entry of x having the largest absolute value. Since $Ax = kx$ we have

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n \end{bmatrix} = \begin{bmatrix} kx_1 \\ \vdots \\ kx_n \end{bmatrix}$$

$$a_{j1}x_1 + a_{j2}x_2 + ... + a_{jn}x_n = kx_j$$

$$(Ax)_j = kx_j$$

where $(Ax)_j$ denote jth entry of vector Ax. So $\sum' x_i = kx_j$ where the summation is over those k vertices x_i which are adjacent to x_j. By the maximal property of x_j, it follows that $x_i = x_j$ for all these vertices. If Γ is connected we may proceed successively in this way, eventually showing that all entries of x are equal. Thus x is a multiple of $\boldsymbol{j}$, and the space of eigenvectors associated with the eigenvalue k has dimension one.

(iii) Suppose that $Ay = \lambda y$, $y \neq 0$, and let y_j denote an entry of y which is largest in absolute value. By the same argument as in (ii), we have $\sum' y_i = \lambda y_j$, where the summation is over those k vertices y_i which are adjacent to y_j, and so

$$|\lambda||y_j| = |\sum' y_i| \leq \sum' |y_i| \leq k|y_j|.$$

Thus $|\lambda| \leq k$, as required. □

We conclude: Since Γ is connected, $\boldsymbol{A}$ is an irreducible nonnegative matrix. Then, by the Perron-Frobenius theorem, the maximum eigenvalue λ_0 is simple, positive (in fact, it coincides with the spectral radius of $\boldsymbol{A}$), and has a positive eigenvector $\boldsymbol{v}$, say, which is useful to normalize in such a way that $\min_{u \in V} \boldsymbol{v}_u = 1$. Moreover, Γ is regular if and only if $\boldsymbol{v} = \boldsymbol{j}$, the all-1 vector (then $\lambda_0 = \delta$, the degree of Γ).

3 The number of walks of a given length between two vertices

(3.01) Lemma

Let $\Gamma = (V, E)$ denote a simple graph and let $\boldsymbol{A}$ be the adjacency matrix of Γ. The number of walks of length $l \geq 0$ in Γ, joining u to v is the (u, v)-entry of the matrix $\boldsymbol{A}^l$.

Proof: We will prove this lemma by mathematical induction.

BASIS OF INDUCTION

If $l = 0$ we have $\boldsymbol{A}^0 = \boldsymbol{I}$ (1 in position (u, u) for all $u \in V$), and the claim is true because walks of length 0 are of form $[u]$ for all $u \in V$. If $l = 1$ we have $\boldsymbol{A}^1 = \boldsymbol{A}$, and so (u, v)-entry of $\boldsymbol{A}^1$ is 1 (resp. 0) if and only if u and v are (resp. are not) adjacent. The claim is true because walks of length 1 are $[u, v]$ iff u and v are adjacent.

INDUCTION STEP

Denote the (u, v)-entry of $\boldsymbol{A}$ by a_{uv} and denote the (u, v)-entry of $\boldsymbol{A}^L$ by a_{uv}^L.

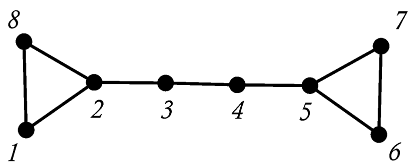
Suppose that the result is true for $l = L$, that is, there is a_{uv}^L walks of length L in Γ between u and v. Consider identity $\boldsymbol{A}^{L+1} = \boldsymbol{A}^L\boldsymbol{A}$. We have

$$(\boldsymbol{A}^{L+1})_{uv} = a_{uv}^{L+1} = \sum_{z\in V} a_{uz}^L a_{zv}.$$

We know, by assumption, that a_{uz}^L is number of walks of length L in Γ joining u and z. If $a_{zv} = 0$ we know by definition of adjacency matrix that z and v are not neighbors, and because of that there is no walk of length $L+1$ between u and v, which contains z as its penultimate vertex. For every $a_{zv} = 1$ we know that there is a_{uz}^L walks of length $L+1$ between u and v, which contains z as its penultimate vertex (there is a_{uz}^L walks of length L between u and z, and because $a_{zv} = 1$, z and v are adjacent, which means that we can use a_{uz}^L walks between u and z and then use edge between z and v). When we sum up these numbers, we deduce that a_{uv}^{L+1} is the number of walks of length $L+1$ joining u to v. Therefore, the result for all l follows by induction. □

(3.02) Example

Consider graph Γ_3 given in Figure 10. Let's say that we want to find number of walks of length 4 and 5, between vertices 3 and 7. Then first that we need to do is to find adjacency matrix for Γ_3. After that we need to find $(3, 7)$-entry (or $(7, 3)$-entry) of $\boldsymbol{A}^4$ and $\boldsymbol{A}^5$.



Γ_3

$$\mathbf{A}= \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1&2&3&4&5&6&7&8 \end{array} \\ \begin{bmatrix} 0&1&0&0&0&0&0&1\\ 1&0&1&0&0&0&0&1\\ 0&1&0&1&0&0&0&0\\ 0&0&1&0&1&0&0&0\\ 0&0&0&1&0&1&1&0\\ 0&0&0&0&1&0&1&0\\ 0&0&0&0&1&1&0&0\\ 1&1&0&0&0&0&0&0 \end{bmatrix} \end{array}$$

FIGURE 10
Simple graph Γ_3 and its adjacency matrix.

We have $A^4 = \begin{bmatrix} 7&6&5&1&1&0&0&6\\ 6&12&2&5&0&1&1&6\\ 5&2&7&0&5&1&1&5\\ 1&5&0&7&2&5&5&1\\ 1&0&5&2&12&6&6&1\\ 0&1&1&5&6&7&6&0\\ 0&1&1&5&6&6&7&0\\ 6&6&5&1&1&0&0&7 \end{bmatrix}$ and $A^5 = \begin{bmatrix} 12&18&7&6&1&1&1&13\\ 18&14&17&2&7&1&1&18\\ 7&17&2&12&2&6&6&7\\ 6&2&12&2&17&7&7&6\\ 1&7&2&17&14&18&18&1\\ 1&1&6&7&18&12&13&1\\ 1&1&6&7&18&13&12&1\\ 13&18&7&6&1&1&1&12 \end{bmatrix}$. ◊

4 The total number of (rooted) closed walks of a given length

(4.01) Definition (functions of diagonalizable matrices)

Let $A = PDP^{-1}$ be a diagonalizable matrix with k distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$, where the eigenvalues in $D = \mathrm{diag}(\lambda_1 I, \lambda_2 I, ..., \lambda_k I)$ are grouped by repetition. For a function $f(x)$ that have finite value at each $\lambda_i \in \sigma(A)$, define

$$f(A) = Pf(D)P^{-1} = P \begin{pmatrix} f(\lambda_1)I & 0 & ... & 0 \\ 0 & f(\lambda_2)I & ... & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & ... & f(\lambda_k)I \end{pmatrix} P^{-1}.$$

◇

(4.02) Lemma

Let A be an $n \times n$ matrix with entries in $\mathbb{R}$ and suppose that A has r different eigenvalues $\sigma(A) = \{\lambda_1, \lambda_2, ..., \lambda_r\}$. Let $\mathcal{E}_i$ denote eigenspace that correspondent to eigenvalue λ_i :

$$\mathcal{E}_i := \ker(A - \lambda_i I) = \{x \in \mathbb{R}^n \mid (A - \lambda_i I)x = \mathbf{0}\} = \{x \in \mathbb{R}^n \mid Ax = \lambda_i x\}.$$

Suppose further that $\dim(\mathcal{E}_i) = m_i$ for all $1 \leq i \leq r$. Then matrix A is diagonalizable if and only if $m_1 + m_2 + ... + m_r = n$.

Proof: The geometric multiplicity of λ is $\dim\ker(A - \lambda I)$. In other words, $\mathrm{geo\,mult}_A(\lambda)$ is the maximal number of linearly independent eigenvectors associated with λ. By assumption $\mathrm{geo\,mult}_A(\lambda_i) = \dim(\mathcal{E}_i) = m_i$ $(1 \leq i \leq r)$. Let $\mathcal{B}_i$ denote a basis for eigenspace $\mathcal{E}_i$. Consider set $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup ... \cup \mathcal{B}_r$. Before we begin with proof of this lemma we want to answer the following question: How much of vectors are in set $\mathcal{B}$?

For eigenspaces $\mathcal{E}_1$, $\mathcal{E}_2$, ...,$\mathcal{E}_r$ we have $\mathcal{E}_i \cap \mathcal{E}_j = \{\mathbf{0}\}$ for $i \neq j$. Why? Because, if there is some nonzero vector $u \in \mathrm{span}(\mathcal{B})$ such that $u \in \mathcal{E}_i$ and $u \in \mathcal{E}_j$ for $i \neq j$ we will have

$$Au = \lambda_i u \text{ and } Au = \lambda_j u \quad \Longrightarrow \quad \lambda_i u = \lambda_j u \quad \Longrightarrow \quad (\lambda_i - \lambda_j)u = \mathbf{0}$$

from which it follows that $\lambda_i = \lambda_j$, a contradiction (with $\lambda_i \neq \lambda_j$). Therefore

$$\mathcal{E}_i \cap \mathcal{E}_j = \{\mathbf{0}\} \text{ for } i \neq j \quad \Longrightarrow \quad \mathcal{B}_i \cap \mathcal{B}_j = \emptyset \text{ for } i \neq j,$$

and since $\dim(\mathcal{E}_i) = |\mathcal{B}_i| = m_i$ we can conclude that set $\mathcal{B}$ have $m_1 + m_2 + ... + m_r$ elements, i.e.

$$|\mathcal{B}| = m_1 + m_2 + ... + m_r.$$

($\Rightarrow$) Assume that A is diagonalizable. Then A has n linearly independent eigenvectors, say $\{u_1, u_2, ..., u_n\}$, in $\mathbb{R}^n$ (Proposition 2.08). Since for every u_i $(1 \leq i \leq n)$ there exist some $\mathcal{E}_j$ $(j \in \{1, 2, ..., r\})$ such that $u_i \in \mathcal{E}_j$, and since $\mathcal{E}_i \cap \mathcal{E}_j = \{\mathbf{0}\}$ we must have $\dim(\mathcal{E}_1) + \dim(\mathcal{E}_2) + ... + \dim(\mathcal{E}_r) \geq n$ that is

$$m_1 + m_2 + ... + m_r \geq n.$$

On the other hand, since $\mathcal{B}_i$ is basis for $\mathcal{E}_i \subseteq \mathbb{R}^n$, and since $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup ... \cup \mathcal{B}_r$ we must have $\mathrm{span}(\mathcal{B}) \subseteq \mathbb{R}^n$, that is $|\mathcal{B}| \leq n$. With another words

$$m_1 + m_2 + ... + m_r \leq n.$$

Therefore

$$m_1 + m_2 + ... + m_r = n.$$

($\Leftarrow$) Assume that $m_1 + m_2 + ... + m_r = n$ where
$m_i = \dim(\mathcal{E}_i) = \dim(\ker(A - \lambda_i I)) = \operatorname{geo\,mult}_A(\lambda_i)$. Every nonzero vector from $\mathcal{E}_i$ is eigenvector of matrix A, and this mean that every vector from $\mathcal{B}_i$ ($1 \leq i \leq r$) is eigenvector of A ($\mathcal{B}_i$ is basis for $\mathcal{E}_i$). Since $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$ for $i \neq j$, we obtain that

matrix A have n linearly independent eigenvectors.

By Proposition 2.07, this mean that A is diagonalizable. □

Let $\Gamma = (V, E)$ denote a simple graph with adjacency matrix $\boldsymbol{A}$ and with $d+1$ distinct eigenvalues $\lambda_0, \lambda_1, ..., \lambda_d$. Let $\mathcal{E}_i$ denote the eigenspace $\mathcal{E}_i = \ker(\boldsymbol{A} - \lambda_i I)$, and let $\dim(\mathcal{E}_i) = m_i$, for $0 \leq i \leq d$. Since $\boldsymbol{A}$ is real symmetric matrix, it is diagonalizable (Proposition 2.09), and for diagonalizable matrices we have

$$m_0 + m_1 + ... + m_d = n \tag{6}$$

by Lemma 4.02.

Matrix $\boldsymbol{A}$ is symmetric $n \times n$ matrix, so $\boldsymbol{A}$ have n distinct eigenvectors $U = \{u_1, u_2, ..., u_n\}$ which form orthonormal basis for $\mathbb{R}^n$ (Lemma 2.06). Notice that for every vector $u_i \in U$ there exist $\mathcal{E}_j$ such that $u_i \in \mathcal{E}_j$. Since $\mathcal{E}_i \cap \mathcal{E}_j = \emptyset$ for $i \neq j$, it is not possible that eigenvector u_i ($1 \leq i \leq n$) belongs to different eigenspaces. So, by Equation (6), we can divide set U to sets U_0, U_1, ...,U_d such that

$$U_i \text{ is a basis for } \mathcal{E}_i, \quad U = U_0 \cup U_1 \cup ... \cup U_d \quad \text{and} \quad U_i \cap U_j = \emptyset.$$

(4.03) Definition (principal idempotents)

Let $\Gamma = (V, E)$ denote simple graph with adjacency matrix $\boldsymbol{A}$, and let $\lambda_0 \geq \lambda_1 \geq ... \geq \lambda_d$ be distinct eigenvalues. For each eigenvalue λ_i, $0 \leq i \leq d$, let U_i be the matrix whose columns form an orthonormal basis of its eigenspace $\mathcal{E}_i := \ker(A - \lambda_i I)$. The *principal idempotents* of $\boldsymbol{A}$ are matrices $\boldsymbol{E}_i := U_i U_i^\top$. ◇

(4.04) Lemma

Let $\Gamma = (V, E)$ denote a simple graph with adjacency matrix $\boldsymbol{A}$ and with $d+1$ distinct eigenvalues $\lambda_0, \lambda_1, ..., \lambda_d$. Then there exist matrices $\boldsymbol{E}_0$, $\boldsymbol{E}_1$, ..., $\boldsymbol{E}_d$ such that for every function $f(x)$ that have finite value on $\sigma(\boldsymbol{A})$ we have

$$f(\boldsymbol{A}) = f(\lambda_0)\boldsymbol{E}_0 + f(\lambda_1)\boldsymbol{E}_1 + ... + f(\lambda_d)\boldsymbol{E}_d.$$

Proof: By Proposition 2.09, there exist matrix $P^\top$ such that

$$P^\top \boldsymbol{A} P = D, \quad \text{where } P = \begin{bmatrix} u_1 & u_2 & ... & u_n \end{bmatrix}, \ D = \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ \vdots & \vdots & ... & \vdots \\ 0 & 0 & ... & \lambda_n \end{bmatrix}$$

and diagonal entries λ_i of D are eigenvalues of $\boldsymbol{A}$, that don't need to be all distinct. Now, we can permute columns of matrix P so that P looks like $P = [U_0|U_1|...|U_d]$ (recall, U_i's are matrices which columns are orthonormal basis for $\ker(\boldsymbol{A} - \lambda_i I)$). Then $P^\top = \begin{bmatrix} U_0^\top \\ \hline U_1^\top \\ \hline \vdots \\ \hline U_d^\top \end{bmatrix}$ and

$$\boldsymbol{A} = PDP^\top = [U_0|U_1|...|U_d] \begin{bmatrix} \lambda_0 I & 0 & ... & 0 \\ 0 & \lambda_1 I & ... & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & ... & \lambda_d I \end{bmatrix} \begin{bmatrix} U_0^\top \\ \hline U_1^\top \\ \hline \vdots \\ \hline U_d^\top \end{bmatrix}.$$

Finally, from definition of function for diagonalizable matrices (Definition 4.01)

$$f(\boldsymbol{A}) = Pf(D)P^{-1} = [U_0|U_1|...|U_d] \begin{bmatrix} f(\lambda_0)I & 0 & ... & 0 \\ 0 & f(\lambda_1)I & ... & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & ... & f(\lambda_d)I \end{bmatrix} \begin{bmatrix} U_0^\top \\ \hline U_1^\top \\ \hline \vdots \\ \hline U_d^\top \end{bmatrix} =$$

$$= f(\lambda_0)U_0U_0^\top + f(\lambda_1)U_1U_1^\top + ... + f(\lambda_d)U_dU_d^\top$$
$$= f(\lambda_0)\boldsymbol{E}_0 + f(\lambda_1)\boldsymbol{E}_1 + ... + f(\lambda_d)\boldsymbol{E}_d.$$

□

(4.05) Proposition

Let $\Gamma = (V, E)$ denote a simple graph with adjacency matrix $\boldsymbol{A}$, with $d+1$ distinct eigenvalues $\lambda_0, \lambda_1, ..., \lambda_d$ and let $\boldsymbol{E}_0, \boldsymbol{E}_1, ..., \boldsymbol{E}_d$ be principal idempotents of Γ. Then each power of $\boldsymbol{A}$ can be expressed as a linear combination of the idempotents $\boldsymbol{E}_i$

$$\boldsymbol{A}^k = \sum_{i=0}^{d} \lambda_i^k \boldsymbol{E}_i;$$

Proof: We have $p(\boldsymbol{A}) = \sum_{i=0}^{d} p(\lambda_i)\boldsymbol{E}_i$, for every polynomial $p \in \mathbb{R}[x]$, where $\lambda_i \in \sigma(\boldsymbol{A})$ (Lemma 4.04). If for polynomial $p(x)$ we pick $p(x) = x^k$ we have

$$\boldsymbol{A}^k = \sum_{i=0}^{d} \lambda_i^k \boldsymbol{E}_i.$$

□

(4.06) Proposition

Let $\Gamma = (V, E)$ denote a simple graph with adjacency matrix $\boldsymbol{A}$, spectrum $spec(\Gamma) = \{\lambda_0^{m(\lambda_0)}, \lambda_1^{m(\lambda_1)}, ..., \lambda_d^{m(\lambda_d)}\}$ and let $\boldsymbol{E}_0$, $\boldsymbol{E}_1$, ..., $\boldsymbol{E}_d$ be principal idempotents of Γ. Then

$$\operatorname{trace}(\boldsymbol{E}_i) = m(\lambda_i), \;\; i = 0, 1, ...d.$$

Proof: For each eigenvalue λ_i, $0 \le i \le d$, we know that $\boldsymbol{E}_i = U_iU_i^\top$ where U_i is matrix whose columns form an orthonormal basis for the eigenspace $\mathcal{E}_i = ker(A - \lambda_i I)$. From linear algebra we also know that

$$\operatorname{trace}(AB) = \operatorname{trace}(BA),$$

where A and B are appropriate matrices for which product exist - proof of this is easy:

$$\operatorname{trace}(A_{m\times n}B_{n\times m}) = \sum_{i=1}^{m}(AB)_{ii} = \sum_{i=1}^{m}\sum_{k=1}^{n}(A)_{ik}(B)_{ki} =$$
$$= \sum_{k=1}^{n}\sum_{i=1}^{m}(B)_{ki}(A)_{ik} = \sum_{k=1}^{n}(BA)_{kk} = \operatorname{trace}(B_{n\times m}A_{m\times n}).$$

Therefore,

$$\operatorname{trace}(\boldsymbol{E}_i) = \operatorname{trace}(U_iU_i^\top) = \operatorname{trace}(U_i^\top U_i) = \operatorname{trace}(\begin{bmatrix} u_1 \\ \hline u_2 \\ \hline \vdots \\ \hline u_{m_i} \end{bmatrix} [u_1|u_2|...|u_{m_i}]) = \operatorname{trace}(I_{m_i\times m_i}),$$

where $m_i = m(\lambda_i)$. Therefore, $\operatorname{trace}(\boldsymbol{E}_i) = m(\lambda_i)$. □

(4.07) Theorem

If $\Gamma = (V, E)$ *has spectrum* $\mathrm{spec}(\Gamma) = \{\lambda_0^{m(\lambda_0)}, \lambda_1^{m(\lambda_1)}, ..., \lambda_d^{m(\lambda_d)}\}$ *then the total number of (rooted) closed walks of length* $l \geq 0$ *is* $\mathrm{trace}(\boldsymbol{A}^l) = \sum_{i=0}^{d} m(\lambda_i)\lambda_i^l$.

Proof: Number of closed walks of length k from vertex i to i is $(\boldsymbol{A}^k)_{ii}$. Therefore, to obtain the number of all closed walks of length k, we have to add values $(\boldsymbol{A}^k)_{ii}$ over all i, that is, we have to take the trace of $\boldsymbol{A}^k$. From Proposition 4.05, we have $\boldsymbol{A}^k = \sum_{i=0}^{d} \lambda_i^k \boldsymbol{E}_i$. If we take traces we get $\mathrm{trace}(\boldsymbol{A}^k) = \mathrm{trace}(\sum_{i=0}^{d} \lambda_i^k \boldsymbol{E}_i) = \sum_{i=0}^{d} \lambda_i^k \,\mathrm{trace}(\boldsymbol{E}_i)$. Therefore,

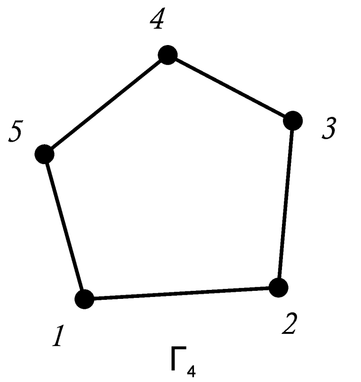
$$\mathrm{trace}(\boldsymbol{A}^k) = \sum_{i=0}^{d} m(\lambda_i)\lambda_i^k$$

(see Proposition 4.06), and result follows. □

(4.08) Example

Consider graph Γ_4 given in Figure 11. This graph has three eigenvalues $\lambda_0 = 2$, $\lambda_1 = \frac{\sqrt{5}}{2} - \frac{1}{2}$, $\lambda_2 = -\frac{\sqrt{5}}{2} - \frac{1}{2}$, and spectrum:

$$\mathrm{spec}(\Gamma_4) = \{2^1, \left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^2, \left(\frac{-\sqrt{5}}{2} - \frac{1}{2}\right)^2\}$$



$$\mathbf{A} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

FIGURE 11
Simple graph Γ_4 and its adjacency matrix.

Total number of rooted closed walks of lengths 3, 4 and 5 is

$$\mathrm{trace}(A^3) = 1 \cdot 2^3 + 2 \cdot \left(\tfrac{\sqrt{5}}{2} - \tfrac{1}{2}\right)^3 + 2 \cdot \left(\tfrac{-\sqrt{5}}{2} - \tfrac{1}{2}\right)^3 = 0,$$

$$\mathrm{trace}(A^4) = 1 \cdot 2^4 + 2 \cdot \left(\tfrac{\sqrt{5}}{2} - \tfrac{1}{2}\right)^4 + 2 \cdot \left(\tfrac{-\sqrt{5}}{2} - \tfrac{1}{2}\right)^4 = 30$$

and

$$\mathrm{trace}(A^5) = 1 \cdot 2^5 + 2 \cdot \left(\tfrac{\sqrt{5}}{2} - \tfrac{1}{2}\right)^5 + 2 \cdot \left(\tfrac{-\sqrt{5}}{2} - \tfrac{1}{2}\right)^5 = 10.$$ ◊

5 The adjacency (Bose-Mesner) algebra $\mathcal{A}(\Gamma)$

(5.01) Definition (adjacency algebra)

The *adjacency* (or *Bose-Mesner*) *algebra* of a graph Γ is algebra of matrices which are polynomials in $\boldsymbol{A}$ under the usual matrix operations. We shall denote this algebra by $\mathcal{A} = \mathcal{A}(\Gamma)$. Therefore

$$\mathcal{A}(\Gamma) = \{p(\boldsymbol{A}) : p \in \mathbb{F}[x]\}$$

(elements in $\mathcal{A}$ are matrices). ◇

(5.02) Proposition

Let $\Gamma = (V, E)$ denote a simple graph with adjacency matrix $\boldsymbol{A}$ and with $d+1$ distinct eigenvalues $\lambda_0, \lambda_1, ..., \lambda_d$. Principal idempotents $\boldsymbol{E}_0$, $\boldsymbol{E}_1$, ..., $\boldsymbol{E}_d$ satisfy the following properties:

(i) $\boldsymbol{E}_i\boldsymbol{E}_j = \delta_{ij}\boldsymbol{E}_i = \begin{cases} \boldsymbol{E}_i & \textit{if } i = j \\ 0 & \textit{if } i \neq j \end{cases}$;

(ii) $\boldsymbol{A}\boldsymbol{E}_i = \lambda_i\boldsymbol{E}_i$, *where* $\lambda_i \in \sigma(\boldsymbol{A})$;

(iii) $p(\boldsymbol{A}) = \sum_{i=0}^{d} p(\lambda_i)\boldsymbol{E}_i$, *for every polynomial* $p \in \mathbb{R}[x]$, *where* $\lambda_i \in \sigma(\boldsymbol{A})$;

(iv) $\boldsymbol{E}_0 + \boldsymbol{E}_1 + ... + \boldsymbol{E}_d = \sum_{i=0}^{d} \boldsymbol{E}_i = I$;

(v) $\sum_{i=0}^{d} \lambda_i\boldsymbol{E}_i = \boldsymbol{A}$, *where* $\lambda_i \in \sigma(\boldsymbol{A})$.

Proof: From Definition 4.03 we have that $\boldsymbol{E}_i = U_iU_i^\top$, where U_i is a matrix which columns form an orthonormal basis for eigenspace $\mathcal{E}_i = ker(\boldsymbol{A} - \lambda_i I)$, $(0 \le i \le d)$. We know that $\begin{bmatrix} U_1^\top \\ \hline U_2^\top \\ \hline \vdots \\ \hline U_d^\top \end{bmatrix} [U_1|U_2|...|U_d] = I$, so $U_i^\top U_j = \begin{cases} I & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$ Therefore,

$$\boldsymbol{E}_i\boldsymbol{E}_j = U_iU_i^\top U_jU_j^\top = \begin{cases} U_iU_j^\top & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} = \delta_{ij}\boldsymbol{E}_i,$$

and (i) follows.

For (ii) we have

$$\boldsymbol{A}\boldsymbol{E}_i = \boldsymbol{A}U_iU_i^\top = \boldsymbol{A}[u_1|u_2|...|u_{m_i}]U_i^\top = [\boldsymbol{A}u_1|\boldsymbol{A}u_2|...|\boldsymbol{A}u_{m_i}]U_i^\top =$$

$$= [\lambda_i u_1|\lambda_i u_2|...|\lambda_i u_{m_i}]U_i^\top = \lambda_i U_iU_i^\top = \lambda_i\boldsymbol{E}_i,$$

and (ii) follows.

Proofs for (iii), (iv) and (v) easy follow from Lemma 4.04. □

(5.03) Proposition

Suppose that non-zero vectors $v_1, ..., v_r$ in a finite-dimensional real inner product space are pairwise orthogonal. Then they are linearly independent.

Proof: Suppose that $\alpha_1, ..., \alpha_r \in \mathbb{R}$ and

$$\alpha_1 v_1 + ... + \alpha_r v_r = \mathbf{0}.$$

Then for every $i = 1, ..., r$, we have

$$0 = \langle \mathbf{0}, v_i \rangle = \langle \alpha_1 v_1, ..., \alpha_r v_r, v_i \rangle = \alpha_1 \langle v_1, v_i \rangle + ... + \alpha_r \langle v_r, v_i \rangle = \alpha_i \langle v_i, v_i \rangle$$

since $\langle v_j, v_i \rangle = 0$ if $j \neq i$. From assumption $v_i \neq \mathbf{0}$, so that $\langle v_i, v_i \rangle \neq 0$, and so we must have $\alpha_i = 0$ for every $i = 1, ..., r$. It follows that vectors $v_1, ..., v_r$ are linearly independent. □

(5.04) Proposition

If simple graph Γ has $d+1$ distinct eigenvalues, then $\{I, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^d\}$ is a basis of the adjacency (Bose-Mesner) algebra $\mathcal{A}(\Gamma)$.

Proof: We have that the set $\{\boldsymbol{E}_0, \boldsymbol{E}_1, ..., \boldsymbol{E}_d\}$ form orthogonal set (Proposition 5.02(i)). That means that set $\{\boldsymbol{E}_0, \boldsymbol{E}_1, ..., \boldsymbol{E}_d\}$ is linearly independent (Proposition 5.03).

Next, from Proposition 5.02(iii) we see that if for polinomyal p we pick 1, x, x^2, ..., x^d, then we can write $\boldsymbol{A}^i$, for every $i \in \{0, 1, 2, ..., d\}$, like linear combination of $\boldsymbol{E}_0$, $\boldsymbol{E}_1$, ..., $\boldsymbol{E}_d$:

$$\begin{aligned} I &= \boldsymbol{E}_0 + \boldsymbol{E}_1 + ... + \boldsymbol{E}_d, \\ \boldsymbol{A} &= \lambda_0 \boldsymbol{E}_0 + \lambda_1 \boldsymbol{E}_1 + ... + \lambda_d \boldsymbol{E}_d, \\ \boldsymbol{A}^2 &= \lambda_0^2 \boldsymbol{E}_0 + \lambda_1^2 \boldsymbol{E}_1 + ... + \lambda_d^2 \boldsymbol{E}_d, \\ &\ \ ... \\ \boldsymbol{A}^d &= \lambda_0^d \boldsymbol{E}_0 + \lambda_1^d \boldsymbol{E}_1 + ... + \lambda_d^d \boldsymbol{E}_d. \end{aligned}$$

Notice that the above equations we can write in matrix form

$$\begin{bmatrix} I \\ \boldsymbol{A} \\ \boldsymbol{A}^2 \\ \vdots \\ \boldsymbol{A}^d \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & ... & 1 \\ \lambda_0 & \lambda_1 & ... & \lambda_d \\ \lambda_0^2 & \lambda_1^2 & ... & \lambda_d^2 \\ \vdots & \vdots & & \vdots \\ \lambda_0^d & \lambda_1^d & ... & \lambda_d^d \end{bmatrix}}_{=B} \begin{bmatrix} \boldsymbol{E}_0 \\ \boldsymbol{E}_1 \\ \boldsymbol{E}_2 \\ \vdots \\ \boldsymbol{E}_d \end{bmatrix}.$$

Matrix $B^\top$ above is Vandermonde matrix, and it is not hard to prove that columns in $B^\top$ constitute a linearly independent set (see [37], page 185) (hint: columns of $B^\top$ form a linearly independent set if and only if $\ker(B^\top) = \{\boldsymbol{0}\}$).

Now, we set up question: Is it $\{I, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^d\}$ linearly independent set? Assume it is not. Then, they would be some numbers $\alpha_0, \alpha_1, ..., \alpha_d \in \mathbb{R}$ such that
$\alpha_0 I + \alpha_1 \boldsymbol{A} + \alpha_2 \boldsymbol{A}^2 + ... + \alpha_d \boldsymbol{A}^d = 0$. We would then obtain that

$$\beta_0 \boldsymbol{E}_0 + \beta_1 \boldsymbol{E}_1 + ... + \beta_d \boldsymbol{E}_d = 0$$

where

$$\beta_i = \alpha_0 + \alpha_1 \lambda_i + ... + \alpha_d \lambda_i^d, \quad 0 \le i \le d.$$

In general, it may happen that $\beta_i = 0$ for all i, even if some of α_i are not zero. But, from fact that

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_d \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \lambda_0 & \lambda_0^2 & ... & \lambda_0^d \\ 1 & \lambda_1 & \lambda_1^2 & ... & \lambda_1^d \\ 1 & \lambda_2 & \lambda_2^2 & ... & \lambda_2^2 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_d & \lambda_d^2 & ... & \lambda_d^d \end{bmatrix}}_{=B^\top} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_d \end{bmatrix}$$

where $B^\top$ is actually a Vandermonde matrix, above system have unique solution, and this imply that if some of α_i are nonzero, then also some of β_i are nonzero. We obtain that $\{\boldsymbol{E}_0, \boldsymbol{E}_1, ..., \boldsymbol{E}_d\}$ is linearly dependent set, a contradiction.

Since matrix B is invertible, every $\boldsymbol{E}_k$, $k = 0, 1, 2, ..., d$ can be expressed as linear combination of I, $\boldsymbol{A}$, $\boldsymbol{A}^2$, ..., $\boldsymbol{A}^d$. Matrices $\boldsymbol{A}^{d+1}$, $\boldsymbol{A}^{d+2}$,... we can write as linear combination of I, $\boldsymbol{A}$, $\boldsymbol{A}^2$,..., $\boldsymbol{A}^d$ because $\boldsymbol{A}^\ell = \sum_{i=0}^d \lambda_i^\ell \boldsymbol{E}_i$ for every ℓ (Proposition 4.05). So $\{I, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^d\}$ is maximal linearly independent set.

Therefore, $\{I, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^d\}$ is a basis of the adjacency algebra $\mathcal{A}(\Gamma)$. □

(5.05) Observation

From the last part of the proof of Proposition 5.04 we have that that principal idempotents are in fact also elements of Bose-Mesner algebra. ◇

(5.06) Proposition

Let $\Gamma = (V, E)$ denote a graph with diameter D. Prove that the set $\{I, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^D\}$ is linearly independent.

Proof: Assume that $\alpha_0 I + \alpha_1 \boldsymbol{A} + ... + \alpha_D \boldsymbol{A}^D = 0$ for some real scalars $\alpha_0, ..., \alpha_D$, not all 0. Let $i = \max\{0 \le j \le D : \alpha_j \neq 0\}$. Then

$$\boldsymbol{A}^i = \frac{1}{\alpha_i}(\alpha_0 I + \alpha_1 \boldsymbol{A} + ... + \alpha_{i-1}\boldsymbol{A}^{i-1}). \tag{7}$$

Pick $x, y \in V$ with $\partial(x, y) = i$. Recall that for $0 \le j \le D$, the (x, y)-entry of $\boldsymbol{A}^j$ is equal to the number of all walks from x to y that are of length j (see Lemma 3.01). Therefore the (x, y)-entry of $\boldsymbol{A}^j$ is 0 for $0 \le j \le i - 1$, and the (x, y)-entry of $\boldsymbol{A}^i$ is nonzero. But this contradicts Equation (7). □

(5.07) Proposition

In simple graph Γ with $d + 1$ distinct eigenvalues and the diameter D, the diameter is always less than the number of distinct eigenvalues: $D \le d$.

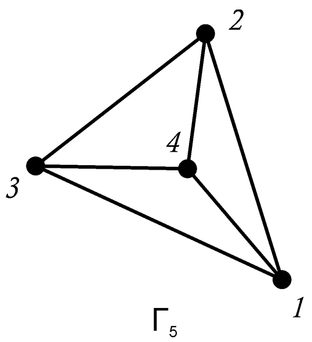
Proof: If Γ has $d + 1$ distinct eigenvalues, then $\{I, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^d\}$ is a basis of the adjacency or Bose-Mesner algebra $\mathcal{A}(\Gamma)$ of matrices which are polynomials in A (Proposition 5.04). Moreover, if Γ has diameter D,

$\dim \mathcal{A}(\Gamma) = d + 1 \ge D + 1$,

because $\{I, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^D\}$ is a linearly independent set of $\mathcal{A}(\Gamma)$ (Proposition 5.06). Hence, the diameter is always less than the number of distinct eigenvalues: $D \le d$. □

(5.08) Example

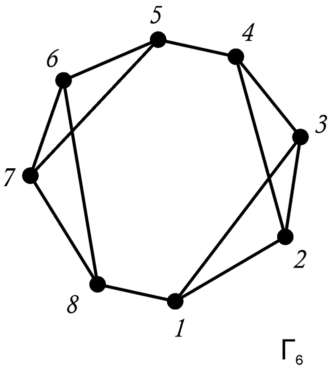
a) Consider graph Γ_5 given in Figure 12. Eigenvalues of Γ_5 are $\lambda_0 = 3$ and $\lambda_1 = -1$, so $d + 1 = 2$. Diameter is $D = 1$. Therefore $D = d$.



$$\mathbf{A} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

FIGURE 12
Simple graph Γ_5 and its adjacency matrix.

b) Consider graph Γ_6 given in Figure 13. Eigenvalues of Γ_6 are $\lambda_0 = 3$, $\lambda_1 = \sqrt{5}$, $\lambda_2 = 1$, $\lambda_3 = -1$ and $\lambda_4 = -\sqrt{5}$, so $d + 1 = 5$. Diameter of Γ_6 is $D = 3$. Therefore $D < d$.



$$\mathbf{A}= \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1&2&3&4&5&6&7&8 \end{array} \\ \left[\begin{array}{cccccccc} 0&1&1&0&0&0&0&1\\ 1&0&1&1&0&0&0&0\\ 1&1&0&1&0&0&0&0\\ 0&1&1&0&1&0&0&0\\ 0&0&0&1&0&1&1&0\\ 0&0&0&0&1&0&1&1\\ 0&0&0&0&1&1&0&1\\ 1&0&0&0&0&1&1&0 \end{array}\right] \end{array}$$

FIGURE 13
Simple graph Γ_6 and its adjacency matrix. ◊

6 Hoffman polynomial

Matrices P and Q with dimension $n \times n$ are said to be <u>similar matrices</u> whenever there exists a nonsingular matrix R such that $P = R^{-1}QR$. We write $P \simeq Q$ to denote that P and Q are similar.

(6.01) Lemma (similarity preserves eigenvalues)
Similar matrices have the same eigenvalues with the same multiplicities.

Proof: Product rules for determinants are

$$\det(AB) = \det(A)\det(B), \text{ for all } n \times n \text{ matrices } A,\, B$$

and

$$\det\begin{pmatrix} A & B \\ \mathbf{0} & D \end{pmatrix} = \det(A)\det(D) \text{ if } A \text{ and } D \text{ are square.}$$

Use the product rule for determinants in conjunction with the fact that $\det(P^{-1}) = \dfrac{1}{\det(P)}$ to write

$$\det(A - \lambda I) = \det(P^{-1}BP - \lambda I) = \det(P^{-1}(B - \lambda I)P) = \\ = \det(P^{-1})\det(B - \lambda I)\det(P) = \det(B - \lambda I).$$

Similar matrices have the same characteristic polynomial, so they have the same eigenvalues with the same multiplicities. □

(6.02) Lemma
If A and B are similar matrices and if A is diagonalizble, then B is diagonalizable.

Proof: Since A and B are similar, there exists a nonsingular matrix P, such that $B = P^{-1}AP$. Since matrix A is diagonalizable, we know that there exist an invertible matrix R, with entries in $\mathbb{R}$, such that $A = RA_0R^{-1}$, where A_0 is diagonal matrix, with entries in $\mathbb{R}$. Now we have

$$B = P^{-1}AP = P^{-1}RA_0R^{-1}P = (P^{-1}R)A_0(P^{-1}R)^{-1}.$$

If we define $D := P^{-1}R$ we have

$$B = DA_0D^{-1}.$$

Therefore, B is diagonalizable. □

(6.03) Lemma

Let A and B be a diagonalizable matrices. Then $AB = BA$ if and only if A and B can be simultaneously diagonalized i.e.,

$$A = UA_0U^{-1} \quad \text{and} \quad B = UB_0U^{-1}$$

for some invertible matrix U, where A_0 and B_0 are diagonal matrices.

Proof: ($\Rightarrow$) Assume that matrices A and B commutes, and assume that $\sigma(A) = \{\lambda_{k_0}, \lambda_{k_1}, ..., \lambda_{k_d}\}$, with multiplicities $m(\lambda_{k_0})$, $m(\lambda_{k_1})$, $...m(\lambda_{k_d})$. Since A is diagonalizable there exist invertible matrix P such that $A_0 = P^{-1}AP$ where columns of P are eigenvectors of A and $A_0 = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$ (λ's not necessary distinct). We can reorder columns in matrix P so that P produces $A_0 = \begin{bmatrix} \lambda_{k_0}I & \mathbf{0} & ... & \mathbf{0} \\ \mathbf{0} & \lambda_{k_1}I & ... & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & ... & \lambda_{k_d}I \end{bmatrix}$ where each I is the identity matrix of appropriate size.

Now consider matrix $D = P^{-1}BP$ (from which it follow that $B = PDP^{-1}$). We have

$$\begin{aligned} AB &= BA \\ PA_0P^{-1}\,PDP^{-1} &= PDP^{-1}\,PA_0P^{-1} \\ PA_0DP^{-1} &= PDA_0P^{-1} \\ A_0D &= DA_0 \\ \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & ... & \lambda_n \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} & ... & d_{1n} \\ d_{21} & d_{22} & ... & d_{2n} \\ \vdots & \vdots & & \vdots \\ d_{n1} & d_{n2} & ... & d_{nn} \end{bmatrix} &= \begin{bmatrix} d_{11} & d_{12} & ... & d_{1n} \\ d_{21} & d_{22} & ... & d_{2n} \\ \vdots & \vdots & & \vdots \\ d_{n1} & d_{n2} & ... & d_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & ... & \lambda_n \end{bmatrix} \\ \lambda_i d_{ij} &= d_{ij}\lambda_j \\ (\lambda_i - \lambda_j)d_{ij} &= 0. \end{aligned}$$

So, if $\lambda_i \neq \lambda_j$ we have $d_{ij} = 0$, and from this it follow $D = \begin{bmatrix} B_1 & \mathbf{0} & ... & \mathbf{0} \\ \mathbf{0} & B_2 & ... & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & ... & B_d \end{bmatrix}$ for some matrices B_1, B_2, ..., B_d, where each B_i is of the dimension $m(\lambda_{k_i}) \times m(\lambda_{k_i})$. Since B is diagonalizable and $D = P^{-1}BP$ it follows that D is diagonalizable (Lemma 6.02), so there exists an invertible matrix R, with entries in $\mathbb{R}$ such that $R^{-1}DR$ is a diagonal matrix, with entries in $\mathbb{R}$, that is

$$D = RD_0R^{-1}.$$

Similar matrices have the same eigenvalues with the same multiplicities (Lemma 6.01), so we have that $D_0 = B_0$ that is

$$D = RB_0R^{-1}$$

and from form of matrix D we can notice that R have form

$$R = \begin{bmatrix} R_1 & \mathbf{0} & ... & \mathbf{0} \\ \mathbf{0} & R_2 & ... & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & ... & R_d \end{bmatrix}.$$

Now we have

$$B = PDP^{-1} = PRB_0R^{-1}P^{-1}.$$

Notice that RA_0R^{-1} is equal to A_0 because $R_i\lambda_i IR_i^{-1} = \lambda_i I$. Therefore, we have found matrix $U := PR$ such that

$$A = UA_0U^{-1} \quad \text{and} \quad B = UB_0U^{-1}.$$

($\Leftarrow$) Conversely, assume that there exist matrix U such that

$$A = UA_0U^{-1} \quad \text{and} \quad B = UB_0U^{-1}$$

where A_0 and B_0 are diagonal matrices, for example

$$A_0 = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} \quad \text{and} \quad B_0 = \begin{bmatrix} b_{11} & 0 & \dots & 0 \\ 0 & b_{22} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & b_{nn} \end{bmatrix}.$$

Notice that $A_0B_0 = B_0A_0$. Then

$$AB = UA_0U^{-1}UB_0U^{-1} = UA_0B_0U^{-1} = UB_0A_0U^{-1} = UB_0U^{-1}UA_0U^{-1} = BA.$$

Therefore, matrices A and B commute. □

(6.04) Corollary

Let A and B be symmetric matrices. Then A and B are commuting matrices if and only if there exists an orthogonal matrix U such that

$$A = UA_0U^\top, \quad B = UB_0U^\top,$$

where A_0 is a diagonal matrix whose diagonal entries are the eigenvalues of A, and B_0 is a diagonal matrix whose diagonal entries are the eigenvalues of B.

Proof: Proof follow from Lemma 2.09 and from proof of Lemma 6.03. If we use Lemma 2.09, in the proof of Lemma 6.03 matrices P^{-1}, R^{-1} and U^{-1} we can replace with $P^\top$, $R^\top$ and $U^\top$, respectively. □

(6.05) Theorem (Hoffman polynomial)

Let $\Gamma = (V, E)$ denote simple graph with n vertices, $\boldsymbol{A}$ be the adjacency matrix of Γ and let $\boldsymbol{J}$ be the square matrix of order n, every entry of which is unity. There exists a polynomial $H(x)$ such that

$$\boldsymbol{J} = H(\boldsymbol{A})$$

if and only if Γ is regular and connected.

Proof: ($\Rightarrow$) Assume that there exist polynomial $H(x) = h_0 + h_1x + h_2x^2 + ... + h_kx^k$ such that $\boldsymbol{J} = H(\boldsymbol{A})$ that is $\boldsymbol{J} = h_0I + h_1\boldsymbol{A} + h_2\boldsymbol{A}^2 + ... + h_k\boldsymbol{A}^k$. Then we have

$$\boldsymbol{AJ} = h_0\boldsymbol{A} + h_1\boldsymbol{A}^2 + h_2\boldsymbol{A}^3 + ... + h_k\boldsymbol{A}^{k+1} \quad \text{and}$$

$$\boldsymbol{JA} = h_0\boldsymbol{A} + h_1\boldsymbol{A}^2 + h_2\boldsymbol{A}^3 + ... + h_k\boldsymbol{A}^{k+1},$$

that is $\boldsymbol{AJ} = \boldsymbol{JA}$ ($\boldsymbol{A}$ commutes with $\boldsymbol{J}$). With another words (since $\boldsymbol{A}$ is symmetric)

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{12} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{12} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix}.$$

We denoted valency of vertex u by δ_u, and it is not hard to see that $\delta_u = (\boldsymbol{AJ})_{uv}$ for arbitrary v, and that $\delta_v = (\boldsymbol{JA})_{uv}$ for arbitrary u. Since $(\boldsymbol{AJ})_{uv} = (\boldsymbol{JA})_{uv}$ we have $\delta_u = \delta_v$ for every $u, v \in V$. So Γ is regular.

Next, we want to prove that graph Γ is connected. It is not hard to see that, if u and v are any vertices of Γ, there is, for some t, a nonzero number as the (u, v)-th entry of $\boldsymbol{A}^t$; otherwise, no linear combination of the powers of $\boldsymbol{A}$ could have 1 as the (u, v)-th entry, and $\boldsymbol{J} = H(\boldsymbol{A})$ would be false. Thus, for some t, there is at least one path of length t from u to v. But this means Γ is connected.

($\Leftarrow$) Conversely, assume that Γ is regular (of degree k) and connected. As we saw in the proof on necessity, because Γ is regular, $\boldsymbol{A}$ commutes with $\boldsymbol{J}$. Thus, since $\boldsymbol{A}$ and $\boldsymbol{J}$ are symmetric commuting matrices, there exists an orthogonal matrix U such that

$$\boldsymbol{J} = UJ_0U^\top, \quad \boldsymbol{A} = UA_0U^\top,$$

where J_0 is a diagonal matrix whose diagonal entries are the eigenvalues of $\boldsymbol{J}$, namely $J_0 = \operatorname{diag}(n, 0, 0, ..., 0)$, and A_0 is a diagonal matrix whose diagonal entries are the eigenvalues of $\boldsymbol{A}$, namely $A_0 = \operatorname{diag}(\lambda_{t_1}, \lambda_{t_2}, ..., \lambda_{t_n})$. Now $\boldsymbol{j} = [1\ 1\ ...\ 1]^\top$ is an eigenvector of both $\boldsymbol{A}$ and $\boldsymbol{J}$, with k and n the corresponding eigenvalues, a consequence of the fact that Γ is regular of degree k. Because Γ is connected, k is an eigenvalue of $\boldsymbol{A}$ of multiplicity 1 (Proposition 2.15) (also, from the same proposition, an eigenvalue of largest absolute value; see also Theorem 2.13). Let $\lambda_0 = k$, λ_1, ..., λ_d be the distinct eigenvalues of $\boldsymbol{A}$, and let

$$H(x) = \frac{n \prod_{i=1}^{d} (x - \lambda_i)}{\prod_{i=1}^{d} (\lambda_0 - \lambda_i)}$$

where n is order of $\boldsymbol{A}$. We can always reorder columns of matrix U in $\boldsymbol{A} = UA_0U^\top$ and obtain that A_0 is of form $A_0 = \operatorname{diag}(\lambda_0, \lambda_{s_2}, ..., \lambda_{s_n})$. Then

$$H(A_0) = H(\begin{bmatrix} \lambda_0 & 0 & ... & 0 \\ 0 & \lambda_{s_2} & ... & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & ... & \lambda_{s_n} \end{bmatrix}) = \begin{bmatrix} H(\lambda_0) & 0 & ... & 0 \\ 0 & H(\lambda_{s_2}) & ... & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & ... & H(\lambda_{s_n}) \end{bmatrix} = \begin{bmatrix} n & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & ... & 0 \end{bmatrix} = J_0,$$

that is

$$H(A_0) = J_0,$$

say

$$J_0 = h_0 I + h_1 A_0 + h_2 A_0^2 + ... + h_d A_0^d.$$

Notice that

$$\boldsymbol{A} = UA_0U^\top,$$

$$\boldsymbol{A}^2 = \boldsymbol{A} \cdot \boldsymbol{A} = UA_0U^\top UA_0U^\top = UA_0^2U^\top,$$

$$...$$

$$\boldsymbol{A}^d = \underbrace{\boldsymbol{A} \cdot \boldsymbol{A} \cdot ... \cdot \boldsymbol{A}}_{d \text{ times}} = UA_0U^\top\, UA_0U^\top ... UA_0U^\top = UA_0^dU^\top,$$

so

$$\boldsymbol{J} = UJ_0U^\top = U(h_0 I + h_1 A_0 + h_2 A_0^2 + ... + h_d A_0^d)U^\top = H(\boldsymbol{A}).$$

□

Let us call

$$H(x) = \frac{n \prod_{i=1}^{d} (x - \lambda_i)}{\prod_{i=1}^{d} (\lambda_0 - \lambda_i)}$$

the *Hoffman polynomial* of graph Γ, and say that the polynomial and graph are associated with each other. It is clear from this formula that this polynomial is of smallest degree for which $\boldsymbol{J} = H(\boldsymbol{A})$ holds. Further, the distinct eigenvalues of $\boldsymbol{A}$, other than λ_0, are roots of $H(x)$.

Chapter II

Distance-regular graphs

In this chapter we will define distance-regular graphs and show some examples of graphs that are distance-regular. Main results will be the following characterizations:

(A) Γ is distance-regular if and only if it is distance-regular around each of its vertices and with the same intersection array.

(B) A graph $\Gamma = (V, E)$ with diameter D is distance-regular if and only if, for any integers $0 \le i, j \le D$, its distance matrices satisfy

$$\boldsymbol{A}_i \boldsymbol{A}_j = \sum_{k=0}^{D} p_{ij}^k \boldsymbol{A}_k \quad (0 \le i, j \le D)$$

for some constants p_{ij}^k.

(B') A graph $\Gamma = (V, E)$ with diameter D is distance-regular if and only if, for some constants a_h, b_h, c_h $(0 \le h \le D)$, $c_0 = b_D = 0$, its distance matrices satisfy the three-term recurrence

$$\boldsymbol{A}_h \boldsymbol{A} = b_{h-1} \boldsymbol{A}_{h-1} + a_h \boldsymbol{A}_h + c_{h+1} \boldsymbol{A}_{h+1} \quad (0 \le h \le D),$$

where, by convention, $b_{-1} = c_{D+1} = 0$.

(C) A graph $\Gamma = (V, E)$ with diameter D is distance-regular if and only if $\{I, \boldsymbol{A}, ..., \boldsymbol{A}_D\}$ is a basis of the adjacency algebra $\mathcal{A}(\Gamma)$.

(C') Let Γ be a graph of diameter D and $\boldsymbol{A}_i$, the distance-i matrix of Γ. Then Γ is distance-regular if and only if $\boldsymbol{A}$ acts by right (or left) multiplication as a linear operator on the vector space $\text{span}\{I, \boldsymbol{A}_1, \boldsymbol{A}_2, ..., \boldsymbol{A}_D\}$.

(D) A graph $\Gamma = (V, E)$ with diameter D is distance-regular if and only if, for any integer h, $0 \le h \le D$, the distance-h matrix $\boldsymbol{A}_h$ is a polynomial of degree h in $\boldsymbol{A}$; that is:

$$\boldsymbol{A}_h = p_h(\boldsymbol{A}) \quad (0 \le h \le D).$$

(D') A graph $\Gamma = (V, E)$ with diameter D and $d + 1$ distinct eigenvalues is distance-regular if and only if Γ is regular, has spectrally maximum diameter $(D = d)$ and the matrix $\boldsymbol{A}_D$ is polynomial in $\boldsymbol{A}$.

(E) A graph $\Gamma = (V, E)$ is distance-regular if and only if, for each non-negative integer ℓ, the number a_{uv}^{ℓ} of walks of length ℓ between two vertices $u, v \in V$ only depends on $h = \partial(u, v)$.

(E') A regular graph $\Gamma = (V, E)$ with diameter D is distance-regular if and only if there are constants a_h^h and a_h^{h+1} such that, for any two vertices $u, v \in V$ at distance h, we have $a_{uv}^h = a_h^h$ and $a_{uv}^{h+1} = a_h^{h+1}$ for any $0 \le h \le D-1$, and $a_{uv}^D = a_D^D$ for $h = D$.

Characterizations (F), (H) and (I) have two terms which are maybe unfamiliar: predistance polynomials and distance $\circ$-algebra $\mathcal{D}$. Predistance polynomials are defined in Definition 11.07 and distance $\circ$-algebra $\mathcal{D}$ in Definition 8.07. Here we can say that predistance polynomials $\{p_i\}_{0\le i\le d}$, $\operatorname{dgr} p_i = i$, are a sequence of orthogonal polynomials with respect to the scalar product $\langle p,q\rangle = \frac{1}{n}\operatorname{trace}(p(\boldsymbol{A})q(\boldsymbol{A}))$ normalized in such a way that $\|p_i\|^2 = p_i(\lambda_0)$, where $\operatorname{spec}(\boldsymbol{A}) = \{\lambda_0^{m_0}, \lambda_1^{m_1}, ..., \lambda_d^{m_d}\}$, and that vector space $\mathcal{D} = \operatorname{span}\{I, \boldsymbol{A}, \boldsymbol{A}_2, ..., \boldsymbol{A}_D\}$ forms an algebra with the entrywise (Hadamard) product of matrices, defined by $(X\circ Y)_{uv} = (X)_{uv}(Y)_{uv}$.

(F) Let Γ be a graph with diameter D, adjacency matrix $\boldsymbol{A}$ and $d+1$ distinct eigenvalues $\lambda_0 > \lambda_1 > ... > \lambda_d$. Let $\boldsymbol{A}_i$, $i = 0,1,...,D$, be the distance-i matrices of Γ, $\boldsymbol{E}_j$, $j = 0,1,...,d$, be the principal idempotents of Γ, let p_{ji}, $i = 0,1,...,D$, $j = 0,1,...,d$, be constants and p_j, $j = 0,1,...,d$, be the predistance polynomials. Finally, let $\mathcal{A}$ be the adjacency algebra of Γ, and $d = D$. Then

$$
\begin{aligned}
\Gamma \text{ distance-regular} &\iff \boldsymbol{A}_i\boldsymbol{E}_j = p_{ji}\boldsymbol{E}_j, \quad i,j = 0,1,...,d(=D),\\
&\iff \boldsymbol{A}_i = \sum_{j=0}^{d} p_{ji}\boldsymbol{E}_j, \quad i,j = 0,1,...,d(=D),\\
&\iff \boldsymbol{A}_i = \sum_{j=0}^{d} p_i(\lambda_j)\boldsymbol{E}_j, \quad i,j = 0,1,...,d(=D),\\
&\iff \boldsymbol{A}_i \in \mathcal{A}, \quad i = 0,1,...,d(=D).
\end{aligned}
$$

(G) A graph Γ with diameter D and $d+1$ distinct eigenvalues is a distance-regular graph if and only if for every $0 \le i \le d$ and for every pair of vertices u, v of Γ, the (u,v)-entry of $\boldsymbol{E}_i$ depends only on the distance between u and v.

(H) Let Γ be a graph with diameter D, adjacency matrix $\boldsymbol{A}$ and $d+1$ distinct eigenvalues $\lambda_0 > \lambda_1 > ... > \lambda_d$. Let $\boldsymbol{A}_i$, $i = 0,1,...,D$, be the distance-i matrices of Γ, $\boldsymbol{E}_j$, $j = 0,1,...,d$, be the principal idempotents of Γ, let q_{ij}, $i = 0,1,...,D$, $j = 0,1,...,d$, be constants and p_j, $j = 0,1,...,d$, be the predistance polynomials. Finally, let q_j, $j = 0,1,...,d$ be polynomials defined by $q_i(\lambda_j) = m_j\frac{p_i(\lambda_j)}{p_i(\lambda_0)}$, $i,j = 0,1,...,d$, let $\mathcal{A}$ be the adjacency algebra of Γ, $\mathcal{D}$ be distance $\circ$-algebra and $d = D$. Then

$$
\begin{aligned}
\Gamma \text{ distance-regular} &\iff \boldsymbol{E}_j\circ\boldsymbol{A}_i = q_{ij}\boldsymbol{A}_i, \quad i,j = 0,1,...,d(=D),\\
&\iff \boldsymbol{E}_j = \sum_{i=0}^{D} q_{ij}\boldsymbol{A}_i, \quad j = 0,1,...,d(=D),\\
&\iff \boldsymbol{E}_j = \frac{1}{n}\sum_{i=0}^{d} q_i(\lambda_j)\boldsymbol{A}_i, \quad j = 0,1,...,d(=D),\\
&\iff \boldsymbol{E}_j \in \mathcal{D}, \quad j = 0,1,...,d(=D).
\end{aligned}
$$

(I) Let Γ be a graph with diameter D, adjacency matrix $\boldsymbol{A}$ and $d+1$ distinct eigenvalues $\lambda_0 > \lambda_1 > ... > \lambda_d$. Let $\boldsymbol{A}_i$, $i = 0,1,...,D$, be the distance-i matrix of Γ, $\boldsymbol{E}_j$, $j = 0,1,...,d$, be the principal idempotents of Γ, and let $a_i^{(j)}$, $i = 0,1,...,D$, $j = 0,1,...,d$, be constants. Finally, let $\mathcal{A}$ be the adjacency algebra of Γ, $\mathcal{D}$ be distance $\circ$-algebra and $d = D$. Then

$$\begin{aligned}
\Gamma \text{ distance-regular } &\iff \boldsymbol{A}^j \circ \boldsymbol{A}_i = a_i^{(j)} \boldsymbol{A}_i, \quad i, j = 0, 1, ..., d(= D), \\
&\iff \boldsymbol{A}^j = \sum_{i=0}^{d} a_i^{(j)} \boldsymbol{A}_i, \quad i, j = 0, 1, ..., d(= D), \\
&\iff \boldsymbol{A}^j = \sum_{i=0}^{d} \sum_{l=0}^{d} q_{i\ell} \lambda_l^j \boldsymbol{A}_i, \quad j = 0, 1, ..., d(= D), \\
&\iff \boldsymbol{A}^j \in \mathcal{D}, \quad j = 0, 1, ..., d.
\end{aligned}$$

7 Definitions and easy results

Let $\Gamma = (V, E)$ denote a simple connected graph with vertex set V, edge set E and diameter D. Let ∂ denotes the path-length distance function for Γ.

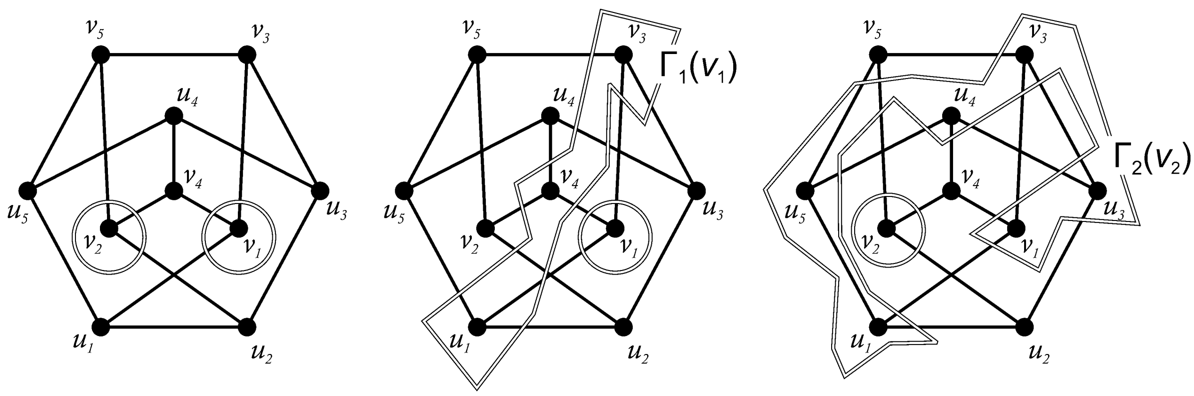


FIGURE 14
Petersen graph. For example, we have $\partial(v_1, v_2) = 2$, $\Gamma_1(v_1) = \{u_1, v_3, v_4\}$, $\Gamma_2(v_2) = \{u_1, u_3, u_4, u_5, v_1, v_3\}$, $|\Gamma_1(v_1) \cap \Gamma_2(v_2)| = |\{u_1, v_3\}| = 2$.

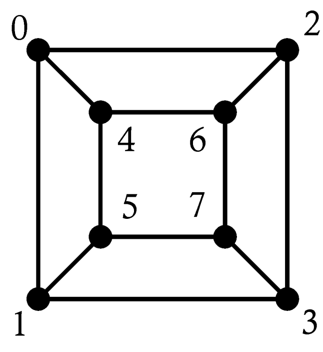


FIGURE 15
The cube.

(7.01) Definition (DRG)

A simple connected graph $\Gamma = (V, E)$ with diameter D is called <u>distance-regular</u> whenever there exist numbers p_{ij}^h $(0 \leq h, i, j \leq D)$ such that for any $x, y \in V$ with $\partial(x, y) = h$ we have

$$|\Gamma_i(x) \cap \Gamma_j(y)| = p_{ij}^h,$$

where $\Gamma_i(x) := \{z \in V : \partial(x, z) = i\}$ and $|\Gamma_i(x) \cap \Gamma_j(y)|$ denote the number of elements of the set $\Gamma_i(x) \cap \Gamma_j(y)$. $\diamond$

(7.02) Example (the cube is distance-regular graph)

The graph that is pictured on Figure 15 is called cube. We will show that the cube is distance-regular, and in this case we want to use only the definition of distance-regular graph.

From Definition 9.01 we will see that the cube is from family of Hamming graphs, and in Lemma 9.08 we will prove that Hamming graphs are distance-regular. If we compare this proof with the proof of Lemma 9.08, the proof of Lemma 9.08 is much more elegant.

Let $V = \{0, 1, ..., 7\}$ denote set of vertices of the cube. Notice that the diameter of graph is 3 ($D = 3$). We must show that there exist numbers p^h_{ij} $(0 \leq h, i, j \leq 3)$ such that for any pair $x, y \in V$ with $\partial(x, y) = h$ we have
$|\Gamma_i(x) \cap \Gamma_j(y)| = |\{z \in V : \partial(x, z) = i \text{ and } \partial(z, y) = j\}| = p^h_{ij}$. Because we want to use only definition, we must to consider all possible numbers p^h_{ij}, and for every of this number we must examine all possible pairs. With another words, since $\partial(x, y) = \partial(y, x)$, we will have to examine

$$|\Gamma_0(x) \cap \Gamma_0(y)|,\ |\Gamma_0(x) \cap \Gamma_1(y)|,\ |\Gamma_0(x) \cap \Gamma_2(y)|,\ |\Gamma_0(x) \cap \Gamma_3(y)|$$
$$|\Gamma_1(x) \cap \Gamma_1(y)|,\ |\Gamma_1(x) \cap \Gamma_2(y)|,\ |\Gamma_1(x) \cap \Gamma_3(y)|$$
$$|\Gamma_2(x) \cap \Gamma_2(y)|,\ |\Gamma_2(x) \cap \Gamma_3(y)|$$
$$|\Gamma_3(x) \cap \Gamma_3(y)|$$

for every pair of vertices x, $y \in V$.

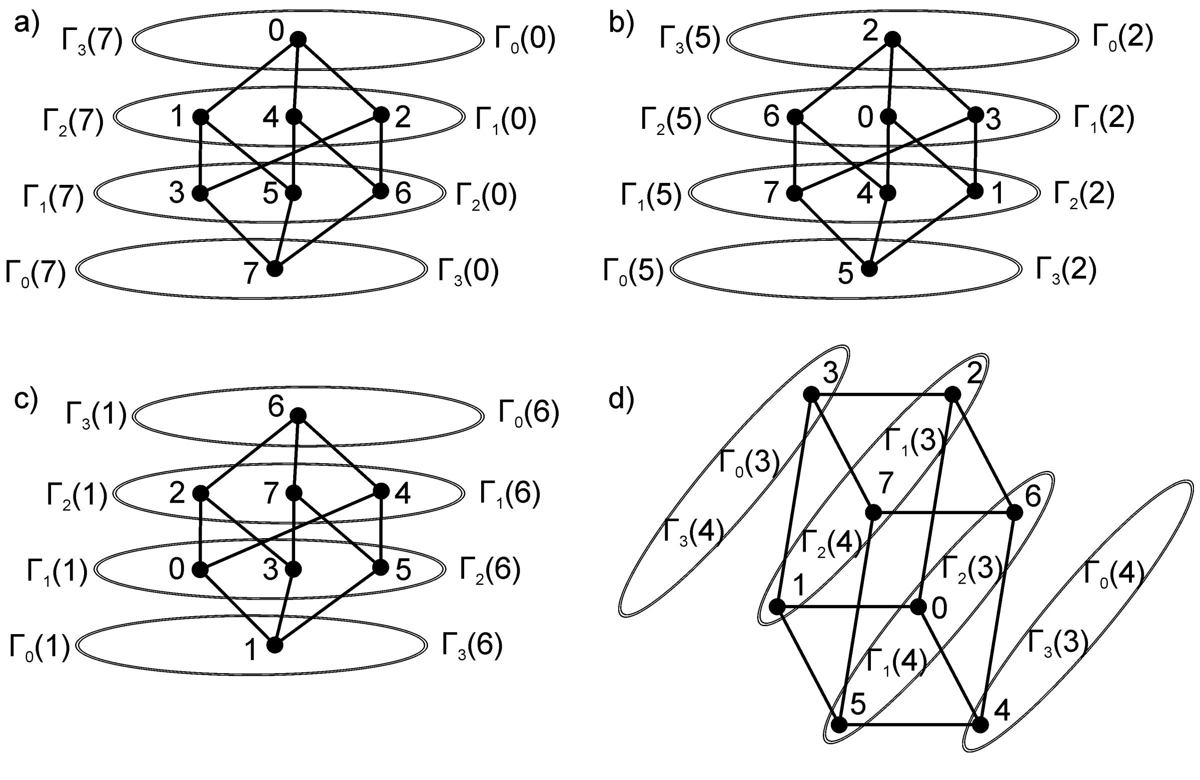


FIGURE 16
The cube drawn on four different way, and subsets of vertices at given distances from the root.

Consider the number p^0_{ii} for $0 \leq i \leq 3$, that is consider $|\Gamma_0(x) \cap \Gamma_0(y)|$, $|\Gamma_1(x) \cap \Gamma_1(y)|$, $|\Gamma_2(x) \cap \Gamma_2(y)|$, $|\Gamma_3(x) \cap \Gamma_3(y)|$ for every two vertices x, y such that $\partial(x, y) = 0$. Note that for $x, y \in V$ we have $\partial(x, y) = 0$ if and only if $x = y$. Therefore $p^0_{ii} = |\Gamma_i(x) \cap \Gamma_i(x)| = |\Gamma_i(x)|$ for every $x \in V$. But it is easy to see that $|\Gamma_i(x)| = 1$ if $i \in \{0, 3\}$, and $|\Gamma_i(x)| = 3$ if $i \in \{1, 2\}$.

Consider the number p^0_{ij} for $0 \leq i, j \leq 3$, where $i \neq j$. Note that for $x, y \in V$ we have $\partial(x, y) = 0$ if and only if $x = y$. Since $i \neq j$ we have $p^0_{ij} = |\Gamma_i(x) \cap \Gamma_j(x)| = |\emptyset| = 0$ for every $x \in V$.

Next we want to find numbers p^1_{ij} for $0 \le i,j \le 3$. Note that for $x,y \in V$ we have $\partial(x,y) = 1$ if and only if x and y are neighbors. Using Figure 16 one can easily find that $p^1_{00} = 0$, $p^1_{01} = 1 = p^1_{10}$ (for example $|\Gamma_0(0) \cap \Gamma_1(2)| = |\{0\} \cap \{0,3,6\}| = |\{0\}| = 1$), $p^1_{02} = 0 = p^1_{20}$, $p^1_{03} = 0 = p^1_{30}$ (for example $|\Gamma_0(0) \cap \Gamma_3(2)| = |\{0\} \cap \{5\}| = |\emptyset| = 0$), $p^1_{11} = 0$, $p^1_{12} = 2 = p^1_{21}$, $p^1_{13} = 0 = p^1_{31}$, $p^1_{22} = 0$, $p^1_{23} = 3 = p^1_{32}$, $p^1_{33} = 0$.

We will left to reader, like an easy exercise, to evaluate p^2_{ij} for $0 \le i,j \le 3$ and p^3_{ij} for $0 \le i,j \le 3$ ($p^2_{00} = 0$, $p^2_{01} = 0 = p^2_{10}$, $p^2_{02} = 1 = p^2_{20}$, $p^2_{03} = 0 = p^2_{30}$, $p^2_{11} = 2$, ...) ◊

It is clear from the solution of Example 7.02, that the given definition of distance-regular graphs is very inconvenient if we want to check whether a given graph is distance-regular or not. Therefore, we want to obtain characterizations of distance-regular graphs, which will relieve check whether a given graph is distance-regular or not. In Theorem 8.12 (Characterization A), Theorem 8.15 (Characterization B), Theorem 8.22 (Characterization C) and so on, we will obtain statements that are equivalent with definition of distance-regular graph and which are "easier" to apply.

(7.03) Proposition

Let $\Gamma = (V,E)$ be a distance-regular graph with diameter D. Then:

(i) For $0 \le h,i,j \le D$ we have $p^h_{ij} = 0$ whenever one of h,i,j is greater than the sum of the other two.

(ii) For $0 \le h,i,j \le D$ we have $p^h_{ij} \neq 0$ whenever one of h,i,j is equal to the sum of the other two.

(iii) For every $x \in V$ and for every integer $0 \le i \le D$ we have $p^0_{ii} = |\Gamma_i(x)|$.

(iv) Γ is regular with valency p^0_{11}.

Proof: (i) Pick $x,y \in V$ with $\partial(x,y) = h$ and assume $p^h_{ij} \neq 0$. This means that there is $z \in V$ such that $\partial(x,z) = i$ and $\partial(y,z) = j$. By the triangle inequality of path-length distance ∂ we have $h \le i+j$, $i \le h+j$ and $j \le i+h$. It follows that none of h,i,j is greater of the sum of the other two.

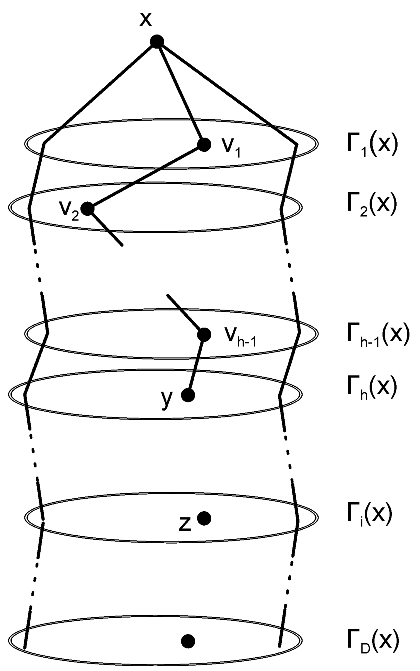


FIGURE 17
Illustration for sets $\Gamma_i(x)$ in connected graph Γ

(ii) Assume that one of h,i,j is the sum of the other two. If $h = i+j$, then pick $x,y \in V$

with $\partial(x, y) = h$, and let z denote a vertex which is at distance i from x and which lies on some shortest path between x and y. Note that z is at distance j from y, and so $p^h_{ij} \neq 0$.

If $i = h + j$, then pick $x, z \in V$ with $\partial(x, z) = i$. Let y denote a vertex which is at distance h from x and which lies on some shortest path between x and z. Note that $\partial(y, z) = j$, and so $z \in \Gamma_i(x) \cap \Gamma_j(y)$. Therefore, $p^h_{ij} = |\Gamma_i(x) \cap \Gamma_j(y)| \neq 0$. The case $j = h + i$ is done analogously.

(iii) Pick $x \in V$ and note that $p^0_{ii} = |\Gamma_i(x) \cap \Gamma_i(x)| = |\Gamma_i(x)|$.

(iv) Immediately from (iii) above. □

From now on we will abbreviate $k_i = p^0_{ii}$.

(7.04) Lemma

Let $\Gamma = (V, E)$ be a distance-regular graph with diameter D, and let $k_i = p^0_{ii}$. Then:
(i) $k_h p^h_{ij} = k_j p^j_{ih}$ for $1 \leq i, j, h \leq D$;
(ii) $p^h_{1,h-1} + p^h_{1h} + p^h_{1,h+1} = k_1$ for $0 \leq h \leq D$;
(iii) if $h + i \leq D$ then $p^h_{1,h-1} \leq p^i_{1,i+1}$.

Proof: (i) Fix $x \in V$. Let us count the number of pairs $y, z \in V$ such that $\partial(x, y) = h$, $\partial(x, z) = j$ and $\partial(y, z) = i$. We can choose y in k_h different ways ($k_h = p^0_{hh} = |\Gamma_h(x)|$), and for every such y, there is p^h_{ij} vertices z ($\partial(x, z) = j$ and $\partial(y, z) = i$). Therefore, there is $k_h p^h_{ij}$ such pairs.

On the other hand, we can choose z in k_j different ways, and for every such z, there is p^j_{ih} vertices y ($\partial(x, y) = h$ and $\partial(y, z) = i$). Therefore, there is $k_j p^j_{ih}$ such pairs.

It follows that $k_h p^h_{ij} = k_j p^j_{ih}$.

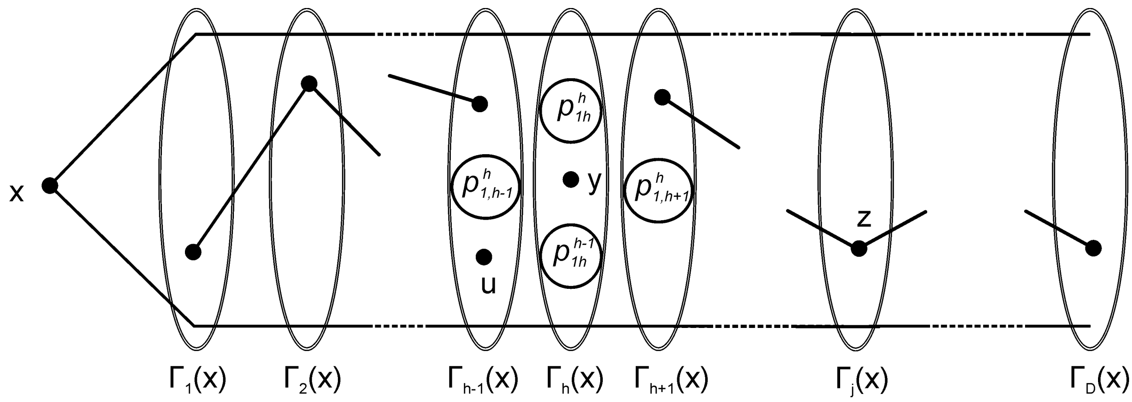


FIGURE 18
Illustration for numbers p^h_{ij} and for sets $\Gamma_h(x)$ (vertices that are on distance h from x) of distance-regular graph.

(ii) $p^h_{1,h-1} + p^h_{1h} + p^h_{1,h+1}$ is the number of neighbors of arbitrary vertex from $\Gamma_h(x)$, $1 \leq h \leq D$. Since Γ is regular with valency k_1 (by Proposition 7.03(iii)), this number is equal to k_1.

(iii) Pick arbitrary $y \in \Gamma_h(x)$ and arbitrary $z \in \Gamma_{h+i}(x) \cap \Gamma_i(y)$ (such z exist because $h + i \leq D$). Notice that z is on distance i from y. We have $\Gamma_{h-1}(x) \cap \Gamma_1(y) \subseteq \Gamma_{i+1}(z) \cap \Gamma_1(y)$, because all vertices that are in $\Gamma_{h-1}(x) \cap \Gamma_1(y)$ are on distance $i + 1$ from z, and maybe there are some vertices in $\Gamma_{i+1}(z) \cap \Gamma_1(y)$, which are not in $\Gamma_{h-1}(x)$. Therefore
$p^h_{1,h-1} = |\Gamma_1(y) \cap \Gamma_{h-1}(x)| \leq |\Gamma_1(y) \cap \Gamma_{i+1}(z)| = p^i_{1,i+1}$. □

For better understanding of distance-regular graphs we will next introduce concept of local distance-regular graphs.

(7.05) Definition (local distance-regular graph)

Let $y \in V$ be a vertex with eccentricity $\mathrm{ecc}(y) = \varepsilon$ of a regular graph Γ. Let $V_k := \Gamma_k(y)$ and consider the numbers

$$c_k(x) := |\Gamma_1(x) \cap V_{k-1}|,$$
$$a_k(x) := |\Gamma_1(x) \cap V_k|,$$
$$b_k(x) := |\Gamma_1(x) \cap V_{k+1}|,$$

defined for any $x \in V_k$ and $0 \le k \le \varepsilon$ (where, by convention, $c_0(x) = 0$ and $b_\varepsilon(x) = 0$ for any $x \in V_\varepsilon$). We say that Γ is *distance-regular around y* whenever $c_k(x)$, $a_k(x)$, $b_k(x)$ do not depend on the considered vertex $x \in V_k$ but only on the value of k. In such a case, we simply denote them by c_k, a_k and b_k, respectively, and we call them the *intersection numbers around y*. The matrix

$$\mathcal{I}(y) := \begin{pmatrix} 0 & c_1 & \dots & c_{\varepsilon-1} & c_\varepsilon \\ a_0 & a_1 & \dots & a_{\varepsilon-1} & a_\varepsilon \\ b_0 & b_1 & \dots & b_{\varepsilon-1} & 0 \end{pmatrix}$$

is called the *intersection array around vertex y*. ◇

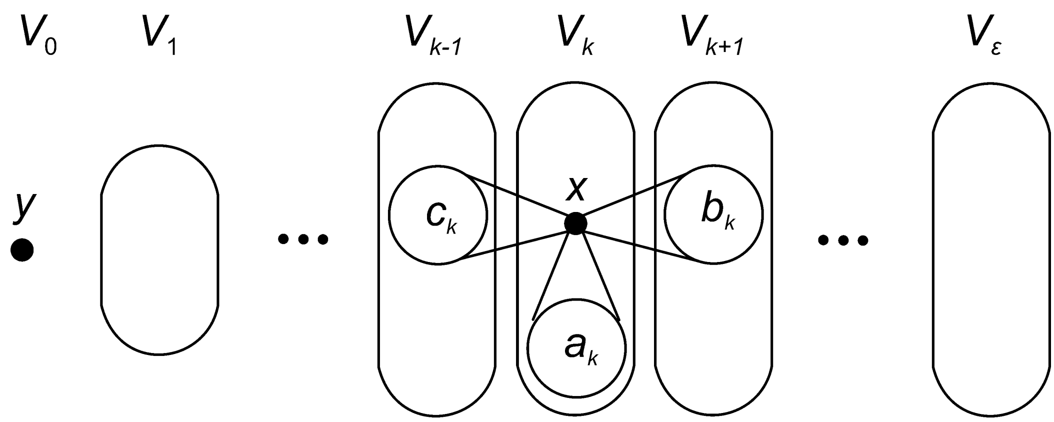


FIGURE 19
Intersection numbers around y.

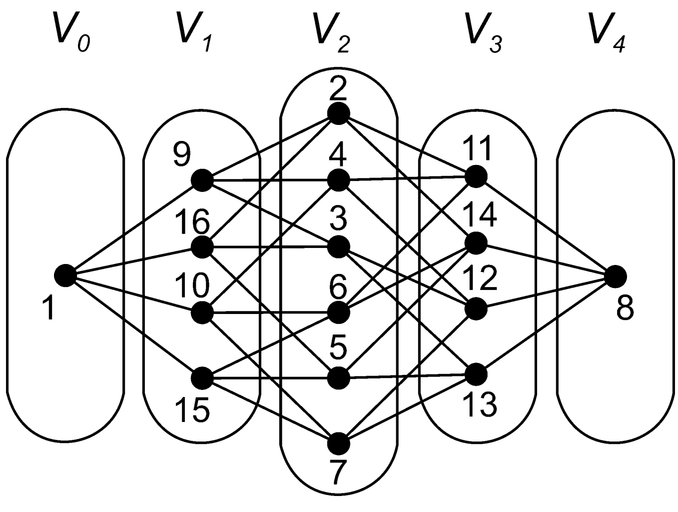


FIGURE 20
Simple connected regular graph that is distance-regular around vertices 1 and 8 (intersection numbers around 1 are $c_0 = 0$, $a_0 = 0$, $b_0 = 4$, $c_1 = 1$, $a_1 = 0$, $b_1 = 3$, $c_2 = 2$, $a_2 = 0$, $b_2 = 2$, $c_3 = 3$, $a_3 = 0$, $b_3 = 1$, $c_4 = 4$, $a_4 = 0$, $b_4 = 4$). This graph is known as Hoffman graph.

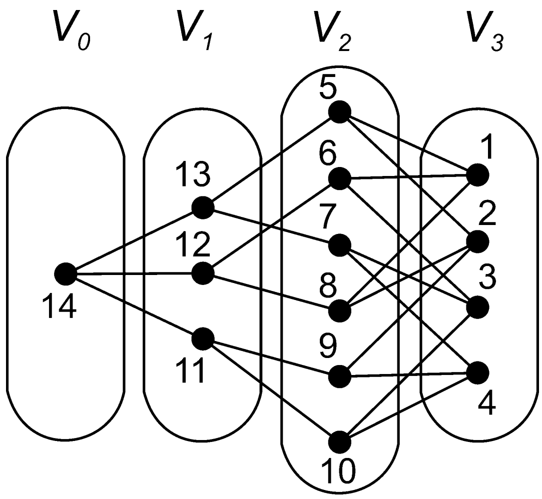


FIGURE 21
Simple connected graph that is distance-regular around vertex 14.

(7.06) Comment (Γ distance-regular $\Rightarrow$ Γ is distance-regular around every vertex)

It follows directly from a definition of distance-regular graph we see that if a graph Γ is distance-regular then it is distance-regular around each of its vertices and with the same intersection array. In other words, if we consider the partition $\Pi(i)$ of $V(\Gamma)$ (where Γ is distance-regular) defined by the sets $\Gamma_k(i)$, $k = 0, 1, ..., \text{ecc}(i)$, the corresponding quotient $\Gamma/\Pi(i)$ is a weighted path with structure independent of the chosen vertex i.

Locally distance-regular graphs shown in Figure 20 and 21 are not distance-regular. $\Diamond$

(7.07) Comment (intersection numbers)

Let $\Gamma = (V, E)$ denote arbitrary connected graph which is distance-regular around each of its vertices and with the same intersection array

$$\mathcal{I} = \begin{pmatrix} 0 & c_1 & ... & c_{\varepsilon-1} & c_\varepsilon \\ a_0 & a_1 & ... & a_{\varepsilon-1} & a_\varepsilon \\ b_0 & b_1 & ... & b_{\varepsilon-1} & 0 \end{pmatrix}.$$

Then every vertex has the same eccentricity ε and diameter of Γ is $D = \varepsilon$. Directly from definition of distance-regularity around vertex it follow that for every $0 \le h \le D$ there exist numbers c_h, a_h and b_h such that for any pair of vertices $x, y \in \Gamma$ with $\partial(x, y) = h$, we have

$$\begin{aligned} c_h &= |\Gamma_1(x) \cap \Gamma_{h-1}(y)| \text{ for } h = 1, 2, ..., D, \\ a_h &= |\Gamma_1(x) \cap \Gamma_h(y)| \text{ for } h = 0, 1, ..., D, \\ b_h &= |\Gamma_1(x) \cap \Gamma_{h+1}(y)| \text{ for } h = 0, 1, ..., D-1. \end{aligned}$$

where $b_D = c_0 = 0$. We will call these numbers the *intersection numbers* of Γ. $\Diamond$

(7.08) Comment

Let x, y be any pair of vertices with $\partial(x, y) = h$ and let $\Gamma = (V, E)$ denote arbitrary connected graph which is distance-regular around each of its vertices and with the same intersection array. Then the intersection number a_h is equal to the number of neighbors of vertex x that are on distance h from y, coefficient b_h presents the number of neighbors of vertex x that are on distance $h + 1$ from y and coefficient c_h presents the number of neighbors of vertex x that are on distance $h - 1$ from y. $\Diamond$

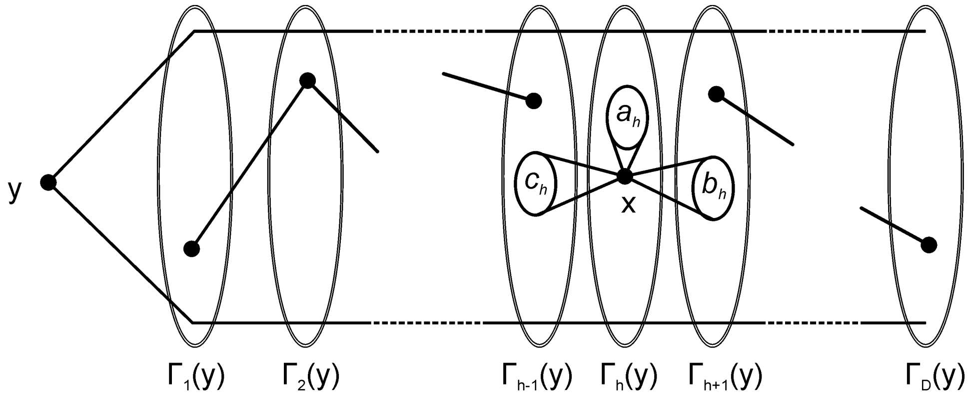


FIGURE 22
Illustration for coefficient a_h, b_h and c_h in connected graph which is distance-regular around each of its vertices and with the same intersection array. ◊

(7.09) Lemma

Let $\Gamma = (V, E)$ denote arbitrary connected graph which is distance-regular around each of its vertices and with the same intersection array. Then the following (i)-(iii) hold.

(i) Γ is regular with valency $k = b_0$.

(ii) $a_0 = 0$ and $c_1 = 1$.

(iii) $a_i + b_i + c_i = k$ for $0 \leq i \leq D$.

Proof: Consider graph pictured on Figure 22. Where are edges of this graph? We can notice that it is not possible to have edge between sets $\Gamma_i(y)$ and $\Gamma_{i+2}(y)$ for some $1 \leq i \leq D-2$ (Why?). So every edge in this graph is between $\Gamma_i(y)$ and $\Gamma_{i+1}(y)$, and because of that $a_h + b_h + c_h$ is valency of vertex x.

(i) Pick $x \in V$. We have $|\Gamma_1(x)| = |\Gamma_1(x) \cap \Gamma_1(x)| = b_0$ (see Comment 7.07). It follows that Γ is regular with valency $k = b_0$.

(ii) Pick $x \in V$ and note that we have $a_0 = |\Gamma_1(x) \cap \Gamma_0(x)| = |\emptyset| = 0$. Pick $y \in V$ such that $\partial(x, y) = 1$ and note that we have $c_1 = |\Gamma_1(x) \cap \Gamma_0(y)| = |\{y\}| = 1$.

(iii) Pick $x \in V$, $0 \leq i \leq D$ and $y \in \Gamma_i(x)$. Note that, by definition of path-length distance, all neighbors of y are at distance either $i-1$ from x, or i form x, or $i+1$ from x. Therefore, $\Gamma_1(y)$ is a disjoint union of $\Gamma_1(y) \cap \Gamma_{i-1}(x)$, $\Gamma_1(y) \cap \Gamma_i(x)$ and $\Gamma_1(y) \cap \Gamma_{i+1}(x)$, and so we have $k = b_0 = |\Gamma_1(y)| = |\Gamma_1(y) \cap \Gamma_{i-1}(x)| + |\Gamma_1(y) \cap \Gamma_i(x)| + |\Gamma_1(y) \cap \Gamma_{i+1}(x)| = c_i + a_i + b_i$. □

Let Γ be distance-regular graph with diameter D. By Lemma 7.09 Γ is regular with valency $k = b_0$, and because of (iii) of the same lemma we have $a_i = k - b_i - c_i$ for $0 \leq i \leq D$.

(7.10) Proposition

Let $\Gamma = (V, E)$ denote arbitrary connected graph which is distance-regular around each of its vertices and with the same intersection array and let $k_i = p^0_{ii}$. Then

(i) $b_0 \geq b_1 \geq b_2 \geq ... \geq b_{D-1}$;

(ii) $c_1 \leq c_2 \leq c_3 \leq ... \leq c_D$;

(iii) $k_{i-1}b_{i-1} = k_i c_i$ for $1 \leq i \leq D$;

(iv) $k_i = (b_0 b_1 ... b_{i-1})/(c_1 c_2 ... c_i)$ for $1 \leq i \leq D$.

Proof: (i) Pick $x, y \in V$ with $\partial(x, y) = i$. Consider a shortest path $[x, z_1, z_2, ..., z_{i-2}, z_{i-1}, y]$ from x to y. Consider the distance-partitions of Γ with respect to vertices x and z_1 (see Figure 23 for illustration). Denote by B_i set $B_i = \Gamma_{i+1}(x) \cap \Gamma_1(y)$, by B_{i-1} set $B_{i-1} = \Gamma_i(z_1) \cap \Gamma_1(y)$ and notice that $b_i = |B_i|$, $b_{i-1} = |B_{i-1}|$. Pick arbitrary vertex $w \in B_i$. We have that $w \sim y$ and $\partial(z_1, w) = i$. This mean that $w \in B_{i-1}$. We conclude that $B_i \subseteq B_{i-1}$, and therefore $b_i = |B_i| \leq |B_{i-1}| = b_{i-1}$.

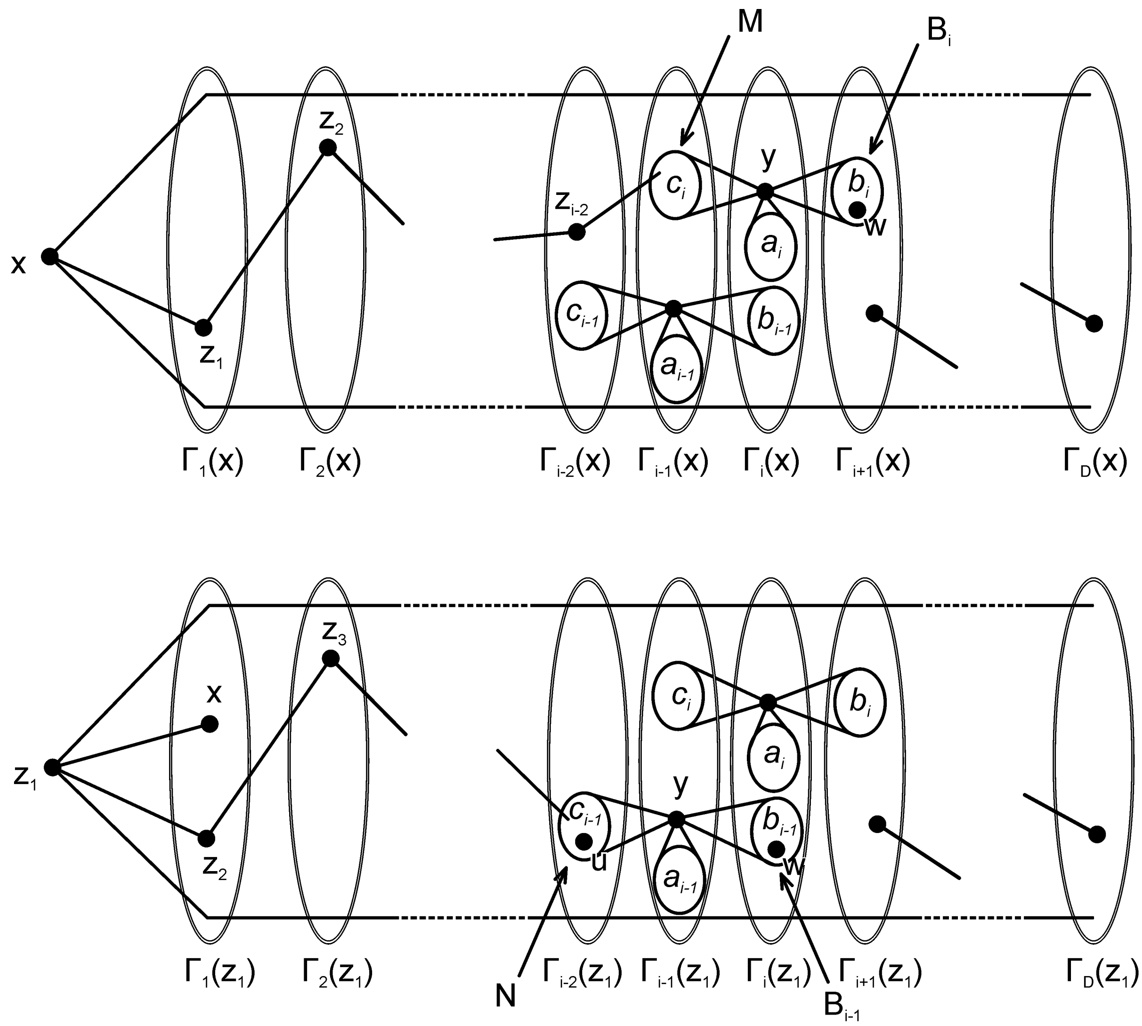


FIGURE 23
Illustration for coefficient a_i, b_i and c_i in connected graph Γ with two different partition.

(ii) We will keep all notations from (i). Notice that y must be in $\Gamma_{i-1}(z_1)$. Now, denote by M set $M = \Gamma_{i-1}(x) \cap \Gamma_1(y)$, by N set $N = \Gamma_{i-2}(z_1) \cap \Gamma_1(y)$, and notice that $|M| = c_i$, $|N| = c_{i-1}$. Pick arbitrary $u \in N$. Note that since u is a neighbor of y, we have $\partial(x, u) \in \{i-1, i, i+1\}$. But on the other hand, $\partial(u, x) \leq i-1$, since $\partial(u, z_1) = i-2$ and z_1 is a neighbor of x. Therefore $\partial(x, u) = i-1$, and so $u \in M$. Therefore $N \subseteq M$ and so $c_{i-1} = |N| \leq |M| = c_i$.

(iii) In Lemma 7.04(i) we had shown that $k_{i-1}p^{i-1}_{1i} = k_i p^i_{1,(i-1)}$, $1 \leq i \leq D$; but in new symbols that means precisely $k_{i-1}b_{i-1} = k_i c_i$ for $1 \leq i \leq D$.

(iv) Pick $y \in V$ and consider distance partition with respect with y. We claim that $|\Gamma_i(y)| = (b_0 b_1 ... b_{i-1})/(c_1 c_2 ... c_i)$. We will prove the result using induction on n.

BASIS OF INDUCTION

Observe that $b_0 = k_1$, so the formula holds for $i = 1$.

INDUCTION STEP

Assume now that formula holds for $i < D$. We will show that formula holds also for $i+1$. Note that by (iii) we have $k_{i+1} = b_i k_i / c_{i+1}$. Since by the induction hypothesis we have $k_i = (b_0 b_1 ... b_{i-1})/(c_1 c_2 ... c_i)$, the result follows. □

8 Characterization of DRG involving the distance matrices

In the texts that follow we want to obtain some characterizations of distance-regular graphs, which depend on information retrieved from their adjacency and distance-i matrices.

(8.01) Definition (distance-i matrix)

Let $\Gamma = (V, E)$ denote a graph with diameter D, adjacency matrix $\boldsymbol{A}$ and let $\mathrm{Mat}_\Gamma(\mathbb{R})$ denote the $\mathbb{R}$-algebra consisting of the matrices with entries in $\mathbb{R}$, and rows and columns indexed by the vertices of Γ. For $0 \leq i \leq D$ we define *distance-i matrix* $\boldsymbol{A}_i \in \mathrm{Mat}_\Gamma(\mathbb{R})$ with entries $(\boldsymbol{A}_i)_{uv} = 1$ if $\partial(u, v) = i$ and $(\boldsymbol{A}_i)_{uv} = 0$ otherwise. Note that $\boldsymbol{A}_0$ is the identity matrix and $\boldsymbol{A}_1 = \boldsymbol{A}$ is the usual adjacency matrix of Γ. $\diamond$

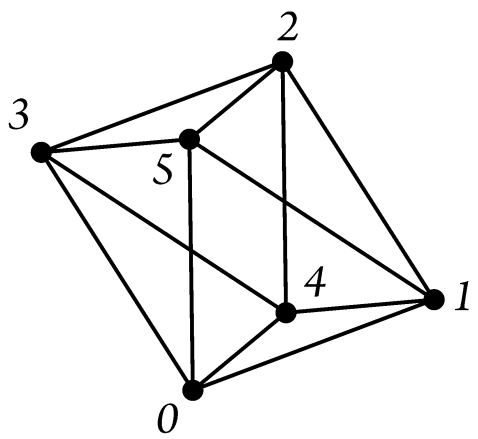


FIGURE 24
Octahedron.

Distance-i matrices for octahedron (that is for graph which is pictured on Figure 24) are

$$\boldsymbol{A}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{A}_1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{A}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

(8.02) Theorem

For arbitrary graph $\Gamma = (V, E)$ which is distance-regular around each of its vertices and with the same intersection array, the distance-i matrices of Γ satisfies

$$\boldsymbol{A}_i \boldsymbol{A} (= \boldsymbol{A}\boldsymbol{A}_i) = b_{i-1}\boldsymbol{A}_{i-1} + a_i \boldsymbol{A}_i + c_{i+1}\boldsymbol{A}_{i+1}, \quad 0 \leq i \leq D$$

where a_i, b_i and c_i are the intersection numbers of Γ (see Comment 7.07) and $\boldsymbol{A}_{-1}$, $\boldsymbol{A}_{D+1}$ are the zero matrices.

Proof: (1°) Let $\Gamma = (V, E)$ be a distance-regular around each of its vertices with diameter D. Then $\forall x, y_1, y_2, y_3 \in V$ for which $\partial(x, y_1) = h - 1$, $\partial(x, y_2) = h$ and $\partial(x, y_3) = h + 1$, there exist constants a_h, b_h and c_h $(0 \leq h < D)$ (known as intersection numbers) such that

$a_h = |\Gamma_1(y_2) \cap \Gamma_h(x)|$, $a_{h-1} = |\Gamma_1(y_1) \cap \Gamma_{h-1}(x)|$, $a_{h+1} = |\Gamma_1(y_3) \cap \Gamma_{h+1}(x)|$,
$b_h = |\Gamma_1(y_2) \cap \Gamma_{h+1}(x)|$, $b_{h-1} = |\Gamma_1(y_1) \cap \Gamma_h(x)|$, $b_{h+1} = |\Gamma_1(y_3) \cap \Gamma_{h+2}(x)|$,
$c_h = |\Gamma_1(y_2) \cap \Gamma_{h-1}(x)|$, $c_{h-1} = |\Gamma_1(y_1) \cap \Gamma_{h-2}(x)|$, $c_{h+1} = |\Gamma_1(y_3) \cap \Gamma_h(x)|$,

(see Figure 25 for illustration).

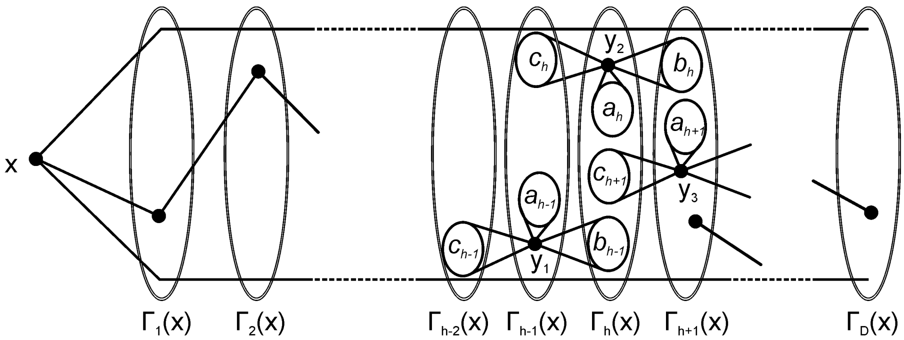


FIGURE 25
Illustration for numbers a_h, b_h and c_h.

Now, if we consider arbitrary vertices $u, v \in V$, for uv-entry of $\boldsymbol{A}_h\boldsymbol{A}$ (of course $\boldsymbol{A} = \boldsymbol{A}_1$) we have

$$(\boldsymbol{A}_h\boldsymbol{A})_{uv} = \sum_{y\in V}(\boldsymbol{A}_h)_{uy}(\boldsymbol{A})_{yv} = |\Gamma_h(u) \cap \Gamma_1(v)| = \begin{cases} a_h, & \text{if } \partial(u,v) = h \\ b_{h-1}, & \text{if } \partial(u,v) = h-1 \\ c_{h+1}, & \text{if } \partial(u,v) = h+1 \\ 0, & \text{otherwise} \end{cases}.$$

Similarly

$$(b_{h-1}\boldsymbol{A}_{h-1} + a_h\boldsymbol{A}_h + c_{h+1}\boldsymbol{A}_{h+1})_{uv} = \begin{cases} a_h, & \text{if } \partial(u,v) = h \\ b_{h-1}, & \text{if } \partial(u,v) = h-1 \\ c_{h+1}, & \text{if } \partial(u,v) = h+1 \\ 0, & \text{otherwise} \end{cases}.$$

Therefore, $\boldsymbol{A}_h\boldsymbol{A} = b_{h-1}\boldsymbol{A}_{h-1} + a_h\boldsymbol{A}_h + c_{h+1}\boldsymbol{A}_{h+1}$ $(0 \le h < D)$.

(2°) Notice that

$$(\boldsymbol{A}\boldsymbol{A}_D)_{uv} = \sum_{z\in V}(\boldsymbol{A})_{uz}(\boldsymbol{A}_D)_{zv} = |\Gamma_1(u) \cap \Gamma_D(v)| = \begin{cases} b_{D-1}, & \text{if } \partial(u,v) = D-1 \\ a_D, & \text{if } \partial(u,v) = D \end{cases} =$$

$$= (b_{D-1}\boldsymbol{A}_{D-1} + a_D\boldsymbol{A}_D)_{uv}.$$

□

(8.03) Definition (mapping $\boldsymbol{\rho}$)

Let $\Gamma = (V, E)$ denote a simple graph with diameter D, and for any vertex $x \in V$ let $\boldsymbol{e}_x = (0, ..., 0, 1, 0, ..., 0)^\top$ denote the x-th unitary vector of the canonical basis of $\mathbb{R}^n$. Then for regular graph Γ, and for subset U of the vertices of Γ, we define *mapping* $\boldsymbol{\rho}$ by

$$\boldsymbol{\rho}U := \sum_{z\in U}\boldsymbol{e}_z.$$

($\boldsymbol{\rho}U$ turns out to be the characteristic vector of U, that is, $(\boldsymbol{\rho}U)_x = 1$ if $x \in U$ and $(\boldsymbol{\rho}U)_x = 0$ otherwise). ◇

(8.04) Proposition

Let $\Gamma = (V, E)$ denote connected graph which is distance-regular around vertex y, and let c_k, a_k and b_k be the intersection numbers around y ($k = 0, 1, ..., \mathrm{ecc}(y)$). Then the polynomials obtained from the recurrence

$$xr_k = b_{k-1}r_{k-1} + a_k r_k + c_{k+1}r_{k+1}, \qquad \textit{with} \quad r_0 = 1,\, r_1 = x,$$

satisfy

$$r_k(\boldsymbol{A})\boldsymbol{e}_y = \boldsymbol{\rho} V_k = \boldsymbol{A}_k \boldsymbol{e}_y$$

where $k = 0, 1, ..., \mathrm{ecc}(y)$ and $V_k := \Gamma_k(y)$.

Proof: Let $\Gamma = (V, E)$ denote connected graph which is distance-regular around vertex y. Since

$$(\boldsymbol{\rho} V_k)_x = \begin{cases} 1, & \text{if } x \in V_k \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & \text{if } x \in \Gamma_k(y) \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & \text{if } \partial(x,y) = k \\ 0, & \text{otherwise} \end{cases} \quad (= (\boldsymbol{A}_k)_{xy})$$

and

$$\boldsymbol{\rho} V_k := \sum_{z \in V_k} \boldsymbol{e}_z = \sum_{z \in \Gamma_k(y)} \boldsymbol{e}_z = \begin{bmatrix} \square = \begin{cases} 1, & \text{if } \partial(1,y) = k \\ 0, & \text{otherwise} \end{cases} \\ \square = \begin{cases} 1, & \text{if } \partial(2,y) = k \\ 0, & \text{otherwise} \end{cases} \\ \vdots \\ \square = \begin{cases} 1, & \text{if } \partial(n,y) = k \\ 0, & \text{otherwise} \end{cases} \end{bmatrix} = (\boldsymbol{A}_k)_{*y} \tag{8}$$

notice that

$$(\boldsymbol{A}\boldsymbol{\rho} V_k)_x = \left(\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} | \\ \boldsymbol{\rho} V_k \\ | \end{bmatrix} \right)_x =$$

$$= \begin{bmatrix} a_{x1} & a_{x2} & \dots & a_{xn} \end{bmatrix} \begin{bmatrix} \square(=1 \text{ or } 0) \\ \square(=1 \text{ or } 0) \\ \vdots \\ \square(=1 \text{ or } 0) \end{bmatrix} = |\Gamma(x) \cap \Gamma_k(y)|,$$

that is

$$(\boldsymbol{A}\boldsymbol{\rho} V_k)_x = |\Gamma(x) \cap V_k| = \begin{cases} a_k, & \text{if } \partial(y,x) = k \\ b_{k-1}, & \text{if } \partial(y,x) = k-1 \\ c_{k+1}, & \text{if } \partial(y,x) = k+1 \\ 0, & \text{otherwise} \end{cases},$$

so we have

$$\boldsymbol{A}\boldsymbol{\rho} V_k = b_{k-1}\boldsymbol{\rho} V_{k-1} + a_k \boldsymbol{\rho} V_k + c_{k+1}\boldsymbol{\rho} V_{k+1}.$$

On the other hand, since $\boldsymbol{A}_k = \begin{bmatrix} | & | & & | & & | \\ (\boldsymbol{A}_k)_{*1} & (\boldsymbol{A}_k)_{*2} & \dots & (\boldsymbol{A}_k)_{*y} & \dots & (\boldsymbol{A}_k)_{*n} \\ | & | & & | & & | \end{bmatrix}$

$$\boldsymbol{\rho} V_k \overset{(8)}{=} (\boldsymbol{A}_k)_{*y} = \boldsymbol{A}_k \boldsymbol{e}_y, \quad 1 \le k \le D \tag{9}$$

and the previous recurrence reads

$$\boldsymbol{A}\boldsymbol{A}_k \boldsymbol{e}_y = b_{k-1}\boldsymbol{A}_{k-1}\boldsymbol{e}_y + a_k \boldsymbol{A}_k \boldsymbol{e}_y + c_{k+1}\boldsymbol{A}_{k+1}\boldsymbol{e}_y, \tag{10}$$

or in details

$$\begin{aligned}
\boldsymbol{A}\boldsymbol{A}_0\boldsymbol{e}_y &= 0 + a_0\boldsymbol{A}_0\boldsymbol{e}_y + c_1\boldsymbol{A}_1\boldsymbol{e}_y,\\
\boldsymbol{A}\boldsymbol{A}_1\boldsymbol{e}_y &= b_0\boldsymbol{A}_0\boldsymbol{e}_y + a_1\boldsymbol{A}_1\boldsymbol{e}_y + c_2\boldsymbol{A}_2\boldsymbol{e}_y,\\
\boldsymbol{A}\boldsymbol{A}_2\boldsymbol{e}_y &= b_1\boldsymbol{A}_1\boldsymbol{e}_y + a_2\boldsymbol{A}_2\boldsymbol{e}_y + c_3\boldsymbol{A}_3\boldsymbol{e}_y,\\
&\dots\\
\boldsymbol{A}\boldsymbol{A}_m\boldsymbol{e}_y &= b_{m-1}\boldsymbol{A}_{m-1}\boldsymbol{e}_y + a_m\boldsymbol{A}_m\boldsymbol{e}_y + 0,
\end{aligned}$$

where, $m = \mathrm{ecc}(y)$, $b_{-1} = c_{m+1} = 0$. On the other hand the polynomials obtained from the recurrence

$$xr_k = b_{k-1}r_{k-1} + a_kr_k + c_{k+1}r_{k+1}, \qquad \text{with} \qquad r_0 = 1,\, r_1 = x,$$

satisfy

$$\boldsymbol{A}r_k(\boldsymbol{A}) = b_{k-1}r_{k-1}(\boldsymbol{A}) + a_kr_k(\boldsymbol{A}) + c_{k+1}r_{k+1}(\boldsymbol{A}),$$

$$\boldsymbol{A}r_k(\boldsymbol{A})\boldsymbol{e}_y = b_{k-1}r_{k-1}(\boldsymbol{A})\boldsymbol{e}_y + a_kr_k(\boldsymbol{A})\boldsymbol{e}_y + c_{k+1}r_{k+1}(\boldsymbol{A})\boldsymbol{e}_y, \tag{11}$$

or in details

$$\begin{aligned}
\boldsymbol{A}r_0(\boldsymbol{A})\boldsymbol{e}_y &= 0 + a_0r_0(\boldsymbol{A})\boldsymbol{e}_y + c_1r_1(\boldsymbol{A})\boldsymbol{e}_y\\
\boldsymbol{A}r_1(\boldsymbol{A})\boldsymbol{e}_y &= b_0r_0(\boldsymbol{A})\boldsymbol{e}_y + a_1r_1(\boldsymbol{A})\boldsymbol{e}_y + c_2r_2(\boldsymbol{A})\boldsymbol{e}_y\\
\boldsymbol{A}r_2(\boldsymbol{A})\boldsymbol{e}_y &= b_1r_1(\boldsymbol{A})\boldsymbol{e}_y + a_2r_2(\boldsymbol{A})\boldsymbol{e}_y + c_3r_3(\boldsymbol{A})\boldsymbol{e}_y\\
&\dots\\
\boldsymbol{A}r_m(\boldsymbol{A})\boldsymbol{e}_y &= b_{m-1}r_{m-1}(\boldsymbol{A})\boldsymbol{e}_y + a_mr_m(\boldsymbol{A})\boldsymbol{e}_y + 0.
\end{aligned}$$

In the end, if we consider equations (9), (10) and (11), since $r_0(\boldsymbol{A}) = I$ and $r_1(\boldsymbol{A}) = \boldsymbol{A}$, with help of mathematical induction on k, we have $r_k(\boldsymbol{A})\boldsymbol{e}_y = \boldsymbol{\rho}V_k$. □

(8.05) Proposition

Let $\Gamma = (V, E)$ denote arbitrary connected graph with diameter D which is distance-regular around each of its vertices and with the same intersection array. Then for $0 \le i \le D$ there exists a polynomial p_i of degree i such that

$$\boldsymbol{A}_i = p_i(\boldsymbol{A}).$$

Moreover, if $p_i(\boldsymbol{A}) = \beta_0^iI + \beta_1^i\boldsymbol{A} + ... + \beta_i^i\boldsymbol{A}^i$, then $\beta_0^i, \beta_1^i, ..., \beta_i^i$ depends only on a_j, b_j, c_j.

Proof: We prove the result using induction on i.

BASIS OF INDUCTION

It is clear that the result holds for $i = 0$ ($\boldsymbol{A}_0 = I$, $p_0(x) = 1$) and for $i = 1$ ($\boldsymbol{A}_1 = \boldsymbol{A}$, $p_1(x) = x$).

INDUCTION STEP

Assume that $\boldsymbol{A}_j = p_j(\boldsymbol{A})$ for $0 < j \le i$ for some $i < D$. By Theorem 8.02 we have

$$c_{i+1}\boldsymbol{A}_{i+1} = \boldsymbol{A}\boldsymbol{A}_i - b_{i-1}\boldsymbol{A}_{i-1} - a_i\boldsymbol{A}_i.$$

From the induction hypothesis we know that for $\boldsymbol{A}_i$ and $\boldsymbol{A}_{i-1}$ there exists a polynomials p_i and p_{i-1} of degree i and $i-1$ such that $\boldsymbol{A}_i = p_i(\boldsymbol{A})$ and $\boldsymbol{A}_{i-1} = p_{i-1}(\boldsymbol{A})$. The result now follows from equation $\boldsymbol{A}_{i+1} = \frac{1}{c_{i+1}}(\boldsymbol{A}\boldsymbol{A}_i - b_{i-1}\boldsymbol{A}_{i-1} - a_i\boldsymbol{A}_i)$ and induction hypothesis. □

(8.06) Lemma

Let $\boldsymbol{A}_i \in Mat_\Gamma(\mathbb{R})$ $(1 \le i \le D)$ denote a distance-i matrices. Vector space $\mathcal{D}$ defined by

$$\mathcal{D} = \text{span}\{I, \boldsymbol{A}, \boldsymbol{A}_2, ..., \boldsymbol{A}_D\}$$

forms an algebra with the entrywise (Hadamard) product of matrices, defined by $(X \circ Y)_{uv} = (X)_{uv}(Y)_{uv}$.

Proof: If we want to prove this lemma, we must to show that all condition from definition of algebra[1] are satisfied. Here we will only show that for arbitrary X, Y in $\mathcal{D}$ we have $X \circ Y \in \mathcal{D}$.

First notice that $\boldsymbol{A}_i \circ \boldsymbol{A}_j \in \mathcal{D}$ for $0 \le i, j \le D$ since $\boldsymbol{A}_i \circ \boldsymbol{A}_j = \boldsymbol{0}$ if $i \ne j$ and is $\boldsymbol{A}_i$ if $i = j$. But now, the general proof that for X, Y in $\mathcal{D}$ we have $X \circ Y \in \mathcal{D}$ is a consequence of the fact, that $\mathcal{D}$ is a vector space.

Rest of the proof is left to reader like an easy exercise i.e. it is left to show that

$\mathcal{D}$ is a vector space;
$(X \circ Y) \circ Z = X \circ (Y \circ Z)$, $\forall X, Y, Z \in \mathcal{D}$;
$X \circ (Y + Z) = (X \circ Y) + (X \circ Z)$, $\forall X, Y, Z \in \mathcal{D}$;
$(X + Y) \circ Z = (X \circ Y) + (Y \circ Z)$, $\forall X, Y, Z \in \mathcal{D}$;
$\forall X, Y \in \mathcal{D}$ and $\forall \alpha \in \mathbb{R}$ we have $\alpha(X \circ Y) = (\alpha X) \circ Y = X \circ (\alpha Y)$. □

(8.07) Definition (distance ∘-algebra)

Algebra $\mathcal{D}$ from Lemma 8.06 will be called the *distance ∘-algebra* of Γ. ◇

(8.08) Comment ($I, \boldsymbol{A}, \boldsymbol{J} \in \mathcal{A} \cap \mathcal{D}$)

Let Γ denote a regular graph with diameter D and with $d + 1$ distinct eigenvalues. For now (see Proposition 5.04) we have two algebras in game:

adjacency algebra $\mathcal{A} = \text{span}\{I, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^d\}$ and
distance ∘-algebra $\mathcal{D} = \text{span}\{I, \boldsymbol{A}, \boldsymbol{A}_2, ..., \boldsymbol{A}_D\}$.

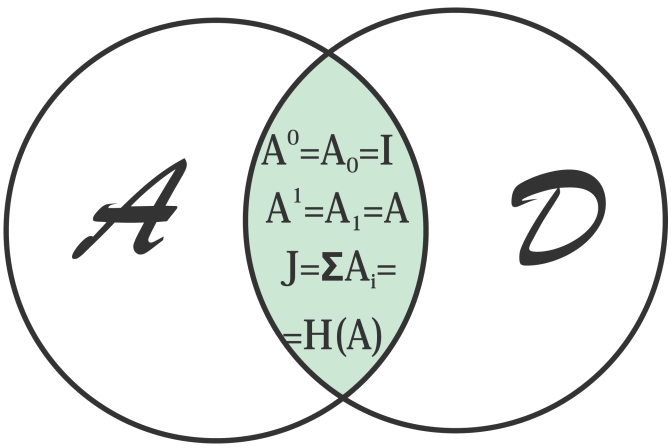


FIGURE 26
Intersection $\mathcal{A} \cap \mathcal{D}$ for regular graphs. ◇

Notice that $I, \boldsymbol{A} \in \mathcal{A}$ and $I, \boldsymbol{A} \in \mathcal{D}$, so $I, \boldsymbol{A} \in \mathcal{A} \cap \mathcal{D}$. For any connected graph Γ it is not hard to see that $\boldsymbol{A}_0 + \boldsymbol{A}_1 + ... + \boldsymbol{A}_D = \boldsymbol{J}$ ($\boldsymbol{J}$ is the all-1 matrix), so $\boldsymbol{J} \in \mathcal{D}$. But Theorem 6.05 say that there exist some polynomial $p(x)$ (Hoffman polynomial) such that $\boldsymbol{J} = p(\boldsymbol{A})$, so also $\boldsymbol{J} \in \mathcal{A}$. Therefore $I, \boldsymbol{A}, \boldsymbol{J} \in \mathcal{A} \cap \mathcal{D}$.

[1]Recall: A vector space $\mathcal{V}$ over a field $\mathbb{F}$ that is also a ring in which holds $\alpha(uv) = (\alpha u)v = u(\alpha v)$ for all vectors $u, v \in \mathcal{V}$ and scalars α, is called an *algebra over* $\mathbb{F}$.

Is this all that we can say about $\mathcal{A} \cap \mathcal{D}$?

(8.09) Corollary

Let $\Gamma = (V, E)$ denote arbitrary connected graph which is distance-regular around each of its vertices and with the same intersection array, and let $\boldsymbol{A}_i$, $1 \leq i \leq D$, be a distance-i matrices. Then

$$\boldsymbol{A}^n \in \mathcal{D}.$$

for arbitrary non-negative integers n. Moreover, if $\boldsymbol{A}^n = \beta_0 \boldsymbol{A}_0 + \beta_1 \boldsymbol{A}_1 + ... + \beta_D \boldsymbol{A}_D$, then β_0, β_1, ..., β_D depends only on a_j, b_j, c_j.

Proof: We will prove the corollary using induction on n.

BASIS OF INDUCTION

It is clear that the result holds for $n = 0$ and $n = 1$ ($\boldsymbol{A}^0 = I \in span\{\boldsymbol{A}_0, \boldsymbol{A}_1, ..., \boldsymbol{A}_D\}$ and $\boldsymbol{A}^1 = \boldsymbol{A} \in span\{\boldsymbol{A}_0, \boldsymbol{A}_1, ..., \boldsymbol{A}_D\}$).

INDUCTION STEP

Assume now that the result holds for n. Then there are scalars $\alpha_0, ..., \alpha_D$ such that $\boldsymbol{A}^n = \alpha_0 \boldsymbol{A}_0 + \alpha_1 \boldsymbol{A}_1 + ... + \alpha_D \boldsymbol{A}_D$. We have

$$\boldsymbol{A}^{n+1} = \boldsymbol{A}\boldsymbol{A}^n = \boldsymbol{A}(\alpha_0 \boldsymbol{A}_0 + \alpha_1 \boldsymbol{A}_1 + ... + \alpha_D \boldsymbol{A}_D) = \alpha_0 \boldsymbol{A}\boldsymbol{A}_0 + \alpha_1 \boldsymbol{A}\boldsymbol{A}_1 + ... + \alpha_D \boldsymbol{A}\boldsymbol{A}_D$$

The result now follows from Theorem 8.02. Result for $\boldsymbol{A}^{n+1}$ is then some linear combination of $\boldsymbol{A}_0, \boldsymbol{A}_1, ..., \boldsymbol{A}_D$, say $\delta_0 \boldsymbol{A}_0 + \delta_1 \boldsymbol{A}_1 + ... + \delta_d \boldsymbol{A}_D$ where $\delta_0, \delta_1 ..., \delta_D$ depends only on a_j, b_j, c_j. □

Recall from Definition 5.01 that the adjacency algebra $\mathcal{A}$ of a graph Γ is the algebra of polynomials in the adjacency matrix $\boldsymbol{A} = \boldsymbol{A}(\Gamma)$. By Proposition 5.04, dimension of $\mathcal{A}$ is d where $d + 1$ is number of distinct eigenvalues of Γ.

(8.10) Corollary

Let $\Gamma = (V, E)$ denote arbitrary connected graph which is distance-regular around each of its vertices and with the same intersection array. Then we have

$$\mathcal{A} = \mathcal{D}.$$

Proof: First we will show that $\mathcal{A} \subseteq \mathcal{D}$. By Corollary 8.09 we have $A^i \in \mathcal{D}$ so $c_0 A^0 + c_1 A^1 + ... + c_m A^m \in \mathcal{D}$ for arbitrary $m \in \mathbb{N}$ and for arbitrary $c_0, c_1, ..., c_m \in \mathbb{F}$. Therefore, $\mathcal{A} \subseteq \mathcal{D}$.

Now we want to show that $\mathcal{D} \subseteq \mathcal{A}$. By Proposition 8.05 there exists polynomials p_i of degree i such that $A_0 = p_0(A)$, $A_1 = p_1(A)$, ..., $A_D = p_D(A)$. Therefore, $\text{span}\{A_0, A_1, ..., A_D\} \subseteq \mathcal{A}$.

The result follow. □

(8.11) Lemma

Let $\Gamma = (V, E)$ denote connected graph which is distance-regular around each of its vertices and with the same intersection array. Then for $0 \leq i, j \leq D$ there exist numbers α_{ij}^h $(0 \leq h \leq D)$ such that

$$\boldsymbol{A}_i \boldsymbol{A}_j = \sum_{h=0}^{D} \alpha_{ij}^h \boldsymbol{A}_h,$$

where for x, y with $\partial(x, y) = h$ and for $0 \leq i, j \leq D$ we have

$$|\Gamma_i(x) \cap \Gamma_j(y)| = \alpha_{ij}^h.$$

Proof: From Corollary 8.10 $\mathcal{A} = \text{span}\{\boldsymbol{A}_0, \boldsymbol{A}_1, ..., \boldsymbol{A}_D\}$. That means that for every $\boldsymbol{A}_i, \boldsymbol{A}_j \in \mathcal{A}$ we have $\boldsymbol{A}_i\boldsymbol{A}_j \in \mathcal{A}$ and so there exist unique scalars α_{ij}^h $(0 \le i, j, h \le D)$ such that

$$\boldsymbol{A}_i\boldsymbol{A}_j = \alpha_{ij}^0\boldsymbol{A}_0 + \alpha_{ij}^1\boldsymbol{A}_1 + ... + \alpha_{ij}^D\boldsymbol{A}_D.$$

Notice that

$$(\alpha_{ij}^0\boldsymbol{A}_0 + \alpha_{ij}^1\boldsymbol{A}_1 + ... + \alpha_{ij}^D\boldsymbol{A}_D)_{xy} = \alpha_{ij}^h \quad \text{if } \partial(x,y) = h.$$

If we consider Comment 7.07, since distance is unique, for $\partial(x,y) = h$, we have

$$\alpha_{1j}^h = (\boldsymbol{A}_1\boldsymbol{A}_j)_{xy} = \sum_{z \in V}(\boldsymbol{A}_1)_{xz}(\boldsymbol{A}_j)_{zy} = |\Gamma_1(x) \cap \Gamma_j(y)| = \begin{cases} a_j, & \text{if } \partial(x,y) = j \\ c_{j+1}, & \text{if } \partial(x,y) = j+1 \\ b_{j-1}, & \text{if } \partial(x,y) = j-1 \end{cases},$$

and

$$\alpha_{ij}^h = \sum_{z \in V}(\boldsymbol{A}_i)_{xz}(\boldsymbol{A}_j)_{zy} = |\Gamma_i(x) \cap \Gamma_j(y)|.$$

□

(8.12) Theorem (characterization A)

Let $\Gamma = (V, E)$ denote a graph with diameter D and let the set $\Gamma_h(u)$ represents the set of vertices at distance h from vertex u. Γ is distance-regular if and only if is distance-regular around each of its vertices and with the same intersection array (with another words if and only if for any two vertices $u, v \in V$ at distance $\partial(u,v) = h$, $0 \le h \le D$, the numbers

$$c_h(u,v){:=}|\Gamma_{h-1}(u) \cap \Gamma(v)|, \ \ a_h(u,v){:=}|\Gamma_h(u) \cap \Gamma(v)|, \ \ b_h(u,v){:=}|\Gamma_{h+1}(u) \cap \Gamma(v)|,$$

do not depend on the chosen vertices u and v, but only on their distance h; in which case they are denoted by c_h, a_h, and b_h, respectively).

Proof: ($\Rightarrow$) If $\Gamma = (V, E)$ is distance-regular then by definition there exist numbers p_{ij}^h $(0 \le i, j, h \le D)$ such that for any $u, v \in V$ with $\partial(u,v) = h$ we have $|\Gamma_i(u) \cap \Gamma_j(v)| = p_{ij}^h$. If we set $j = 1$, $i \in \{h-1, h, h+1\}$ we have that for any two vertices $u, v \in V$ at distance $\partial(u,v) = h$, $0 \le h \le D$, the numbers

$$c_h(u,v){:=}p_{h-1,1}^h = |\Gamma_{h-1}(u) \cap \Gamma(v)|, \quad a_h(u,v){:=}p_{h1}^h = |\Gamma_h(u) \cap \Gamma(v)|,$$

$$b_h(u,v){:=}p_{h+1,1}^h = |\Gamma_{h+1}(u) \cap \Gamma(v)|,$$

do not depend on the chosen vertices u and v, but only on their distance h.

($\Leftarrow$) Conversely, assume that for any two vertices $u, v \in V$ at distance $\partial(u,v) = h$, $0 \le h \le D$, the numbers

$$c_h(u,v){=}|\Gamma_{h-1}(u) \cap \Gamma(v)|, \ \ a_h(u,v){=}|\Gamma_h(u) \cap \Gamma(v)|, \ \ b_h(u,v){=}|\Gamma_{h+1}(u) \cap \Gamma(v)|,$$

do not depend on the chosen vertices u and v, but only on their distance h. With another words we have $c_h(u,v) = c_h$, $a_h(u,v) = a_h$, $b_h(u,v) = b_h$, where numbers c_h, a_h, b_h are intersection numbers from Comment 7.07. From Corollary 8.11 we have

$$|\Gamma_i(x) \cap \Gamma_j(y)| = \alpha_{ij}^h.$$

for x, y with $\partial(x,y) = h$ and for $0 \le i, j \le D$. Therefore, Γ is distance-regular, with $p_{ij}^h = \alpha_{ij}^h$, for $0 \le i, j, h \le D$. □

(8.13) Comment

Thus, one intuitive way of looking at distance-regularity is to "hang" the graph from a given vertex and observe the resulting different "layers" in which the vertex set is partitioned; that is, the subsets of vertices at given distances from the root: If vertices in the same layer are "neighborhood-indistinguishable" from each other, and the whole configuration does not depend on the chosen vertex, the graph is distance-regular (see Figure 16 for illustration, hang of the cube).

Second thing is, that for distance-regular graphs we have (see Corollary 8.10)

$$\mathcal{A} \cap \mathcal{D} = \mathcal{A} = \mathcal{D}.$$

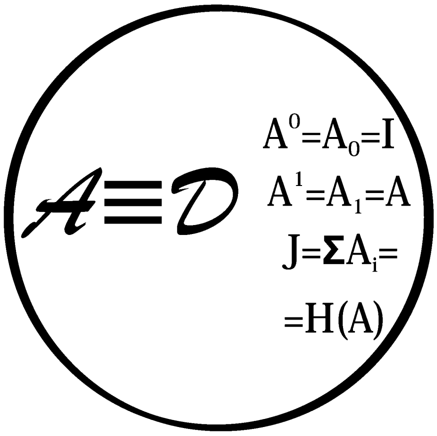


FIGURE 27
Intersection $\mathcal{A} \cap \mathcal{D}$ for distance-regular graphs. ♢

(8.14) Definition (intersection array)

Let Γ be distance-regular graph with diameter D, and let a_i, b_i and c_i be numbers from Theorem 8.12. By the <u>*intersection array*</u> of Γ we mean the following matrix

$$\mathcal{I} := \begin{pmatrix} 0 & c_1 & \dots & c_{D-1} & c_D \\ a_0 & a_1 & \dots & a_{D-1} & a_D \\ b_0 & b_1 & \dots & b_{D-1} & 0 \end{pmatrix}$$

(since $a_i = \delta - b_i - c_i$ where δ is valency of graph Γ (Lemma 7.09), some authors intersection array denote by $\{b_0, b_1, ..., b_{D-1}; c_1, c_2, ..., c_D\}$). ⋄

(8.15) Theorem (characterization B)

A graph $\Gamma = (V, E)$ with diameter D is distance-regular if and only if, for any integers $0 \le i, j \le D$, its distance matrices satisfy

$$\boldsymbol{A}_i \boldsymbol{A}_j = \sum_{k=0}^{D} p_{ij}^k \boldsymbol{A}_k \quad (0 \le i, j \le D)$$

for some constants p_{ij}^k.

Proof: ($\Rightarrow$) Let $\Gamma = (V, E)$ be a distance-regular graph with diameter D. Pick two arbitrary vertices u and v on distance h ($\partial(u, v) = h$), where $0 \le h \le D$. Now, for every $0 \le i, j \le D$ we have

$$(\boldsymbol{A}_i \boldsymbol{A}_j)_{uv} = \sum_{x \in V} (\boldsymbol{A}_i)_{ux} (\boldsymbol{A}_j)_{xv} = |\Gamma_i(u) \cap \Gamma_j(v)| = p_{ij}^h = (p_{ij}^h \boldsymbol{A}_h)_{uv}$$

where p^h_{ij} are numbers from definition of DRG (Definition 7.01). From uniqueness of distance we have

$$\boldsymbol{A}_i\boldsymbol{A}_j = \sum_{k=0}^{D} p^k_{ij}\boldsymbol{A}_k.$$

($\Leftarrow$) Assume that for any integers $0 \le i, j \le D$, distance matrices of a graph $\Gamma = (V, E)$ satisfy

$$\boldsymbol{A}_i\boldsymbol{A}_j = \sum_{k=0}^{D} p^k_{ij}\boldsymbol{A}_k \quad (0 \le i, j \le D)$$

for some constants p^k_{ij}. Pick two arbitrary vertices u and v on distance h ($\partial(u, v) = h$), where $0 \le h \le D$. Consider the following equations

$$|\Gamma_1(u) \cap \Gamma_h(v)| = \sum_{x \in V}(\boldsymbol{A}_1)_{ux}(\boldsymbol{A}_h)_{xv} = (\boldsymbol{A}_1\boldsymbol{A}_h)_{uv} = (\sum_{k=0}^{D} p^k_{1h}\boldsymbol{A}_k)_{uv} = p^h_{1h},$$

$$|\Gamma_1(u) \cap \Gamma_{h-1}(v)| = \sum_{x \in V}(\boldsymbol{A}_1)_{ux}(\boldsymbol{A}_{h-1})_{xv} = (\boldsymbol{A}_1\boldsymbol{A}_{h-1})_{uv} = (\sum_{k=0}^{D} p^k_{1,h-1}\boldsymbol{A}_k)_{uv} = p^h_{1,h-1},$$

$$|\Gamma_1(u) \cap \Gamma_{h+1}(v)| = \sum_{x \in V}(\boldsymbol{A}_1)_{ux}(\boldsymbol{A}_{h+1})_{xv} = (\boldsymbol{A}_1\boldsymbol{A}_{h+1})_{uv} = (\sum_{k=0}^{D} p^k_{1,h+1}\boldsymbol{A}_k)_{uv} = p^h_{1,h+1}.$$

Now, we see that the numbers $|\Gamma_1(u) \cap \Gamma_h(v)|$, $|\Gamma_1(u) \cap \Gamma_{h-1}(v)|$, $|\Gamma_1(u) \cap \Gamma_{h+1}(v)|$ depend only on distance between u and v, so the result follows from Theorem 8.12 (Characterization A). □

(8.16) Exercise

Show that graph pictured on Figure 24 is distance-regular, and find numbers p^h_{ij} $(0 \le i, j, h \le D)$ from definition of DRG (Definition 7.01).

Solution: It is not hard to compute the distance matrices for a given graph

$$\boldsymbol{A}_0 = \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}, \quad \boldsymbol{A}_1 = \begin{bmatrix} 0&1&0&1&1&1\\ 1&0&1&0&1&1\\ 0&1&0&1&1&1\\ 1&0&1&0&1&1\\ 1&1&1&1&0&0\\ 1&1&1&1&0&0 \end{bmatrix}, \quad \boldsymbol{A}_2 = \begin{bmatrix} 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&1&0 \end{bmatrix}.$$

Now we have $\boldsymbol{A}_0\boldsymbol{A}_0 = \boldsymbol{A}_0$, $\boldsymbol{A}_0\boldsymbol{A}_1 = \boldsymbol{A}_1\boldsymbol{A}_0 = \boldsymbol{A}_1$, $\boldsymbol{A}_0\boldsymbol{A}_2 = \boldsymbol{A}_2\boldsymbol{A}_0 = \boldsymbol{A}_2$,

$$\boldsymbol{A}_1\boldsymbol{A}_1 = 4\boldsymbol{A}_0 + 2\boldsymbol{A}_1 + 4\boldsymbol{A}_2,$$

$$\boldsymbol{A}_1\boldsymbol{A}_2 = \boldsymbol{A}_2\boldsymbol{A}_1 = \boldsymbol{A}_1,$$

$$\boldsymbol{A}_2\boldsymbol{A}_2 = \boldsymbol{A}_0,$$

so from Theorem 8.15 (Characterization B) we can conclude that given graph is distance-regular. From obtained equations we have $p^0_{00} = 1$, $p^1_{01} = p^1_{10} = 1$, $p^2_{02} = p^2_{20} = 1$, $p^0_{11} = 4$, $p^1_{11} = 2$, $p^2_{11} = 4$, $p^1_{12} = p^1_{21} = 1$, $p^0_{22} = 1$, and all the rest numbers are equal to 0. ◊

(8.17) Theorem (characterization B')

A graph $\Gamma = (V, E)$ with diameter D is distance-regular if and only if, for some constants a_h, b_h, c_h $(0 \le h \le D)$, $c_0 = b_D = 0$, its distance matrices satisfy the three-term recurrence

$$\boldsymbol{A}_h\boldsymbol{A} = b_{h-1}\boldsymbol{A}_{h-1} + a_h\boldsymbol{A}_h + c_{h+1}\boldsymbol{A}_{h+1} \quad (0 \le h \le D),$$

where, by convention, $b_{-1} = c_{D+1} = 0$.

Proof: $(\Rightarrow)$ This direction follows from Theorem 8.02.

$(\Leftarrow)$ Assume that for some constants a_h, b_h, c_h $(0 \le h \le D)$, $c_0 = b_D = 0$, distance matrices of graph $\Gamma = (V, E)$, satisfy the three-term recurrence

$$\boldsymbol{A}_h\boldsymbol{A} = b_{h-1}\boldsymbol{A}_{h-1} + a_h\boldsymbol{A}_h + c_{h+1}\boldsymbol{A}_{h+1} \quad (0 \le h \le D),$$

where, by convention, $b_{-1} = c_{D+1} = 0$. Now, pick two arbitrary vertices $u, v \in V$ on distance h $(\partial(u, v) = h)$ where $0 \le h \le D$. Consider equations that follows

$$|\Gamma_h(u) \cap \Gamma_1(v)| = \sum_{x \in V} (\boldsymbol{A}_h)_{ux}(\boldsymbol{A})_{xv} = (\boldsymbol{A}_h\boldsymbol{A})_{uv} = (b_{h-1}\boldsymbol{A}_{h-1} + a_h\boldsymbol{A}_h + c_{h+1}\boldsymbol{A}_{h+1})_{uv} =$$

$$= \left\{ \begin{array}{rl} a_h, & \text{if } \partial(u,v) = h \\ b_{h-1}, & \text{if } \partial(u,v) = h-1 \\ c_{h+1}, & \text{if } \partial(u,v) = h+1 \end{array} \right\} = a_h,$$

$$|\Gamma_{h-1}(u) \cap \Gamma_1(v)| = \sum_{x \in V} (\boldsymbol{A}_{h-1})_{ux}(\boldsymbol{A})_{xv} = (\boldsymbol{A}_{h-1}\boldsymbol{A})_{uv} = (b_{h-2}\boldsymbol{A}_{h-2} + a_{h-1}\boldsymbol{A}_{h-1} + c_h\boldsymbol{A}_h)_{uv} =$$

$$= \left\{ \begin{array}{rl} a_{h-1}, & \text{if } \partial(u,v) = h-1 \\ b_{h-2}, & \text{if } \partial(u,v) = h-2 \\ c_h, & \text{if } \partial(u,v) = h \end{array} \right\} = c_h,$$

$$|\Gamma_{h+1}(u) \cap \Gamma_1(v)| = \sum_{x \in V} (\boldsymbol{A}_{h+1})_{ux}(\boldsymbol{A})_{xv} = (\boldsymbol{A}_{h+1}\boldsymbol{A})_{uv} = (b_h\boldsymbol{A}_h + a_{h+1}\boldsymbol{A}_{h+1} + c_{h+2}\boldsymbol{A}_{h+2})_{uv} =$$

$$= \left\{ \begin{array}{rl} a_{h+1}, & \text{if } \partial(u,v) = h+1 \\ b_h, & \text{if } \partial(u,v) = h \\ c_{h+2}, & \text{if } \partial(u,v) = h+2 \end{array} \right\} = b_h.$$

Therefore, the result follow from Theorem 8.12 (Characterization A). □

(8.18) Example

We want to show that graph pictured on Figure 15 is distance-regular, and we want to find his intersection array.

First we will compute the distance matrices of given graph:

$$\boldsymbol{A}_0 = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{bmatrix}, \quad \boldsymbol{A}_1 = \begin{bmatrix} 0&1&1&0&1&0&0&0 \\ 1&0&0&1&0&1&0&0 \\ 1&0&0&1&0&0&1&0 \\ 0&1&1&0&0&0&0&1 \\ 1&0&0&0&0&1&1&0 \\ 0&1&0&0&1&0&0&1 \\ 0&0&1&0&1&0&0&1 \\ 0&0&0&1&0&1&1&0 \end{bmatrix},$$

$$\boldsymbol{A}_2 = \begin{bmatrix} 0&0&0&1&0&1&1&0\\ 0&0&1&0&1&0&0&1\\ 0&1&0&0&1&0&0&1\\ 1&0&0&0&0&1&1&0\\ 0&1&1&0&0&0&0&1\\ 1&0&0&1&0&0&1&0\\ 1&0&0&1&0&1&0&0\\ 0&1&1&0&1&0&0&0 \end{bmatrix}, \quad \boldsymbol{A}_3 = \begin{bmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{bmatrix}.$$

Now, with little help of computer, it is not hard to compute

$$\boldsymbol{A}_0\boldsymbol{A} = 0 + 0\,\boldsymbol{A}_0 + 1\,\boldsymbol{A}_1,$$

$$\boldsymbol{A}_1\boldsymbol{A} = 3\,\boldsymbol{A}_0 + 0\,\boldsymbol{A}_1 + 2\,\boldsymbol{A}_2,$$

$$\boldsymbol{A}_2\boldsymbol{A} = 2\,\boldsymbol{A}_1 + 0\,\boldsymbol{A}_2 + 3\,\boldsymbol{A}_3,$$

$$\boldsymbol{A}_3\boldsymbol{A} = 1\,\boldsymbol{A}_2 + 0\,\boldsymbol{A}_3 + 0,$$

and from the obtain equations (and Theorem 8.17 (characterization B')) we conclude that given graph is distance-regular. From this we also see that

$$a_0 = 0,\ a_1 = 0,\ a_2 = 0,\ a_3 = 0,$$

$$b_0 = 3,\ b_1 = 2,\ b_2 = 1,\ b_3 = 0,$$

$$c_0 = 0,\ c_1 = 1,\ c_2 = 2,\ c_3 = 3.$$

Demanded intersection array is $\{3,2,1;1,2,3\}$. ◊

(8.19) Lemma (d=D)

Let $\Gamma = (V,E)$ denote a distance-regular graph with diameter D. Then

$$\mathcal{A} = \text{span}\{I, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^D\}.$$

Proof: We know that $\mathcal{A} = \text{span}\{I, \boldsymbol{A}, \boldsymbol{A}_2, ..., \boldsymbol{A}_D\}$ for distance-regular graph Γ (Corollary 8.10). Because every distance matrix $\boldsymbol{A}_i$ of Γ can be written as polynomial in $\boldsymbol{A}$ that is of degree i (see Proposition 8.05), it is enough to show that $I, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^D$ are linearly independent. But, from Proposition 5.06, this is true, and it follows

$$\mathcal{A} = \text{span}\{I, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^D\}.$$

□

(8.20) Comment

Since $\mathcal{A} = \mathcal{D}$ (Corollary 8.10), $\{I, \boldsymbol{A},\ \boldsymbol{A}^2, ..., \boldsymbol{A}^d\}$ is basis of the adjacency algebra $\mathcal{A}$ (Proposition 5.04, where $d+1$ is number of distinct eigenvalues) and

$$\mathcal{A} = \text{span}\{I, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^D\},$$
$$\mathcal{D} = \text{span}\{I, \boldsymbol{A}, \boldsymbol{A}_2, ..., \boldsymbol{A}_D\},$$

we have that for any distance-regular graph $\Gamma = (V,E)$ with diameter D, there exist $D+1$ distinct eigenvalues. So, just by realizing that a graph is distance-regular, we automatically know how many eigenvalues it's adjacency matrix has! ◊

(8.21) Exercise

Compare number of distinct eigenvalues of distance-regular graphs given in Figure 15, Figure 24, Figure 41 and Figure 28, with diameter of these graphs.

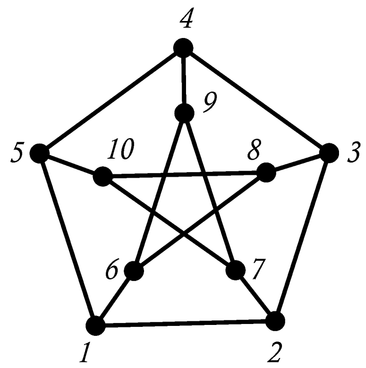


FIGURE 28
Petersen graph.

Solution: Eigenvalues for octahedron (Figure 24) are -2, 0, 4 (diameter of octahedron is 2). Eigenvalues for the cube (Figure 15) are -3, -1, 1, 3 (diameter of the cube is 3). Eigenvalues of Heawood graph (Figure 41) are -3, $-\sqrt{2}$, $\sqrt{2}$ and 3 (diameter of Heawood graph is 3). Eigenvalues of Petersen graph (Figure 28) are -2, 1, 3 (diameter of Petersen graph is 2). ◊

(8.22) Theorem (characterization C)

A graph $\Gamma = (V, E)$ *with diameter* D *is distance-regular if and only if* $\{I, \boldsymbol{A}, ..., \boldsymbol{A}_D\}$ *is a basis of the adjacency algebra* $\mathcal{A}(\Gamma)$.

Proof: This theorem can be proved on many different ways, but in our case, we want to use Lemma 8.19.

($\Rightarrow$) Assume that a graph $\Gamma = (V, E)$ with diameter D is distance-regular. Notice that the set $\{\boldsymbol{A}_0, \boldsymbol{A}_1, ..., \boldsymbol{A}_D\}$ is linearly independent because no two vertices u, v can have two different distances from each other, so for any position (u, v) in the set of distance matrices, there is only one matrix with a one entry in that position, and all the other matrices have zero. So this set is a linearly independent set of $D + 1$ elements. Since any distance-i matrix of distance-regular graph Γ can be written as a polynomial in $\boldsymbol{A}$ that is of degree i (Proposition 8.05) (we have $\boldsymbol{A}_i \in \mathcal{A}$ for any $i = 0, 1, ..., D$) and since $\dim(\mathcal{A}) = D + 1$ (for example see Lemma 8.19), the set $\{\boldsymbol{A}_0, \boldsymbol{A}_1, ..., \boldsymbol{A}_D\}$ must span arbitrary polynomial $p(A)$, and be a basis for $\mathcal{A}(\Gamma)$.

($\Leftarrow$) Assume that the set $\{I, \boldsymbol{A}, ..., \boldsymbol{A}_D\}$ is a basis of the adjacency algebra $\mathcal{A}(\Gamma)$. Because $\mathcal{A}$ is algebra and by assumption $\mathcal{A} = \text{span}\{I, \boldsymbol{A}, \boldsymbol{A}_2, ..., \boldsymbol{A}_D\}$ it follow that $\boldsymbol{A}_i\boldsymbol{A}_j \in \mathcal{A}$ for every i, j. Now, there are unique $\alpha_{ij}^k \in \mathbb{R}$ such that

$$\boldsymbol{A}_i\boldsymbol{A}_j = \alpha_{ij}^0\boldsymbol{A}_0 + \alpha_{ij}^1\boldsymbol{A}_1 + ... + \alpha_{ij}^D\boldsymbol{A}_D = \sum_{k=0}^{D} \alpha_{ij}^k\boldsymbol{A}_k \quad (0 \leq i, j \leq D).$$

Now, result follows from Theorem 8.15 (Characterization B). □

(8.23) Theorem (characterization C')

Let Γ *be a graph of diameter* D *and let* $\boldsymbol{A}_i$ *be the distance-*i *matrix of* Γ. *Then* Γ *is distance-regular if and only if* $\boldsymbol{A}$ *acts by right (or left) multiplication as a linear operator on the vector space* $\text{span}\{I, \boldsymbol{A}_1, \boldsymbol{A}_2, ..., \boldsymbol{A}_D\}$.

Proof: ($\Rightarrow$) Assume that a graph $\Gamma = (V, E)$ with diameter D is distance-regular. Then by Corollary 8.10 and Lemma 8.19 we have $\mathcal{A} = \text{span}\{I, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^D\} = \text{span}\{I, \boldsymbol{A}, \boldsymbol{A}_2, ..., \boldsymbol{A}_D\}$, and from this it is not hard to see that $\boldsymbol{A}$ acts by right (or left) multiplication as a linear operator on the vector space $\text{span}\{I, \boldsymbol{A}_1, \boldsymbol{A}_2, ..., \boldsymbol{A}_D\}$.

($\Leftarrow$) Now assume that in a graph $\Gamma = (V, E)$ with diameter D, matrix $\boldsymbol{A}$ acts by right multiplication as a linear operator on the vector space span$\{I, \boldsymbol{A}_1, \boldsymbol{A}_2, ..., \boldsymbol{A}_D\}$. That means

$$\boldsymbol{A}, \boldsymbol{A}\boldsymbol{A}, \boldsymbol{A}_2\boldsymbol{A}, ..., \boldsymbol{A}_D\boldsymbol{A} \in \text{span}\{I, \boldsymbol{A}_1, \boldsymbol{A}_2, ..., \boldsymbol{A}_D\}$$

so there exist unique $\beta_k \in \mathbb{R}$ $(1 \le k \le D)$ such that

$$\boldsymbol{A}_h\boldsymbol{A} = \sum_{k=1}^{D} \beta_k \boldsymbol{A}_k \quad (1 \le h \le D).$$

If we consider arbitrary (u, v)-entry of $\boldsymbol{A}_h\boldsymbol{A}$ we have

$$(\boldsymbol{A}_h\boldsymbol{A})_{uv} = \sum_{x \in V} (\boldsymbol{A}_h)_{ux}(\boldsymbol{A})_{xv} = |\Gamma_h(u) \cap \Gamma_1(v)| = \begin{cases} \beta_h, & \text{if } \partial(u,v) = h \\ \beta_{h-1}, & \text{if } \partial(u,v) = h-1 \\ \beta_{h+1}, & \text{if } \partial(u,v) = h+1 \\ 0, & \text{otherwise} \end{cases}$$

so for some constants β_{h-1}, β_h, β_{h+1}, $(0 \le h \le D)$, its distance matrices satisfy the three-term recurrence

$$\boldsymbol{A}_h\boldsymbol{A} = \beta_{h-1}\boldsymbol{A}_{h-1} + \beta_h\boldsymbol{A}_h + \beta_{h+1}\boldsymbol{A}_{h+1}.$$

Result now follows from Theorem 8.17 (Characterization B'). □

9 Examples of distance-regular graphs

(9.01) Definition (Hamming graph)

The Hamming graph $H(n, q)$ is the graph whose vertices are words (sequences or n-tuples) of length n from an alphabet of size $q \ge 2$. Two vertices are considered adjacent if the words (or n-tuples) differ in exactly one term. We observe that $|V(H(n, q))| = q^n$. ◇

(9.02) Example

Fix a set $S = \{a, b\}$ $(|S| = 2)$. Let $V = \{aaa, aab, aba, abb, baa, bab, bba, bbb\}$, and $E = \{\{x, y\} : x, y \in V,\ x \text{ and } y \text{ differ in exactly 1 coordinate}\}$. Then graph pictured on Figure 29 is Hamming graph $H(3, 2)$.

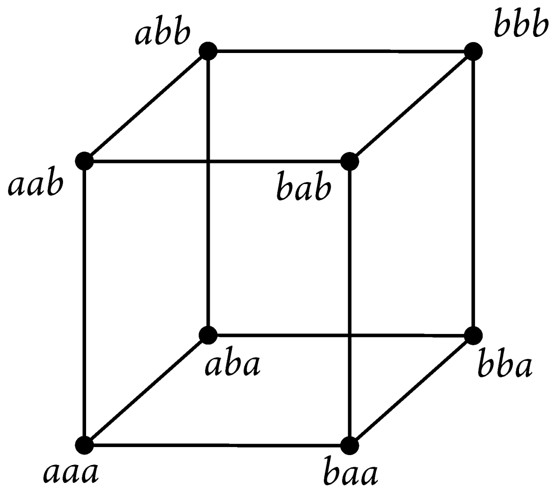


FIGURE 29
Hamming graph $H(3, 2)$. ◇

(9.03) Example

Fix a set $S = \{a, b, c, d\}$ $(|S| = 4)$. Let $V = \{a, b, c, d\}$, and

$E = \{\{x, y\} : x, y \in V, x$ and y differ in exactly 1 coordinate$\}$. Then Hamming graph $H(1, 4)$ pictured on Figure 30 is the complete graph K_4.

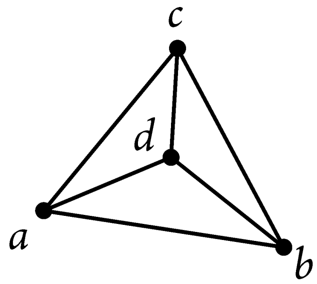


FIGURE 30
Hamming graph $H(1, 4)$. ♢

(9.04) Example

Fix a set $S = \{a, b, c\}$ ($|S| = 3$). Let $V = \{aa, ab, ac, ba, bb, bc, ca, cb, cc\}$, and $E = \{\{x, y\} : x, y \in V, x$ and y differ in exactly 1 coordinate$\}$. Then graph pictured on Figure 31 is Hamming graph $H(2, 3)$.

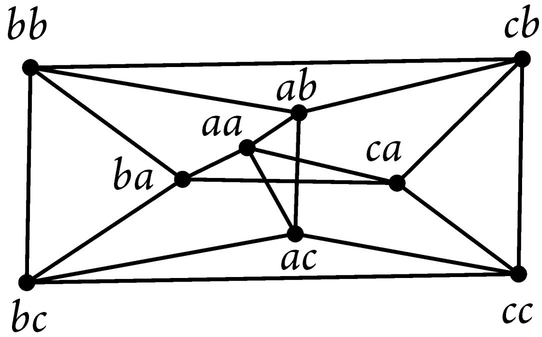


FIGURE 31
Hamming graph $H(2, 3)$. ♢

(9.05) Example

The Hamming graphs $H(n, 2)$ are the n-dimensional hypercubes, Q_n. Q_4 is shown on Figure 32.

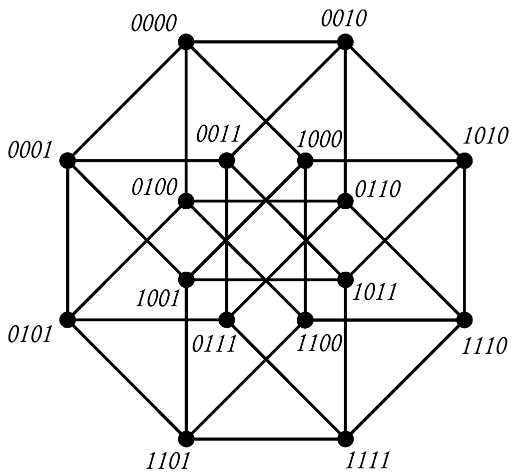


FIGURE 32
Hamming graph $H(4, 2)$. ♢

We will show that the Hamming graphs are distance-regular. First, we need Lemma 9.06 and Lemma 9.07.

(9.06) Lemma

For all vertices x, y of $H(n, q)$, distance $\partial(x, y) = i$ if and only if $\operatorname{num}(x, y) = i$, where $\operatorname{num}(x, y)$ is defined to be the number of coordinates in which vertices x and y are different when considered as words (or n-tuples).

Proof: We will prove this lemma by induction on i.

BASIS OF INDUCTION

Let x and y be vertices of Hamming graph, $H(n, q)$. Then by the adjacency relation, if $\partial(x, y) = 0$ then x and y are the same vertices and therefore differ in 0 coordinates. Similarly, if $\partial(x, y) = 1$ then x and y are adjacent and by the adjacency relation differ in exactly one term.

INDUCTION STEP

Suppose that hypothesis holds for $\partial(x, y) < i$. Consider $\partial(x, y) = i$. Then by definition of distance, there exists a path between x and y of length i say $[x, v_1, v_2, ..., v_{i-2}, z, y]$. So there exists a vertex z, which is on distance $i - 1$ from x and distance 1 from y. Assume that x is word $x_1x_2...x_n$, vertex z is word $z_1z_2...z_n$ and because $\partial(x, z) = i - 1$, by the induction hypothesis, z differs from x in exactly $i - 1$ terms, say in these terms which have indexes $\{h_1, h_2, ..., h_{i-1}\}$. Without loss of generality we can assume that we have $x = c_1c_2...c_{n-i+1}x_{h_1}x_{h_2}...x_{h_{i-1}}$ and $z = c_1c_2...c_{n-i+1}z_{h_1}z_{h_2}...z_{h_{i-1}}$. Vertex z differs from y in exactly 1 term by the adjacency relation. This term can't be one with indexes $\{h_1, h_2, ..., h_{i-1}\}$ because in that case x and y ($y = c_1c_2...c_{n-i+1}z_{h_1}z_{h_2}...y_k...z_{h_{i-1}}$) will have $i - 1$ different terms and by induction hypothesis that induced $\partial(x, y) < i$, which is impossible. Thus, y differs from x in exactly $i - 1 + 1 = i$ terms. □

(9.07) Lemma

The Hamming graphs are vertex-transitive.

Proof: Recall: The simple graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ are *isomorphic* if there is a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in Γ_1 if and only if $f(a)$ and $f(b)$ are adjacent in Γ_2, for all a and b in V_1. Such a function f is called an *isomorphism*. An isomorphism of a graph Γ with itself is called an *automorphism* of Γ. Thus an automorphism f of Γ is a one-one function of Γ onto itself (bijection of Γ) such that $u \sim v$ if and only if $f(u) \sim f(v)$. Two vertices u and v of the graph Γ are *similar* if for some automorphism α of Γ, $\alpha(u) = v$. A *fixed point* is not similar to any other point. A graph is *vertex-transitive* if every pair of vertices are similar.

By definition of vertex-transitivity, $H(n, q)$ is vertex-transitive if for all pairs of vertices x, y there exists an automorphism of the graph that maps x to y. In this proof, we will interpret vertices of $H(n, q)$ like words (sequences) of integers $d_1d_2...d_n$ where each d_i is between 0 and $q - 1$. Why this interpretation? In this way, if we for example consider $H(5, 3)$, we can sum up two vertices termwise modulo q, for example $x = 00122$, $y = 00121$, $z = 11002$ then $x + z = 11121$ and $y + z = 11120$. This interpretation will help us to easier show that the Hamming graph is vertex-transitive.

Let v be a fixed vertex and $x \in V(H(n, q))$. Then the mapping $\rho_v : x \to x + v$, where addition is done termwise modular q, will be an automorphism of the graph since if the words (or n-tuples) x, y differ in exactly 1 term, then the words $x + v$ and $y + v$ will differ in exactly 1 term thus preserving the adjacency relation. And for any two vertices, $x, y \in V(H(n, q))$, the automorphism ρ_{y-x} maps x to y. Thus, Hamming graphs are vertex-transitive. □

(9.08) Lemma

The Hamming graph $H(n, q)$ is distance-regular (with $a_i = i(q - 2)$ $(0 \le i \le n)$,

$b_i = (n-i)(q-1)$ $(0 \le i \le n-1)$ and $c_i = i$ $(1 \le i \le n)$).

Proof: For a graph to be distance-regular, by Theorem 8.12 (Characterization A) it is enough to show that for any vertex, the intersection numbers a_i, b_i, and c_i are independent of choice of vertex. We will prove this lemma on two ways.

FIRST WAY

In the first proof, we will, like in proof of Lemma 9.07, interpret vertices of $H(n,q)$ like words (sequences) of integers $d_1d_2...d_n$ where each d_i is between 0 and $q-1$.

Pick vertices x, y such that $\partial(x,y) = i$. Since $H(n,q)$ is vertex transitive, suppose, without loss of generality, that vertex x is the word 00000...0 ($x = 00...00...00$). By Lemma 9.06, y will have i nonzero entries, and say that $y = y_1y_2...y_i0...0$. Now, a_i is the number of neighbors of y that are also distance i from x. To get neighbor of y we need to pick an term of y, say a ($a \in \{y_1, y_2, ..., y_i, 0\}$), and change it in an element that is different from a, say to b ($b \ne a$). Because we need z such that $\partial(x,z) = i$ we can't pick a to be zero, that is a must be some term from word $y_1y_2...y_i$. Term b can't be 0, and it must be $\ne a$, so for z we have i choices of coordinate in which to differ from y and $q-2$ letters of the alphabet to choose from. Thus, $a_i = \binom{i}{1}(q-2) = i(q-2)$ (see Figure 33 for illustration).

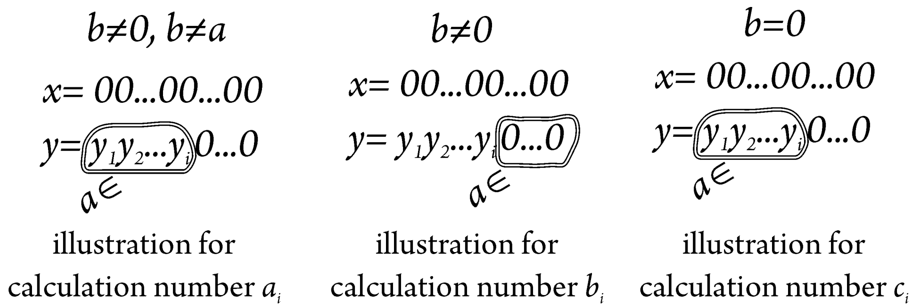


FIGURE 33
To get neighbor of y we need to pick an term of y, say a, and change it in an element that is different from a, say to b.

Number b_i is the number of neighbors of y that are also distance $i+1$ from x. For a we must pick zero and change it to $b \ne 0$. So for vertex u there are $n-i$ places in which to differ from y and $q-1$ letters to choose from. So $b_i = (n-i)(q-1)$.

As for c_i, we are counting the number of vertices that are distance $i-1$ from x and adjacent to y. For term a we will pick one of nonzero terms from $y_1y_2...y_i$, and b must be zero. So we can change any of the i nonzero terms to choose to turn back to zero. So $c_i = i$. Thus the Hamming graph is distance-regular.

SECOND WAY

Pick $x, y \in V$ with $\partial(x,y) = i$. By Lemma 9.06 x and y are differ in i terms and assume that $x = x_1x_2...x_n$, $y = y_1y_2...y_n$ differ in coordinates with indexes $\{h_1, h_2, ..., h_i\}$. Note that $b_i = |\Gamma_1(x) \cap \Gamma_{i+1}(y)|$. Pick $z \in \Gamma_1(x) \cap \Gamma_{i+1}(y)$, and assume that z and x differ in jth coordinate. If $j \in \{h_1, h_2, ..., h_i\}$, then because $\partial(x,y) = i$ we have $\partial(z,y) \in \{i-1, i\}$, a contradiction. Therefore $j \notin \{h_1, h_2, ..., h_i\}$. So we have $n-i$ possibilities for j, and for each of these possibilities we have $q-1$ choices for the jth coordinate of z. Therefore $b_i = (n-i)(q-1)$.

Let us now compute $c_i = |\Gamma_1(x) \cap \Gamma_{i-1}(y)|$ $(1 \le i \le n)$. Pick $z \in \Gamma_1(x) \cap \Gamma_{i-1}(y)$, and assume that z and x differ in jth coordinate. If $j \notin \{h_1, h_2, ..., h_i\}$, then $\partial(z,y) \in \{i, i+1\}$, a contradiction. Therefore $j \in \{h_1, h_2, ..., h_i\}$. So we have i possibilities for j, and for each of these possibilities, the jth coordinate of z must be equal to the jth coordinate of y. Therefore $c_i = i$.

It is an easy exercise to prove that $H(n,q)$ is regular graph. This shows that $H(n,q)$ is distance-regular. □

(9.09) Definition (Johnson graph)

The *Johnson graph* $J(n,r)$, is the graph whose vertices are the r-element subsets of a n-element set S. Two vertices are adjacent if the size of their intersection is exactly $r-1$. To put it on another way, vertices are adjacent if they differ in only one term. We observe that $|V(J(n,r))| = \binom{n}{r}$. ◇

(9.10) Example ($J(4,2)$)

Let S be a set $S = \{a,b,c,d\}$ ($|S| = 4$). Set $\{x,y\}$ in this example we will denoted by xy. The Johnson graph $J(4,2)$ is graph with vertex set $V = \{ab, ac, ad, bc, bc, cd\}$, and edge set $E = \{\{x,y\} : \; x,y \in V,\ x \text{ and } y \text{ are intersect in exactly 1 element } (|x \cap y| = r-1)\}$ (see Figure 34).

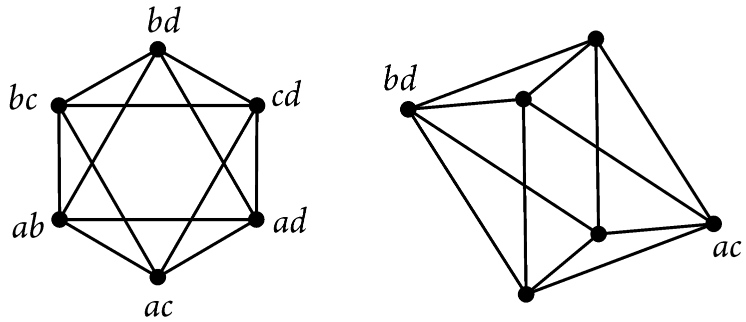


FIGURE 34
Johnson graph $J(4,2)$, drawn in two different ways (this graph is also known as octahedron).◇

(9.11) Example ($J(3,2)$)

Let S be a set $S = \{a,b,c\}$ ($|S| = 3$). Set $\{x,y\}$ in this example we will denoted by xy. The Johnson graph $J(3,2)$ is graph with vertex set $V = \{ab, ac, bc\}$, and edge set $E = \{\{x,y\} : x,y \in V,\ |x \cap y| = r-1\}$ (see Figure 35).

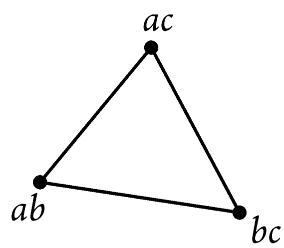


FIGURE 35
Johnson graph $J(3,2)$. ◇

(9.12) Example ($J(5,3)$)

Let S be a set $S = \{0,1,2,3,4\}$ ($|S| = 5$). Set $\{x,y,z\}$ in this example we will denoted by xyz. Edge set is $E = \{\{x,y\} : \; x,y \in V,\ x \text{ and } y \text{ are intersect in exactly 2 elements}\}$. The

Johnson graph $J(5,3)$ is graph pictured on Figure 36.

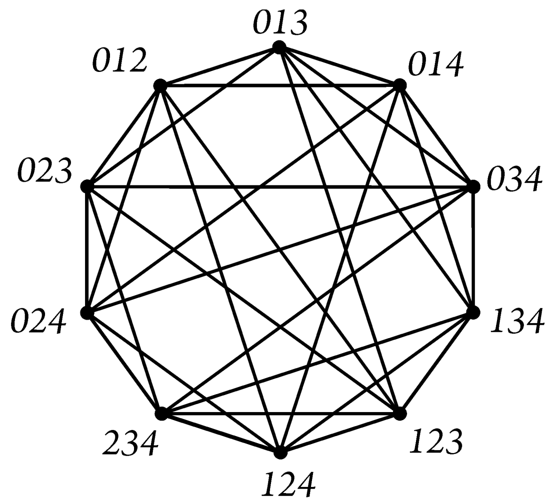


FIGURE 36
Johnson graph $J(5,3)$. ◊

We will show the Johnson graphs are distance-regular but we need the following lemma first.

(9.13) Lemma

If x, y are vertices of the Johnson graph $J(n,r)$, then $\partial(x,y) = i$ if and only if $|x \cap y| = r - i$.

Proof: We will prove this lemma by induction on i.

BASIS OF INDUCTION

Let x, y be vertices of $J(n,r)$. Then $\partial(x,y) = 0$ if and only if x and y are the same vertices, which holds if and only if $|x \cap y| = r = r - 0$. And $\partial(x,y) = 1$ if and only if x and y differ in only one term i.e. $|x \cap y| = r - 1$.

INDUCTION STEP

Suppose the result holds for any x, y with $\partial(x,y) < i$. That is, for any $0 \leq k < i$ assume that $\partial(x,y) = k$ if and only if $|x \cap y| = r - k$. If we write this with details we have

$$\partial(x,y) = 1 \Leftrightarrow |x \cap y| = r - 1 \Leftrightarrow (|x \backslash y| = 1 \text{ and } |y \backslash x| = 1),$$

$$\partial(x,y) = 2 \Leftrightarrow |x \cap y| = r - 2 \Leftrightarrow (|x \backslash y| = 2 \text{ and } |y \backslash x| = 2),$$

$$\ldots$$

$$\partial(x,y) = i-3 \Leftrightarrow |x \cap y| = r - i + 3 \Leftrightarrow (|x \backslash y| = i-3 \text{ and } |y \backslash x| = i-3),$$

$$\partial(x,y) = i-2 \Leftrightarrow |x \cap y| = r - i + 2 \Leftrightarrow (|x \backslash y| = i-2 \text{ and } |y \backslash x| = i-2),$$

$$\partial(x,y) = i-1 \Leftrightarrow |x \cap y| = r - i + 1 \Leftrightarrow (|x \backslash y| = i-1 \text{ and } |y \backslash x| = i-1),$$

where symbol "\" denote difference of sets. Notice that $r-1 > r-2 > \ldots > r-i+3 >$ $> r-i+2 > r-i+1$.

($\Rightarrow$) If $\partial(x,y) = i$, then $\partial(x,y) > i-1$, so $|x \cap y| < r - i + 1$ by the induction hypothesis. So

$$|x \cap y| \leq r - i. \tag{12}$$

By definition of distance, there exists a path of length i from x to y. Thus, there exists a vertex z that is distance $i-1$ from x and adjacent to y ($\partial(x,z)=i-1$, $\partial(z,y)=1$). So by the induction hypothesis

$$|z\backslash x| = i-1 \text{ and } |y\backslash z| = 1.$$

Now we notice that

$$|y\backslash x| = |[(y\backslash x)\cap z]\cup[(y\backslash x)\backslash z]| = |(y\backslash x)\cap z| + |(y\backslash x)\backslash z|.$$

Since $(y\backslash x)\cap z \subseteq z\backslash x$ and $(y\backslash x)\backslash z \subseteq y\backslash z$ we have $|y\backslash x| \le |z\backslash x| + |y\backslash z| = (i-1)+1$, so $|y\backslash x| \le i$ which implies

$$|x\cap y| \ge r-i. \tag{13}$$

From Equations (12) and (13) we conclude $|x\cap y| = r-i$ as desired.

($\Leftarrow$) Now suppose $|x\cap y| = r-i$. We need to show that $\partial(x,y)=i$. If $\partial(x,y)<i$ then, by the induction hypothesis, $|x\cap y| > r-i$, a contradiction. So

$$\partial(x,y) \ge i.$$

On the other hand, if we let

$$x\backslash y = \{x_1,...,x_i\} \text{ and } y\backslash x = \{y_1,...,y_i\},$$

then we can define, for each j ($0\le j\le i$),

$$z_j = (x\backslash\{x_1,...,x_j\})\cup\{y_1,...,y_j\}.$$

If we write this with details, we have

$$\begin{aligned} z_0 &= x,\\ z_1 &= (x\backslash\{x_1\})\cup\{y_1\},\\ z_2 &= (x\backslash\{x_1,x_2\})\cup\{y_1,y_2\},\\ &\ \ ...\\ z_i &= (x\backslash\{x_1,...,x_i\})\cup\{y_1,...,y_i\} \end{aligned}$$

(z_j and z_{j-1} differ in one coordinate for $0\le j\le i$). Then the sequence $[x=z_0,z_1,...,z_i=y]$ is an xy-path of length i. So

$$\partial(x,y)\le i,$$

forcing $\partial(x,y)=i$ as desired. □

(9.14) Lemma

Johnson graph $J(n,r)$ is distance-regular (with intersection numbers $a_i=(r-i)i+i(n-r-i)$, $b_i=(r-i)(n-r-i)$, $c_i=i^2$).

Proof: It is enough to show that the intersection numbers for Johnson graphs are independent of choice of vertex for the graph to be distance-regular. We will prove this lemma on two ways.

FIRST WAY

Let x,y be vertices of $J(n,r)$ such that $\partial(x,y)=i$. By Lemma 9.13 that means $|x\cap y|=r-i$. Say, without lost of generality that $x=\{c_1,...,c_{r-i},x_1,...,x_i\}$ and $y=\{c_1,...,c_{r-i},y_1,...,y_i\}$. To get a neighbor of y, we need to pick an element of y, say a

($a \in \{c_1, ..., c_{r-i}, y_1, ..., y_i\}$), and change it in element that is not in y, say to b ($b \notin \{c_1, ..., c_{r-i}, y_1, ..., y_i\}$). There are four ways this can be done.

Case 1: If a is an element of $x \cap y = \{c_1, ..., c_{r-i}\}$ and b is an element of $x \backslash y = \{x_1, ..., x_i\}$, then z will differ from y in 1 element and from x in i elements because $y_1, ..., y_i \in z$ but $y_1, ..., y_i \notin x$ (a was common to both x and y but b does not belong to y). This gives a neighbor of y such that $\partial(x, z) = i$.

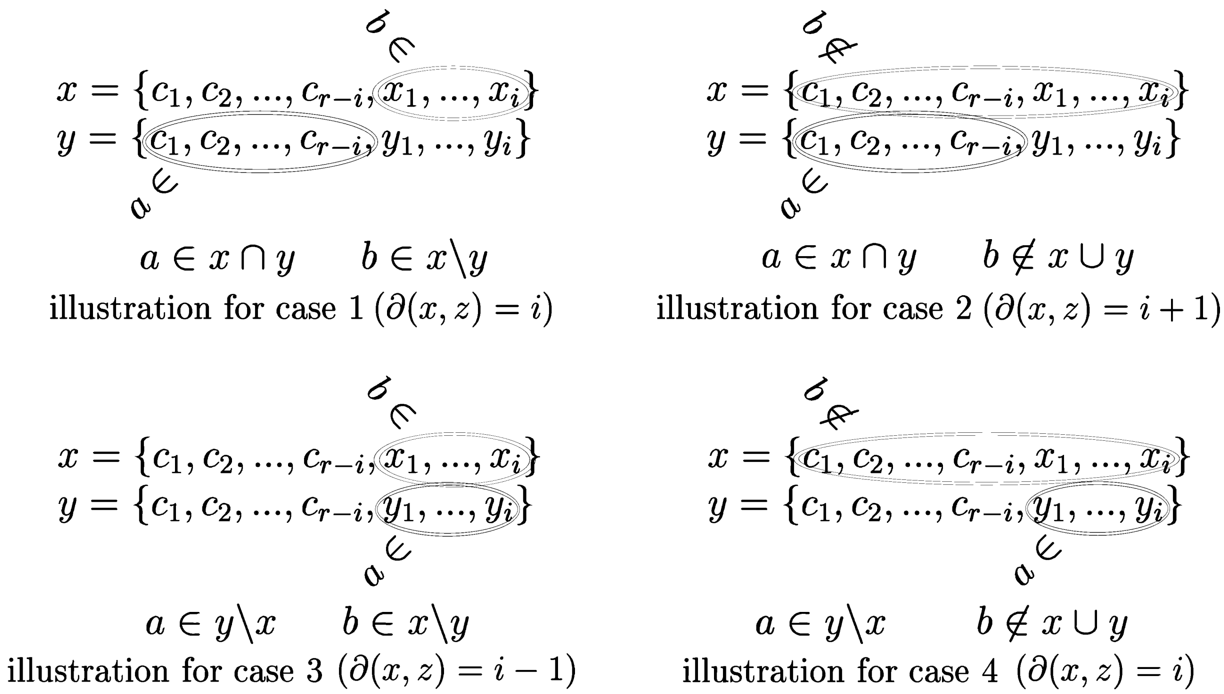


FIGURE 37
To get a neighbor of y, we need to pick an element of y, say a, and change it in element that is not in y, say to b.

Case 2: If a is an element of $x \cap y$ and b is not an element of $x \cup y$, then z will be a neighbor of y that differs from y in 1 element and from x in $i + 1$ elements. So $\partial(x, z) = i + 1$.

Case 3: If a is an element of $y \backslash x$ and b is an element of $x \backslash y$, then z will differ from y in 1 element and from x in only $i - 1$ elements since we are changing a to a element that is already in x. Thus $\partial(x, z) = i - 1$.

Case 4: If a is an element of $y \backslash x$ and b is not an element of $x \cup y$, then z will differ from y by 1 element and from x in i elements since a was not in x and neither is b. Thus $\partial(x, z) = i$.

Now, by definition the intersection number a_i is given by $|\Gamma_i(x) \cap \Gamma_1(y)|$. So we want to count all vertices z, such that $\partial(x, z) = i$ and $\partial(z, y) = 1$. These are given by Case 1 and Case 4. From Case 1, we have that there are $r - i$ choices for a ($a \in x \cap y$, $|x \cap y| = r - i$) and i choices for b ($b \in x \backslash y$, $|x \backslash y| = i$). From Case 4 we have i choices for a ($a \in y \backslash x$, $|y \backslash x| = i$) and $n - r - i$ choices from b ($b \notin x \cup y$, $|x \cup y| = (r - i) + 2i = r + i$. Thus $a_i = (r - i)i + i(n - r - i)$.

The intersection number b_i is given by $|\Gamma_{i+1}(x) \cap \Gamma_1(y)|$. So we want to count all vertices z, such that $\partial(x, z) = i + 1$ and $\partial(z, y) = 1$. These are given by Case 2. We have $r - i$ choices for a ($a \in x \cap y$, $|x \cap y| = r - i$) and since we must pick z not in the union of x and y, we have $n - 2r + (r - i) = n - r - i$ choices for b. Thus $b_i = (r - i)(n - r - i)$.

The intersection number c_i is given by $|\Gamma_{i-1}(x) \cap \Gamma_1(y)|$. So we want to count all vertices z, such that $\partial(x, z) = i - 1$ and $\partial(z, y) = 1$. These are given by Case 3. We have i choices for a ($a \in y \backslash x$, $|y \backslash x| = i$) and i choices for b ($b \in x \backslash y$, $|x \backslash y| = i$), thus $c_i = i^2$. Since the intersection numbers for $J(n, r)$ are independent of choice of vertex, the Johnson graph is distance-regular.

SECOND WAY

Pick $x, y \in V(\Gamma)$ with $\partial(x, y) = h$. Let $x = \{x_1, x_2, ..., x_{r-h}, x_{r-h+1}, ..., x_r\}$ and $y = \{x_1, x_2, ..., x_{r-h}, y_{r-h+1}, ..., y_r\}$ (see Lemma 9.13). Pick $z \in \Gamma_1(x) \cap \Gamma_{h+1}(y)$. Note that z and x differ in exactly one element and assume $x \backslash z = \{x_j\}$. If $j \geq r - h + 1$ then $\{x_1, x_2, ..., x_{r-h}\} \subseteq z$. This implies that $\{x_1, x_2, ..., x_{r-h}\} \subseteq z \cap y$ and therefore $|z \cap y| \geq r - h$. This shows, by Lemma 9.13, that $\partial(z, y) \leq h$, a contradiction.

Therefore $j \in \{1, 2, ..., r - h\}$. So, to get z from x we have to replace any of elements $\{x_1, ..., x_{r-h}\}$. This gives us $(r - h)(n - r - h)$ possibilities for z in total. This shows that $b_h = (r - h)(n - r - h)$ $(0 \leq h \leq D - 1)$.

Pick $z \in \Gamma_1(x) \cap \Gamma_{h-1}(y)$. Hence $|z \backslash x| = 1$ and $|z \cap x| = r - h + 1$. Again, assume $x \backslash z = |x_j|$. If $j \in \{1, 2, ..., r - h\}$, then $|z \cap y| = r - h - 1$. Therefore, to get z from x, we have to replace one of the $\{x_{r-h+1}, ..., x_r\}$ with one of $\{y_{r-h+1}, ..., y_r\}$. This gives us h^2 possibilities in total. Therefore $c_h = h^2$ $(1 \leq h \leq d)$. □

(9.15) Definition (generalized Petersen graph)

Let $n \geq 3$ and $1 \leq k \leq n - 1$, $k \neq \frac{n}{2}$, be integers. A generalized Petersen graph $GPG(n, k)$ is the graph with vertex set $V = \{u_i : i \in \mathbb{Z}_n\} \cup \{v_i : i \in \mathbb{Z}_n\}$ and edge set

$$E = \{\{u_i, u_{i+1}\} \mid i \in \mathbb{Z}_n\} \cup \{\{u_i, v_i\} \mid i \in \mathbb{Z}_n\} \cup \{\{v_i, v_{i+k}\} \mid i \in \mathbb{Z}_n\}. \quad \diamond$$

(9.16) Exercise

Prove that the Petersen graph $GPG(5, 2)$ is distance-regular.

Solution: Consider Theorem 8.12 (Characterization A). We will draw graph with subsets of vertices at given distance from the root, where for a root we will consider all possibilities. If vertices in the same layer are "neighborhood-indistinguishable" from each other, and the whole configuration does not depend on the chosen vertex, the graph is distance-regular. For illustration see Figure 38. Therefore, Petersen graph $GPG(5, 2)$ is distance-regular.

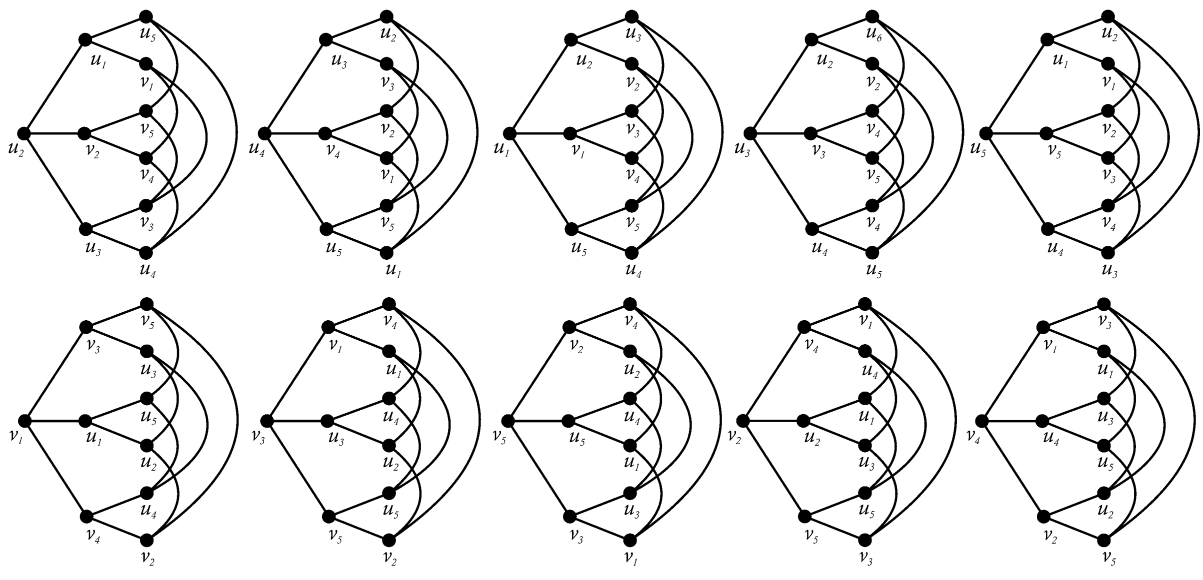


FIGURE 38
Petersen graph $GPG(5, 2)$ drawn on 10 different ways, each with different root. ◊

10 Characterization of DRG involving the distance polynomials

(10.01) Definition (distance-polynomial graphs, distance polynomials)

Graph Γ is called a *distance-polynomial* graph if and only if its distance matrix $\boldsymbol{A}_i$ is a polynomial in $\boldsymbol{A}$ for each $i = 0, 1, ..., D$, where D is the diameter of Γ. Polynomials $\{p_k\}_{0 \leq k \leq D}$ in $\boldsymbol{A}$, such that

$$\boldsymbol{A}_k = p_k(\boldsymbol{A}) \ \ (0 \leq k \leq D),$$

are called the *distance polynomials* (of course, $p_0 = 1$ and $p_1 = x$). ◇

(10.02) Lemma

If the graph Γ is regular, connected and of diameter 2, then Γ is distance-polynomial.

Proof: Consider the sum $I + \boldsymbol{A}_1 + \boldsymbol{A}_2 = \boldsymbol{J}$. Since Γ is regular and connected, $\boldsymbol{J}$ is a polynomial in $\boldsymbol{A}_1$ say $\boldsymbol{J} = q(\boldsymbol{A})$ (Theorem 6.05). Then
$\boldsymbol{A}_2 = \boldsymbol{J} - I - \boldsymbol{A}_1 = \boldsymbol{J} - I - \boldsymbol{A} = q(\boldsymbol{A}) - I - \boldsymbol{A}$, is polynomial in $\boldsymbol{A}$. Thus Γ is distance-polynomial. □

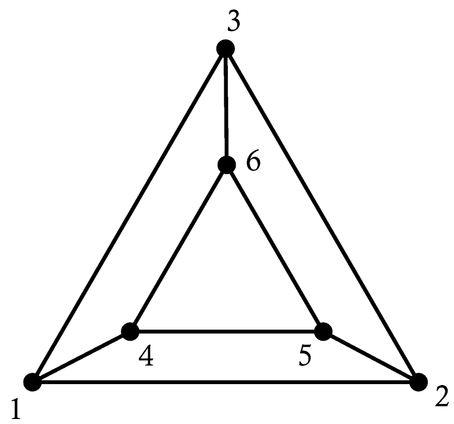


FIGURE 39
The 3-prism (example of distance-polynomial graph which is not distance-regular).

(10.03) Comment

From Proposition 8.05 we see that, distance-regular graphs are distance-polynomial, that is, in a distance-regular graph, each distance matrix $\boldsymbol{A}_h$ is a polynomial of degree h in $\boldsymbol{A}$: $\boldsymbol{A}_h = p_h(\boldsymbol{A}) \in \mathcal{A}(\Gamma)$ $(0 \leq h \leq D)$.

The simplest example (that we took from [48]) of a distance-polynomial graph which is not distance-regular is the 3-prism Γ (Figure 39). Γ clearly has diameter 2, is connected and is regular. Thus Γ is distance-polynomial. It is straightforward to check that Γ is not distance-regular. A distance-polynomial graph which is not distance-regular need not have diameter 2. This example show that classes of distance-regular and distance-polynomial are distinct.

We shall see that distance polynomials satisfies some nice properties which facilitate the computation of the different parameters of Γ.

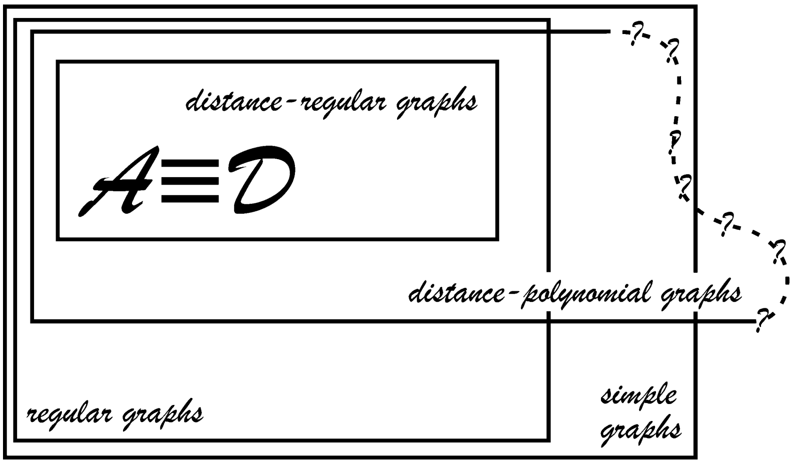


FIGURE 40
Illustration of classes for distance-regular and distance-polynomial graphs. ◊

(10.04) Exercise
Find distance polynomials of distance-regular graph which is given in Figure 41.

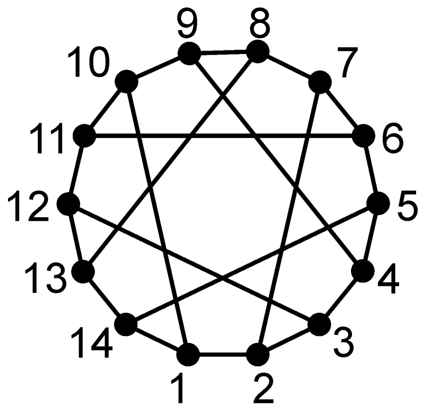


FIGURE 41
Heawood graph.

Solution: First we will find distance matrices:

$$
\boldsymbol{A}_0 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad
\boldsymbol{A}_1 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix},
$$

$$\boldsymbol{A}_2 = \begin{bmatrix} 0&0&1&0&1&0&1&0&1&0&1&0&1&0\\ 0&0&0&1&0&1&0&1&0&1&0&1&0&1\\ 1&0&0&0&1&0&1&0&1&0&1&0&1&0\\ 0&1&0&0&0&1&0&1&0&1&0&1&0&1\\ 1&0&1&0&0&0&1&0&1&0&1&0&1&0\\ 0&1&0&1&0&0&0&1&0&1&0&1&0&1\\ 1&0&1&0&1&0&0&0&1&0&1&0&1&0\\ 0&1&0&1&0&1&0&0&0&1&0&1&0&1\\ 1&0&1&0&1&0&1&0&0&0&1&0&1&0\\ 0&1&0&1&0&1&0&1&0&0&0&1&0&1\\ 1&0&1&0&1&0&1&0&1&0&0&0&1&0\\ 0&1&0&1&0&1&0&1&0&1&0&0&0&1\\ 1&0&1&0&1&0&1&0&1&0&1&0&0&0\\ 0&1&0&1&0&1&0&1&0&1&0&1&0&0 \end{bmatrix}, \quad \boldsymbol{A}_3 = \begin{bmatrix} 0&0&0&1&0&1&0&1&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0&1&0&1&0&1&0\\ 0&0&0&0&0&1&0&1&0&1&0&0&0&1\\ 1&0&0&0&0&0&1&0&0&0&1&0&1&0\\ 0&1&0&0&0&0&0&1&0&1&0&1&0&0\\ 1&0&1&0&0&0&0&0&1&0&0&0&1&0\\ 0&0&0&1&0&0&0&0&0&1&0&1&0&1\\ 1&0&1&0&1&0&0&0&0&0&1&0&0&0\\ 0&1&0&0&0&1&0&0&0&0&0&1&0&1\\ 0&0&1&0&1&0&1&0&0&0&0&0&1&0\\ 0&1&0&1&0&0&0&1&0&0&0&0&0&1\\ 1&0&0&0&1&0&1&0&1&0&0&0&0&0\\ 0&1&0&1&0&1&0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&1&0&1&0&1&0&0&0 \end{bmatrix}.$$

After that we can calculate

$$\boldsymbol{A}_0 = \boldsymbol{A}^0 = I,$$
$$\boldsymbol{A}_1 = \boldsymbol{A}^1 = \boldsymbol{A},$$
$$\boldsymbol{A}_2 = (-3)\boldsymbol{A}^0 + \boldsymbol{A}^2,$$
$$\boldsymbol{A}_3 = (-\frac{5}{3})\boldsymbol{A} + \frac{1}{3}\boldsymbol{A}^3.$$

Distance polynomials of a given graph are $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = -3 + x^2$ and $p_3(x) = -\frac{5}{3}x + \frac{1}{3}x^3$. $\Diamond$

(10.05) Proposition

Let $\Gamma = (V, E)$ be a simple connected graph with adjacency matrix $\boldsymbol{A}$, $|V| = n$ and let $\mathbb{R}[x] = \{a_0 + a_1x + ... + a_mx^m | a_i \in \mathbb{R}\}$ be a set of all polynomials of degree $m \in \mathbb{N}$, with coefficients from $\mathbb{R}$. Define the inner product of two arbitrary elements $p, q \in \mathbb{R}[x]$ with

$$\langle p, q\rangle = \frac{1}{n}\text{trace}(p(\boldsymbol{A})q(\boldsymbol{A})).$$

Prove that $\mathbb{R}[x]$ is inner product space.

Proof: We need to verify that $\mathbb{R}[x]$ is vector space, and that defined product $\langle \cdot, \cdot \rangle$ satisfy axioms from definition of general inner product[2]. We will left this like an easy exercise. $\square$

(10.06) Exercise

Let $\Gamma = (V, E)$ denote regular graph with diameter D, valency λ_0, and let $\{p_h\}_{0\leq h\leq D}$ be distance polynomials. Then

(i) $k_h := |\Gamma_h(u)| = p_h(\lambda_0)$, for arbitrary vertex u (k_h is number independent of u);

(ii) $\|p_h\|^2 = p_h(\lambda_0)$;

for any $0 \leq h \leq D$.

Solution: (i) By $\boldsymbol{j}$ we will denote vector which entries are all ones, $\boldsymbol{j} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$. From Proposition 2.15, $\boldsymbol{j}$ is eigenvector for $\boldsymbol{A}$ with eigenvalue λ_0 so $\boldsymbol{A}\boldsymbol{j} = \lambda_0\boldsymbol{j}$,

[2]Recall: An inner product on a real (or complex) vector space $\mathcal{V}$ is a function that maps each ordered pair of vectors x, y to a real (or complex) scalar $\langle x, y\rangle$ such that the following four properties hold.

$\langle x, x\rangle$ is real with $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0$ if and only if $x = 0$,

$\langle x, \alpha y\rangle = \alpha\langle x, y\rangle$ for all scalars α,

$\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$,

$\langle x, y\rangle = \overline{\langle y, x\rangle}$ (for real spaces, this becomes $\langle x, y\rangle = \langle y, x\rangle$).

Notice that for each fixed value of x, the second and third properties say that $\langle x, y\rangle$ is a linear function of y. Any real or complex vector space that is equipped with an inner product is called an inner-product space.

$\boldsymbol{A}^2\boldsymbol{j} = \boldsymbol{A}\cdot\boldsymbol{A}\boldsymbol{j} = \boldsymbol{A}\cdot\lambda_0\boldsymbol{j} = \lambda_0\boldsymbol{A}\boldsymbol{j} = \lambda_0^2\boldsymbol{j}$,

$$\boldsymbol{A}^k\boldsymbol{j} = \lambda_0^k\boldsymbol{j} \quad (\text{for any } k \in \mathbb{N}). \tag{14}$$

Then for arbitrary vertex $u \in V$

$$|\Gamma_h(u)| = (\boldsymbol{A}_h \cdot \boldsymbol{j})_u = (p_h(\boldsymbol{A}) \cdot \boldsymbol{j})_u = ((c_m\boldsymbol{A}^m + c_{m-1}\boldsymbol{A}^{m-1} + ... + c_1\boldsymbol{A} + c_0 I) \cdot \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix})_u \overset{(14)}{=}$$

$$= (c_m\lambda_0^m\boldsymbol{j} + c_{m-1}\lambda_0^{m-1}\boldsymbol{j} + ... + c_1\lambda_0\boldsymbol{j} + c_0\boldsymbol{j})_u =$$
$$= c_m\lambda_0^m + c_{h-1}\lambda_0^{m-1} + ... + c_1\lambda_0 + c_0 = p_h(\lambda_0).$$

(ii) Let $c_{uv}^h = 1$ if shortest path from u to v is of length h and let $c_{uv}^h = 0$ otherwise. Notice that we have

$$c_{kh}^i c_{hk}^j = \begin{cases} 1, & \text{if } \partial(h,k) = i = j \\ 0, & \text{otherwise} \end{cases}.$$

If we denote vertices of graph Γ with numbers from 1 to n, that is $V = \{1, 2, ..., n\}$, we have

$$\boldsymbol{A}_h = \begin{bmatrix} c_{11}^h & c_{12}^h & \dots & c_{1n}^h \\ c_{21}^h & c_{22}^h & \dots & c_{2n}^h \\ \vdots & \vdots & \dots & \vdots \\ c_{n1}^h & c_{n2}^h & \dots & c_{nn}^h \end{bmatrix}, \quad c_{u1}^h c_{1u}^h + c_{u2}^h c_{2u}^h + ... + c_{un}^h c_{nu}^h = \begin{cases} \text{number of vertices which are} \\ \text{on distance } h \text{ from vertex } u \end{cases},$$

$$\text{trace}(\boldsymbol{A}_h\boldsymbol{A}_h) = \sum_{k=1}^{n}(c_{k1}^h c_{1k}^h + c_{k2}^h c_{2k}^h + ... + c_{kn}^h c_{nk}^h) = |\Gamma_h(1)| + |\Gamma_h(2)| + ... + |\Gamma_h(n)| \overset{(i)}{=} nk_h.$$

Finally

$$\|p_h\|^2 = \langle p_h, p_h\rangle = \frac{1}{n}\text{trace}(p_h(\boldsymbol{A})p_h(\boldsymbol{A})) = \frac{1}{n}\text{trace}(\boldsymbol{A}_h\boldsymbol{A}_h) = k_h = |\Gamma_h(u)| = p_h(\lambda_0).$$

$\Diamond$

In terms of notation from Proposition 7.10, we have $k_h = (b_0b_1...b_{h-1})/(c_1c_2...c_h)$ for $1 \le h \le D$.

(10.07) Proposition

Let $\{p_k\}_{0\le k\le D}$ denote distance polynomials for some regular graph $\Gamma = (V, E)$ which has n vertices, and diameter D. Then

$$\langle p_h, p_l\rangle = \begin{cases} k_h, & \text{if } h = l \\ 0, & \text{otherwise.} \end{cases}$$

where inner product of two polynomials is defined with $\langle p, q\rangle = \frac{1}{n}\text{trace}(p(A)q(A))$, and $k_h = |\Gamma_h(u)|$ is number independent of u.

Proof: Let $c_{uv}^h = 1$ if shortest path from u to v is of length h and let $c_{uv}^h = 0$ otherwise. Notice that we have

$$c_{uv}^h c_{vu}^\ell = \begin{cases} 1, & \text{if } \partial(u,v) = h = \ell \\ 0, & \text{otherwise} \end{cases},$$

that is

$$(\boldsymbol{A}_h)_{uv}(\boldsymbol{A}_\ell)_{uv} = \begin{cases} 1, & \text{if } \partial(u,v) = h = \ell \\ 0, & \text{otherwise} \end{cases}.$$

It follows from Exercise 10.06, that $\langle p_h, p_h \rangle$ is k_h. Now assume that $h \neq \ell$ and compute $\langle p_h, p_\ell \rangle$. We have $\langle p_h, p_\ell \rangle = \frac{1}{n}\text{trace}(p_h(\boldsymbol{A})p_\ell(\boldsymbol{A})) = \frac{1}{n}\text{trace}(\boldsymbol{A}_h\boldsymbol{A}_\ell)$. Pick a vertex u of Γ and compute (u,u)-entry of $\boldsymbol{A}_h\boldsymbol{A}_\ell$:

$$(\boldsymbol{A}_h\boldsymbol{A}_\ell)_{uu} = \left(\begin{bmatrix} c_{11}^h & c_{12}^h & \dots & c_{1n}^h \\ c_{21}^h & c_{22}^h & \dots & c_{2n}^h \\ \vdots & \vdots & \dots & \vdots \\ c_{n1}^h & c_{n2}^h & \dots & c_{nn}^h \end{bmatrix} \begin{bmatrix} c_{11}^\ell & c_{12}^\ell & \dots & c_{1n}^\ell \\ c_{21}^\ell & c_{22}^\ell & \dots & c_{2n}^\ell \\ \vdots & \vdots & \dots & \vdots \\ c_{n1}^\ell & c_{n2}^\ell & \dots & c_{nn}^\ell \end{bmatrix} \right)_{uu} = \sum_{x \in V} (\boldsymbol{A}_h)_{ux}(\boldsymbol{A}_l)_{xu}.$$

As $h \neq \ell$, either $(\boldsymbol{A}_h)_{ux} = 0$, or $(\boldsymbol{A}_\ell)_{xu} = 0$. Therefore $(\boldsymbol{A}_h\boldsymbol{A}_l)_{uu} = 0$, and so $\text{trace}(\boldsymbol{A}_h\boldsymbol{A}_l) = 0$. This shows that $\langle p_h, p_\ell \rangle = 0$. □

(10.08) Theorem (characterization D)

A graph $\Gamma = (V, E)$ with diameter D is distance-regular if and only if, for any integer h, $0 \leq h \leq D$, the distance-h matrix $\boldsymbol{A}_h$ is a polynomial of degree h in $\boldsymbol{A}$; that is:

$$\boldsymbol{A}_h = p_h(\boldsymbol{A}) \quad (0 \leq h \leq D).$$

Proof: ($\Rightarrow$) Let $\Gamma = (V, E)$ be distance-regular graph with diameter D. Then, every condition from Proposition 8.05 is satisfied, so we have

$$\boldsymbol{A}_h = p_h(\boldsymbol{A}) \quad (0 \leq h \leq D).$$

($\Leftarrow$) Now assume that for any integer h, $0 \leq h \leq D$, the distance-h matrix $\boldsymbol{A}_h$ is a polynomial of degree h in $\boldsymbol{A}$; that is $\boldsymbol{A}_h = p_h(\boldsymbol{A})$. If Γ has $d+1$ distinct eigenvalues, then $\{I, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^d\}$ is a basis of the adjacency or Bose-Mesner algebra $\mathcal{A}(\Gamma)$ of matrices which are polynomials in $\boldsymbol{A}$ (Proposition 5.04). Moreover, since Γ has diameter D,

$\dim\mathcal{A}(\Gamma) = d + 1 \geq D + 1,$

because $\{I, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^D\}$ is a linearly independent set of $\mathcal{A}(\Gamma)$ (Proposition 5.06). Hence, the diameter is always less than the number of distinct eigenvalues:

$$D \leq d. \tag{15}$$

Is it true that for any connected graph Γ we have $\boldsymbol{A}_0 + \boldsymbol{A}_1 + ... + \boldsymbol{A}_D = \boldsymbol{J}$, the all-1 matrix? Yes, and it is an easy exercise to explain why. Now, notice that $I + \boldsymbol{A} + ... + \boldsymbol{A}_D = \boldsymbol{J}$, that is $p_0(\boldsymbol{A}) + p_1(\boldsymbol{A}) + ... + p_D(\boldsymbol{A}) = \boldsymbol{J}$, and degree of $h = p_0 + p_1 + ... + p_D$ is D. Comment after Theorem 6.05 say that Hoffman polynomial H is polynomial of smalest degree for which $\boldsymbol{J} = H(\boldsymbol{A})$ and this polynomial has degree d, where $d + 1$ is the number of distinct eigenvalues of Γ. Thus, assuming that Γ has $d + 1$ distinct eigenvalues and using (15) we have

$$D \leq d \leq \text{dgr}\,(h) = \text{dgr}\,(p_0 + p_1 + ... + p_D) = D.$$

The above reasoning's lead to $D = d$, and to conclusion that $\{I, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^D\}$ is a basis of the adjacency algebra $\mathcal{A}(\Gamma)$.

As distance matrices $\boldsymbol{A}_i$ are polynomials in $\boldsymbol{A}$, they belong to the Bose-Mesner algebra. Distance matrices are clearly linearly independent, and since dimension of Bose-Mesner algebra is $d + 1 = D + 1$, they form a basis for Bose-Mesner algebra. By Theorem 8.22 (characterization C), Γ is distance-regular. □

The existence of the first two distance polynomials, p_0 and p_1, is always guaranteed since $\boldsymbol{A}_0 = I$ and $\boldsymbol{A}_1 = \boldsymbol{A}$.

Recall that eccentricity of a vertex u is $\text{ecc}(u) := \max_{v \in V} \partial(u, v)$. Now, if every vertex $u \in V$ has the maximum possible eccentricity allowed by the spectrum (that is, the number of distinct eigenvalues minus one: $\text{ecc}(u) = d$, $\forall u \in V$), the existence of the highest degree distance polynomial suffices:

(10.09) Theorem

A graph $\Gamma = (V, E)$ with diameter D and $d+1$ distinct eigenvalues is distance-regular if and only if all its vertices have spectrally maximum eccentricity d ($\Rightarrow D = d$) and the distance matrix $\boldsymbol{A}_d$ is a polynomial of degree d in $\boldsymbol{A}$:

$$\boldsymbol{A}_d = p_d(\boldsymbol{A}).$$

This was proved by Fiol, Garriga and Yebra [19] in the context of "pseudo-distance-regularity" - a generalization of distance-regularity that makes sense even for non-regular graphs. We will prove similar theorem in Section 11, and our proof will use Lemma 13.07 that we had found in [13] and part of proof from [21].

(10.10) Theorem (characterization E)

A graph $\Gamma = (V, E)$ is distance-regular if and only if, for each non-negative integer ℓ, the number a_{uv}^{ℓ} of walks of length ℓ between two vertices $u, v \in V$ only depends on $h = \partial(u, v)$.

Proof: ($\Rightarrow$) Assume Γ is distance-regular. From Proposition 5.04 we know that $\{I, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^d\}$ is a basis of the adjacency algebra $\mathcal{A}$, where $d+1$ is number of distinct eigenvalues. From Corollary 8.10 and Lemma 8.19 we had that for distance-regular graphs $\mathcal{A} = \text{span}\{I, \boldsymbol{A}, \boldsymbol{A}_2, ..., \boldsymbol{A}_D\} = \text{span}\{I, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^D\}$. So distance matrices $\{I, \boldsymbol{A}, \boldsymbol{A}_2, ..., \boldsymbol{A}_D\}$ are a basis for a Bose-Mesner algebra. It follows that $\boldsymbol{A}^\ell$ is a linear combination of distance matrices for every ℓ, that is for every $\boldsymbol{A}^\ell \in \mathcal{A}$ there are unique constants a_k^ℓ such that

$$\boldsymbol{A}^\ell = \sum_{k=0}^{D} a_k^\ell \boldsymbol{A}_k \quad \Rightarrow \quad (\boldsymbol{A}^\ell)_{uv} = \sum_{k=0}^{D} a_k^\ell (\boldsymbol{A}_k)_{uv} \text{ for arbitrary } u, v \in V.$$

That is, the number $(\boldsymbol{A}^\ell)_{uv} = a_h^\ell$ of walks of length ℓ between two vertices $u, v \in V$ only depends on $h = \partial(u, v)$.

($\Leftarrow$) Conversely, assume that, for a certain graph and any $0 \leq k \leq D$, there are constants a_k^ℓ satisfying $\boldsymbol{A}^\ell = \sum_{k=0}^{D} a_k^\ell \boldsymbol{A}_k$ ($\ell \geq 0$), where a_k^ℓ is number of walks of length ℓ between two vertices on distance k. As a matrix equation,

$$\begin{bmatrix} I \\ \boldsymbol{A} \\ \boldsymbol{A}^2 \\ \vdots \\ \boldsymbol{A}^D \end{bmatrix} = \underbrace{\begin{bmatrix} a_0^0 & 0 & 0 & ... & 0 \\ a_0^1 & a_1^1 & 0 & ... & 0 \\ a_0^2 & a_1^2 & a_2^2 & ... & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_0^D & a_1^D & a_2^D & ... & a_D^D \end{bmatrix}}_{T} \begin{bmatrix} I \\ \boldsymbol{A} \\ \boldsymbol{A}_2 \\ \vdots \\ \boldsymbol{A}_D \end{bmatrix}$$

where the lower triangular matrix T, with rows and columns indexed with the integers 0, 1 ..., D, has entries $(T)_{\ell k} = a_k^\ell$. In particular, note that $a_0^0 = a_1^1 = 1$ and $a_0^1 = 0$. Moreover, since $a_k^k > 0$, such a matrix has an inverse which is also a lower triangular matrix and hence each $\boldsymbol{A}_k$ is a polynomial of degree k in $\boldsymbol{A}$. Therefore, according to Theorem 10.08 (characterization D), we are dealing with a distance-regular graph. (Of course, the entries of T^{-1} are the coefficients of the distance polynomials.) □

We do not need to impose the invariance condition for each value of ℓ. For instance, if Γ is regular we have the following result:

(10.11) Theorem (characterization E')

A regular graph $\Gamma = (V, E)$ with diameter D is distance-regular if and only if there are constants a_h^h and a_h^{h+1} such that, for any two vertices $u, v \in V$ at distance h, we have $a_{uv}^h = a_h^h$

(a^h_{uv} - number of walks of length h) and $a^{h+1}_{uv} = a^{h+1}_h$ for any $0 \le h \le D-1$, and $a^D_{uv} = a^D_D$ for $h = D$.

Proof: To illustrate some typical reasoning's involving the intersection numbers, let us prove characterization E' from the characterization A.

($\Rightarrow$) Assume first that Γ is distance-regular. We shall use induction on k.

BASIS OF INDUCTION

The result clearly holds for $k = 0$ since $a^0_{uu} = 1 = a^0_0$ and $a^1_{uu} = 0 = a^1_0$ (a^0_{uu} is number of walks of length 0 from u to u, and a^1_{uu} is number of walks of length 1 from u to u.)

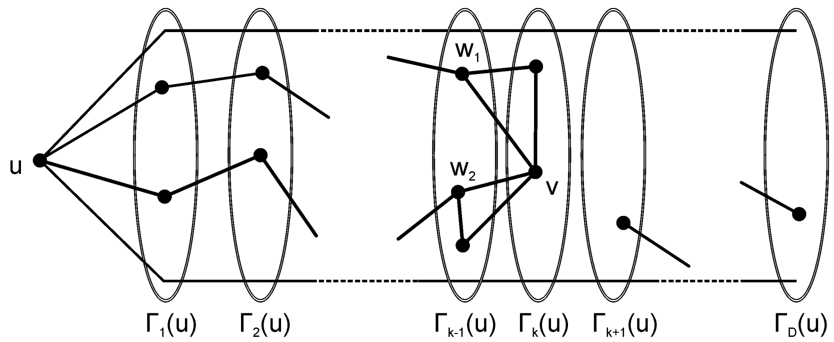


FIGURE 42
Illustration for computing numbers a^k_{uv} and a^{k+1}_{uv}.

INDUCTION STEP

Assume that $a^{k-1}_{uv} = a^{k-1}_{k-1}$ and $a^k_{uv} = a^k_{k-1}$ for any vertices u, v at distance $k-1$. Then, for any vertices u, v at distance k we get equation, say:

$$a^k_{uv} = \sum_{w \in \Gamma_{k-1}(u) \cap \Gamma(v)} a^{k-1}_{uw} = a^{k-1}_{k-1} |\Gamma_{k-1}(u) \cap \Gamma(v)| \tag{16}$$

so we have

$$a^k_{uv} = a^{k-1}_{k-1} c_k \text{ for all } u, v \in V \text{ at distance } k,$$

and from that $a^k_k = a^{k-1}_{k-1} c_k$. Notice that

$$\text{if } \partial(u, w) = k-1 \text{ then by assumption } a^k_{uw} = a^k_{k-1}. \tag{17}$$

Similarly, using equality $a^k_{uv} = a^{k-1}_{k-1} c_k$ and

$$a^{k+1}_{uv} = \sum_{w \in [\Gamma_{k-1}(u) \cup \Gamma_k(u)] \cap \Gamma(v)} a^k_{uw} = \sum_{w \in \Gamma_{k-1}(u) \cap \Gamma(v)} a^k_{uw} + \sum_{w \in \Gamma_k(u) \cap \Gamma(v)} a^k_{uw} =$$

$$\begin{aligned} &\overset{(17)}{=} a^k_{k-1} |\Gamma_{k-1}(u) \cap \Gamma(v)| + a^{k-1}_{k-1} c_k |\Gamma_k(u) \cap \Gamma(v)| = \\ &= a^k_{k-1} |\Gamma_{k-1}(u) \cap \Gamma(v)| + a^k_k |\Gamma_k(u) \cap \Gamma(v)| \end{aligned} \tag{18}$$

we have

$$a^{k+1}_{uv} = a^k_{k-1} c_k + a^{k-1}_{k-1} c_k a_k \text{ for every } u, v \in V.$$

We infer that $a^{k+1}_k = a^k_{k-1} c_k + a^{k-1}_{k-1} c_k a_k$, and the result follows.

($\Leftarrow$) Conversely, suppose that such constants a^k_k and a^k_{k+1} do exist. Now, if $\partial(u, v) = k$, from $a^k_{uv} = a^k_k$ and $a^k_{uv} = a^{k-1}_{k-1} |\Gamma_{k-1}(u) \cap \Gamma(v)|$ (see (16)) we obtain that

$$|\Gamma_{k-1}(u) \cap \Gamma(v)| = \frac{a^k_k}{a^{k-1}_{k-1}}$$

does not depend on the chosen vertices $u \in V$, $v \in \Gamma_k(u)$ and so

$$c_k(u,v) = c_k = \frac{a_k^k}{a_{k-1}^{k-1}}. \tag{19}$$

Analogously, from $a_{uv}^{k+1} = a_k^{k+1}$ and $a_{uv}^{k+1} = a_{k-1}^k|\Gamma_{k-1}(u) \cap \Gamma(v)| + a_k^k|\Gamma_k(u) \cap \Gamma(v)|$ (see (18)) we get

$$a_k^{k+1} = a_{k-1}^k \frac{a_k^k}{a_{k-1}^{k-1}} + a_k^k|\Gamma_k(u) \cap \Gamma(v)|,$$

where we have used the above value of c_k. Consequently, the value

$$|\Gamma_k(u) \cap \Gamma(v)| = \frac{a_k^{k+1}}{a_k^k} - \frac{a_{k-1}^k}{a_{k-1}^{k-1}}$$

is also independent of the vertices u, v, provided that $\partial(u,v) = k$, and

$$a_k(u,v) = a_k = \frac{a_k^{k+1}}{a_k^k} - \frac{a_{k-1}^k}{a_{k-1}^{k-1}}. \tag{20}$$

Finally, since Γ is regular, of degree δ say,

$$b_k(u,v) = |\Gamma_{k+1} \cap \Gamma(v)| = \delta - c_k - a_k,$$

shows that b_k is also independent of u, v and, hence, since Equations (19) and (20) are true, Γ is a distance-regular graph. □

In Proposition 10.05 we have define inner product in $\mathbb{R}[x]$ with $\langle p,q \rangle = \frac{1}{n}\text{trace}(p(A)q(A))$. We also have:

(10.12) Proposition

Let $\Gamma = (V,E)$ be a simple, connected graph with spectrum $\text{spec}(\Gamma) = \{\lambda_0^{m(\lambda_0)}, \lambda_1^{m(\lambda_1)}, ..., \lambda_d^{m(\lambda_d)}\}$, *let p and q be arbitrary polynomials, and let $|V| = n$ (number of vertices in Γ is n). Then*

$$\langle p,q \rangle = \frac{1}{n}\sum_{k=0}^{d} m_k\, p(\lambda_k)q(\lambda_k).$$

where $m_k = m(\lambda_k)$ $(0 \le k \le d)$.

Proof: By Lemma 2.06, there are n orthonormal vectors $v_1, ..., v_n$, that are eigenvectors of the adjacency matrix $\boldsymbol{A}$ of Γ. For these eigenvectors there are some eigenvalues λ_{i_1}, λ_{i_2}, ..., λ_{i_n}, not necessary distinct, and because of Proposition 2.07 we have that $D = P^{-1}AP$ where

$$P = \begin{bmatrix} v_1 & v_2 & ... & v_n \end{bmatrix} \text{ and } D = \begin{bmatrix} \lambda_{i_1} & 0 & ... & 0 \\ 0 & \lambda_{i_2} & ... & 0 \\ \vdots & \vdots & ... & \vdots \\ 0 & 0 & ... & \lambda_{i_n} \end{bmatrix},$$

that is $\boldsymbol{A}$ is diagonalizable. Notice that $PP^\top = I$ (that is $P^{-1} = P^T$),
$\boldsymbol{A}^2 = \boldsymbol{A} \cdot \boldsymbol{A} = PDP^\top PDP^\top = PD^2P^\top$, $\boldsymbol{A}^n = \boldsymbol{A} \cdot \boldsymbol{A} \cdot ... \cdot \boldsymbol{A} = ... = PD^nP^\top$,
$p(\boldsymbol{A}) = \alpha_n\boldsymbol{A}^n + \alpha_{n-1}\boldsymbol{A}^{n-1} + ... + \alpha_1\boldsymbol{A} + \alpha_0 I$,

$$p(\boldsymbol{A}) = P(\alpha_n D^n + \alpha_{n-1}D^{n-1} + ... + \alpha_1 D + \alpha_0)P^\top. \tag{21}$$

For arbitrary matrices A, B for which product AB and BA exist, we know that

$$\operatorname{trace}(AB) = \operatorname{trace}(BA). \tag{22}$$

Now

$$\langle p, q\rangle = \frac{1}{n}\operatorname{trace}(p(\boldsymbol{A})q(\boldsymbol{A})) \overset{(21)}{=}$$

$$= \frac{1}{n}\operatorname{trace}(Pp(D)q(D)P^{\top}) \overset{(22)}{=} \frac{1}{n}\operatorname{trace}(p(D)q(D)P^{\top}P) =$$

$$= \frac{1}{n}\operatorname{trace}(p(D)q(D)) = \frac{1}{n}\sum_{k=0}^{n} p(\lambda_{i_k})q(\lambda_{i_k}) = \frac{1}{n}\sum_{k=0}^{d} m(\lambda_k)p(\lambda_k)q(\lambda_k).$$

□

(10.13) Proposition

Let $\Gamma = (V, E)$ denote a distance-regular graph with adjacency matrix $\boldsymbol{A}$ and with spectrum $\operatorname{spec}(\Gamma) = \{\lambda_0^{m(\lambda_0)}, \lambda_1^{m(\lambda_1)}, ..., \lambda_d^{m(\lambda_d)}\}$. Then multiplicities $m(\lambda_i)$, for any $\lambda_i \in \operatorname{spec}(\Gamma)$, can be computed by using all the distance polynomials $\{p_i\}_{i=0}^d$ of graph Γ:

$$m(\lambda_i) = n\left(\sum_{j=0}^{d}\frac{1}{k_j}p_j(\lambda_i)^2\right)^{-1} \quad (0 \le i \le d)$$

where $k_j := p_j(\lambda_0)$.

Proof: Consider matrix $P = \begin{bmatrix} p_0(\lambda_0) & p_0(\lambda_1) & ... & p_0(\lambda_d) \\ p_1(\lambda_0) & p_1(\lambda_1) & ... & p_1(\lambda_d) \\ \vdots & \vdots & & \vdots \\ p_d(\lambda_0) & p_d(\lambda_1) & ... & p_d(\lambda_d) \end{bmatrix}$, where $p_i(x)$'s are distance polynomials. From Proposition 10.07

$$\langle p_h, p_l\rangle = \begin{cases} k_h, & \text{if } h = l \\ 0, & \text{otherwise} \end{cases},$$

while from Proposition 10.12

$$\langle p, q\rangle = \frac{1}{n}\sum_{k=0}^{d} m(\lambda_k)p(\lambda_k)q(\lambda_k).$$

From this we have

$$\langle p_h, p_h\rangle = \frac{1}{n}(m(\lambda_0)p_h(\lambda_0)^2 + m(\lambda_1)p_h(\lambda_1)^2 + ... + m(\lambda_d)p_h(\lambda_d)^2) = k_h$$

and

$$\langle p_i, p_j\rangle = \frac{1}{n}(m(\lambda_0)p_i(\lambda_0)p_j(\lambda_0) + m(\lambda_1)p_i(\lambda_1)p_j(\lambda_1) + ... + m(\lambda_d)p_i(\lambda_d)p_j(\lambda_d)) = 0, \text{ if } i \neq j.$$

If we use the above equations, it is not hard to see that $PP^{-1} = I$ where

$$P^{-1} = \frac{1}{n}\begin{bmatrix} m(\lambda_0)\frac{p_0(\lambda_0)}{k_0} & m(\lambda_0)\frac{p_1(\lambda_0)}{k_1} & ... & m(\lambda_0)\frac{p_d(\lambda_0)}{k_d} \\ m(\lambda_1)\frac{p_0(\lambda_1)}{k_0} & m(\lambda_1)\frac{p_1(\lambda_1)}{k_1} & ... & m(\lambda_1)\frac{p_d(\lambda_1)}{k_d} \\ \vdots & \vdots & & \vdots \\ m(\lambda_d)\frac{p_0(\lambda_d)}{k_0} & m(\lambda_d)\frac{p_1(\lambda_d)}{k_1} & ... & m(\lambda_d)\frac{p_d(\lambda_d)}{k_d} \end{bmatrix}$$

is inverse of P. Since $P^{-1}P = I$, we have

$$\frac{1}{n}\left(m(\lambda_0)\frac{p_0(\lambda_0)^2}{k_0} + m(\lambda_0)\frac{p_1(\lambda_0)^2}{k_1} + ... + m(\lambda_0)\frac{p_d(\lambda_0)^2}{k_d}\right) = 1$$

$$\vdots$$

$$\frac{1}{n}\left(m(\lambda_d)\frac{p_0(\lambda_d)^2}{k_0} + m(\lambda_d)\frac{p_1(\lambda_d)^2}{k_1} + ... + m(\lambda_d)\frac{p_d(\lambda_d)^2}{k_d}\right) = 1$$

that is

$$\frac{1}{n}m(\lambda_i)\left(\sum_{j=0}^{d}\frac{1}{k_j}p_j(\lambda_i)^2\right) = 1$$

$$\Rightarrow \quad m(\lambda_i) = n\left(\sum_{j=0}^{d}\frac{1}{k_j}p_j(\lambda_i)^2\right)^{-1}$$

where $k_i = |\Gamma_i(x)| = p_i(\lambda_0)$ (see Proposition 10.06). □

11 Characterization of DRG involving the principal idempotent matrices

(11.01) Proposition

Let $\Gamma = (V, E)$ be a (simple and connected) graph with adjacency matrix $\boldsymbol{A}$, and spectrum

$$\text{spec}(\Gamma) = \text{spec}(\boldsymbol{A}) = \{\lambda_0^{m_0}, \lambda_1^{m_1}, ..., \lambda_d^{m_d}\},$$

where the different eigenvalues of Γ are in decreasing order, $\lambda_0 > \lambda_1 > ... > \lambda_d$, and the superscripts stand for their multiplicities $m_i = m(\lambda_i)$. Then all the multiplicities add up to $n = |V|$, the number of vertices of Γ.

Proof: We know that an eigenvalue of $\boldsymbol{A}$ is scalar λ such that $\boldsymbol{A}v = \lambda v$, for some nonzero $v \in \mathbb{R}^n$. From $(\boldsymbol{A} - \lambda I)v = 0$ it follow that $\boldsymbol{A}v = \lambda v$ if and only if $\det(\boldsymbol{A} - \lambda I) = 0$, that is iff $\lambda \in \{\lambda_0, \lambda_1, ..., \lambda_d\}$ where $\det(\boldsymbol{A} - \lambda I) = (\lambda - \lambda_0)^{m_0}(\lambda - \lambda_1)^{m_1}...(\lambda - \lambda_d)^{m_d}$. Recall that number m_i is called algebraic multiplicity of λ_i.

Since $\boldsymbol{A}$ is a real symmetric matrix, it follows from Proposition 2.09 that $\boldsymbol{A}$ is diagonalizable, and (by Theorem 2.12) it follows that $\text{geo mult}_{\boldsymbol{A}}(\lambda) = \text{alg mult}_{\boldsymbol{A}}(\lambda)$ for each $\lambda \in \sigma(\boldsymbol{A}) = \{\lambda_0, \lambda_1, ..., \lambda_d\}$, where $\text{geo mult}_{\boldsymbol{A}}(\lambda_i)$ is geometric multiplicity of λ_i, that is $\dim\ker(\boldsymbol{A} - \lambda_i I) = \dim(\mathcal{E}_i)$. Finaly, from Lemma 4.02, $m_0 + m_1 + ... + m_d = n$, and result follows. □

In Definition 4.03 we have defined principal idempotents of $\boldsymbol{A}$ with $\boldsymbol{E}_i := U_i U_i^\top$ where U_i are the matrices whose columns form an orthonormal basis of eigenspace $\mathcal{E}_i := \ker(A - \lambda_i I)$ ($\lambda_0 \geq \lambda_1 \geq ... \geq \lambda_d$ are distinct eigenvalues of $\boldsymbol{A}$).

(11.02) Example

Let $\Gamma = (V, E)$ denote a regular graph with λ_0 as its largest eigenvalue. Then (from Proposition 2.15) multiplicity of λ_0 is 1 and $\boldsymbol{j} = (1, 1, ..., 1)^\top$ is eigenvector for λ_0. From this it follows

$$\boldsymbol{E}_0 = U_0 U_0^\top = \frac{\boldsymbol{j}}{\|\boldsymbol{j}\|}\frac{\boldsymbol{j}^\top}{\|\boldsymbol{j}\|} = \frac{1}{\|\boldsymbol{j}\|^2}\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}\begin{bmatrix}1 & ... & 1\end{bmatrix} = \frac{1}{n}\begin{bmatrix}1 & 1 & ... & 1\\ 1 & 1 & ... & 1\\ \vdots & \vdots & & \vdots\\ 1 & 1 & ... & 1\end{bmatrix}.$$

◊

(11.03) Exercise

A <u>*path graph*</u> P_n $(n \geq 1)$ is a graph with vertex set $\{1, 2, ..., n\}$ and edge set $\{\{1,2\}, \{2,3\}, ..., \{n-1, n\}\}$ (graph with $n \geq 1$ vertices, that can be drawn so that all of its vertices and edges lie on a single straight line, i.e. two vertices will have degree 1, and other $n-2$ vertices will have degree 2). Illustration for P_3 is on Figure 43 (left).

Cartesian product $\Gamma_1 \times \Gamma_2$ of graphs Γ_1 and Γ_2 is a graph such that

(i) the vertex set of $\Gamma_1 \times \Gamma_2$ is the Cartesian product $V(\Gamma_1) \times V(\Gamma_2)$ and

(ii) any two vertices (u, u') and (v, v') are adjacent in $\Gamma_1 \times \Gamma_2$ if and only if either

(a) $u = v$ and u' is adjacent with v' in Γ_2, or

(b) $u' = v'$ and u is adjacent with v in Γ_1.

Illustration for $P_3 \times P_3$ is on Figure 43 (right).

Determine principal idempotents for graph $\Gamma = P_3 \times P_3$.

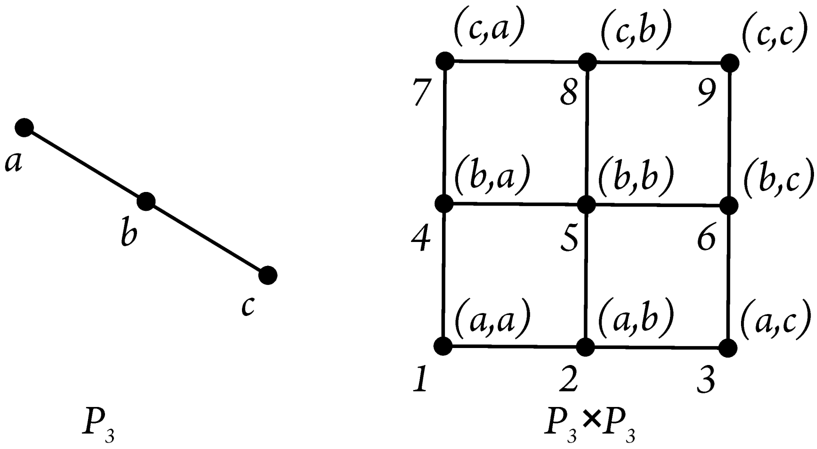


FIGURE 43
Path graph P_3 and graph $P_3 \times P_3$.

Solution: Spectrum of graph $P_3 \times P_3$ is

$$\text{spec}(P_3 \times P_3) = \{2\sqrt{2}^1, \sqrt{2}^2, 0^3, -\sqrt{2}^2, -2\sqrt{2}^1\}.$$

Eigenspace $\mathcal{E}_0$ is spanned by a vector $u_1 = (1, \sqrt{2}, 1, \sqrt{2}, 2, \sqrt{2}, 1, \sqrt{2}, 1)^\top$, eigenspace $\mathcal{E}_1$ is spanned by vectors $u_2 = (1, \sqrt{2}, 1, 0, 0, 0, -1, -\sqrt{2}, -1)^\top$, and $u_3 = (-\sqrt{2}, -1, 0, -1, 0, 1, 0, 1, \sqrt{2})^\top$, eigenspace $\mathcal{E}_2$ is spanned by vectors $u_4 = (0, 0, 1, 0, -1, 0, 1, 0, 0)^\top$, $u_5 = (1, 0, 0, 0, -1, 0, 0, 0, 1)^\top$ and $u_6 = (0, 1, 0, -1, 0, -1, 0, 1, 0)^\top$, eigenspace $\mathcal{E}_3$ is spanned by vectors $u_7 = (\sqrt{2}, -1, 0, -1, 0, 1, 0, 1, -\sqrt{2})^\top$ and $u_8 = (1, -\sqrt{2}, 1, 0, 0, 0, -1, \sqrt{2}, -1)^\top$ and, finelly, eigenspace $\mathcal{E}_4$ is spanned by a vector $u_9 = (1, -\sqrt{2}, 1, -\sqrt{2}, 2, -\sqrt{2}, 1, -\sqrt{2}, 1)^\top$. Now we can use the Gram-Schmidt orthogonalization procedure (if it is necessary) and compute orthonormal vectors:

$$v_1 = (\tfrac{1}{4}, \tfrac{1}{4}\sqrt{2}, \tfrac{1}{4}, \tfrac{1}{4}\sqrt{2}, \tfrac{1}{2}, \tfrac{1}{4}\sqrt{2}, \tfrac{1}{4}, \tfrac{1}{4}\sqrt{2}, \tfrac{1}{4})^\top,$$
$$v_2 = (\tfrac{1}{4}\sqrt{2}, \tfrac{1}{2}, \tfrac{1}{4}\sqrt{2}, 0, 0, 0, -\tfrac{1}{4}\sqrt{2}, -\tfrac{1}{2}, -\tfrac{1}{4}\sqrt{2})^\top,$$
$$v_3 = (-\tfrac{1}{4}\sqrt{2}, 0, \tfrac{1}{4}\sqrt{2}, -\tfrac{1}{2}, 0, \tfrac{1}{2}, -\tfrac{1}{4}\sqrt{2}, 0, \tfrac{1}{4}\sqrt{2})^\top,$$
$$v_4 = (0, 0, \tfrac{1}{3}\sqrt{3}, 0, -\tfrac{1}{3}\sqrt{3}, 0, \tfrac{1}{3}\sqrt{3}, 0, 0)^\top,$$
$$v_5 = (\tfrac{1}{4}\sqrt{6}, 0, -\tfrac{1}{12}\sqrt{6}, 0, -\tfrac{1}{6}\sqrt{6}, 0, -\tfrac{1}{12}\sqrt{6}, 0, \tfrac{1}{4}\sqrt{6})^\top,$$
$$v_6 = (0, \tfrac{1}{2}, 0, -\tfrac{1}{2}, 0, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 0)^\top,$$
$$v_7 = (\tfrac{1}{2}, -\tfrac{1}{4}\sqrt{2}, 0, -\tfrac{1}{4}\sqrt{2}, 0, \tfrac{1}{4}\sqrt{2}, 0, \tfrac{1}{4}\sqrt{2}, -\tfrac{1}{2})^\top,$$
$$v_8 = (0, -\tfrac{1}{4}\sqrt{2}, \tfrac{1}{2}, \tfrac{1}{4}\sqrt{2}, 0, -\tfrac{1}{4}\sqrt{2}, -\tfrac{1}{2}, \tfrac{1}{4}\sqrt{2}, 0)^\top \text{ and}$$
$$v_9 = (\tfrac{1}{4}, -\tfrac{1}{4}\sqrt{2}, \tfrac{1}{4}, -\tfrac{1}{4}\sqrt{2}, \tfrac{1}{2}, -\tfrac{1}{4}\sqrt{2}, \tfrac{1}{4}, -\tfrac{1}{4}\sqrt{2}, \tfrac{1}{4})^\top.$$

After that it is not hard to obtain U_0, U_1, U_2, U_3 and U_4, and evaluate

$$\boldsymbol{E}_0 = \frac{1}{16}\begin{bmatrix} 1 & \sqrt{2} & 1 & \sqrt{2} & 2 & \sqrt{2} & 1 & \sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} & 2 & 2\sqrt{2} & 2 & \sqrt{2} & 2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 & \sqrt{2} & 2 & \sqrt{2} & 1 & \sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} & 2 & 2\sqrt{2} & 2 & \sqrt{2} & 2 & \sqrt{2} \\ 2 & 2\sqrt{2} & 2 & 2\sqrt{2} & 4 & 2\sqrt{2} & 2 & 2\sqrt{2} & 2 \\ \sqrt{2} & 2 & \sqrt{2} & 2 & 2\sqrt{2} & 2 & \sqrt{2} & 2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 & \sqrt{2} & 2 & \sqrt{2} & 1 & \sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} & 2 & 2\sqrt{2} & 2 & \sqrt{2} & 2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 & \sqrt{2} & 2 & \sqrt{2} & 1 & \sqrt{2} & 1 \end{bmatrix},$$

$$\boldsymbol{E}_1 = \frac{1}{8}\begin{bmatrix} 2 & \sqrt{2} & 0 & \sqrt{2} & 0 & -\sqrt{2} & 0 & -\sqrt{2} & -2 \\ \sqrt{2} & 2 & \sqrt{2} & 0 & 0 & 0 & -\sqrt{2} & -2 & -\sqrt{2} \\ 0 & \sqrt{2} & 2 & -\sqrt{2} & 0 & \sqrt{2} & -2 & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} & 2 & 0 & -2 & \sqrt{2} & 0 & -\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{2} & 0 & \sqrt{2} & -2 & 0 & 2 & -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & -2 & \sqrt{2} & 0 & -\sqrt{2} & 2 & \sqrt{2} & 0 \\ -\sqrt{2} & -2 & -\sqrt{2} & 0 & 0 & 0 & \sqrt{2} & 2 & \sqrt{2} \\ -2 & -\sqrt{2} & 0 & -\sqrt{2} & 0 & \sqrt{2} & 0 & \sqrt{2} & 2 \end{bmatrix},$$

$$\boldsymbol{E}_2 = \frac{1}{8}\begin{bmatrix} 3 & 0 & -1 & 0 & -2 & 0 & -1 & 0 & 3 \\ 0 & 2 & 0 & -2 & 0 & -2 & 0 & 2 & 0 \\ -1 & 0 & 3 & 0 & -2 & 0 & 3 & 0 & -1 \\ 0 & -2 & 0 & 2 & 0 & 2 & 0 & -2 & 0 \\ -2 & 0 & -2 & 0 & 4 & 0 & -2 & 0 & -2 \\ 0 & -2 & 0 & 2 & 0 & 2 & 0 & -2 & 0 \\ -1 & 0 & 3 & 0 & -2 & 0 & 3 & 0 & -1 \\ 0 & 2 & 0 & -2 & 0 & -2 & 0 & 2 & 0 \\ 3 & 0 & -1 & 0 & -2 & 0 & -1 & 0 & 3 \end{bmatrix}, \quad \boldsymbol{E}_3 = \frac{1}{8}\begin{bmatrix} 2 & -\sqrt{2} & 0 & -\sqrt{2} & 0 & \sqrt{2} & 0 & \sqrt{2} & -2 \\ -\sqrt{2} & 2 & -\sqrt{2} & 0 & 0 & 0 & \sqrt{2} & -2 & \sqrt{2} \\ 0 & -\sqrt{2} & 2 & \sqrt{2} & 0 & -\sqrt{2} & -2 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} & 2 & 0 & -2 & -\sqrt{2} & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & -\sqrt{2} & -2 & 0 & 2 & \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & -2 & -\sqrt{2} & 0 & \sqrt{2} & 2 & -\sqrt{2} & 0 \\ \sqrt{2} & -2 & \sqrt{2} & 0 & 0 & 0 & -\sqrt{2} & 2 & -\sqrt{2} \\ -2 & \sqrt{2} & 0 & \sqrt{2} & 0 & -\sqrt{2} & 0 & -\sqrt{2} & 2 \end{bmatrix},$$

$$\boldsymbol{E}_4 = \frac{1}{16}\begin{bmatrix} 1 & -\sqrt{2} & 1 & -\sqrt{2} & 2 & -\sqrt{2} & 1 & -\sqrt{2} & 1 \\ -\sqrt{2} & 2 & -\sqrt{2} & 2 & -2\sqrt{2} & 2 & -\sqrt{2} & 2 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 & -\sqrt{2} & 2 & -\sqrt{2} & 1 & -\sqrt{2} & 1 \\ -\sqrt{2} & 2 & -\sqrt{2} & 2 & -2\sqrt{2} & 2 & -\sqrt{2} & 2 & -\sqrt{2} \\ 2 & -2\sqrt{2} & 2 & -2\sqrt{2} & 4 & -2\sqrt{2} & 2 & -2\sqrt{2} & 2 \\ -\sqrt{2} & 2 & -\sqrt{2} & 2 & -2\sqrt{2} & 2 & -\sqrt{2} & 2 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 & -\sqrt{2} & 2 & -\sqrt{2} & 1 & -\sqrt{2} & 1 \\ -\sqrt{2} & 2 & -\sqrt{2} & 2 & -2\sqrt{2} & 2 & -\sqrt{2} & 2 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 & -\sqrt{2} & 2 & -\sqrt{2} & 1 & -\sqrt{2} & 1 \end{bmatrix}.$$

◊

(11.04) Proposition

Set $\{\boldsymbol{E}_0, \boldsymbol{E}_1, ..., \boldsymbol{E}_d\}$ is an orthogonal basis of adjacency algebra $\mathcal{A}(\Gamma)$.

Proof: By Proposition 4.05 we have that $\mathcal{A} = \text{span}\{\boldsymbol{E}_0, \boldsymbol{E}_1, ..., \boldsymbol{E}_d\}$. We have seen that $\boldsymbol{E}_i\boldsymbol{E}_j = \delta_{ij}\boldsymbol{E}_i$ (Proposition 5.02), and since orthogonal set is linearly independent (Proposition 5.03), the result follow. □

(11.05) Proposition

Let $\Gamma = (V, E)$ denote a simple graph with adjacency matrix $\boldsymbol{A}$ and with $d+1$ distinct eigenvalues. Principal idempotents of Γ satisfy the following equation

$$\boldsymbol{E}_i = \frac{1}{\phi_i}\prod_{\substack{j=0 \\ j\neq i}}^{d}(\boldsymbol{A} - \lambda_j I), \quad (0 \leq i \leq d)$$

where $\phi_i = \prod_{j=0(j\neq i)}^{d}(\lambda_i - \lambda_j)$.

Proof: We know that for a set of m points $\mathcal{S} = \{(x_1, y_1), (x_2, y_2), ..., (x_m, y_m)\}$ there is unique polynomial

$$p(x) = \sum_{i=1}^{m} \left(y_i \frac{\prod\limits_{\substack{j=1\\j\neq i}}^{m} (x - x_j)}{\prod\limits_{\substack{j=1\\j\neq i}}^{m} (x_i - x_j)} \right)$$

of degree $m-1$ which pass through every point in $\mathcal{S}$. This polynomial is known as <u>*Lagrange interpolation polynomial*</u>. Let $\sigma(\boldsymbol{A}) = \{\lambda_0, \lambda_1, ..., \lambda_d\}$ be a set of all, distinct, eigenvalues of $\boldsymbol{A}$, and let $f(x)$ be function which has finite value on $\sigma(\boldsymbol{A})$. Consider set $\mathcal{S}_1 = \{(\lambda_0, f(\lambda_0)), (\lambda_1, f(\lambda_1)), ..., (\lambda_d, f(\lambda_d))\}$. Lagrange interpolation polynomial for $\mathcal{S}_1$ is

$$p(x) = \sum_{i=0}^{d} \left(f(\lambda_i) \frac{\prod\limits_{\substack{j=0\\j\neq i}}^{d} (x - \lambda_j)}{\prod\limits_{\substack{j=0\\j\neq i}}^{d} (\lambda_i - \lambda_j)} \right) = \sum_{i=0}^{d} \left(\frac{1}{\phi_i} f(\lambda_i) \prod_{\substack{j=0\\j\neq i}}^{d} (x - \lambda_j) \right).$$

In notation for matrix $\boldsymbol{A}$ this mean

$$p(\boldsymbol{A}) = \sum_{i=0}^{d} \left(f(\lambda_i) \frac{\prod\limits_{\substack{j=0\\j\neq i}}^{d} (\boldsymbol{A} - \lambda_j I)}{\prod\limits_{\substack{j=0\\j\neq i}}^{d} (\lambda_i - \lambda_j)} \right) = \sum_{i=0}^{d} \left(\frac{1}{\phi_i} f(\lambda_i) \prod_{\substack{j=0\\j\neq i}}^{d} (\boldsymbol{A} - \lambda_j I) \right).$$

By Lemma 4.04, we know that $f(\boldsymbol{A}) = f(\lambda_0)\boldsymbol{E}_0 + f(\lambda_1)\boldsymbol{E}_1 + ... + f(\lambda_d)\boldsymbol{E}_d$. If for function f above we pick

$$g_i(x) = \begin{cases} 1, & \text{if } x = \lambda_i \\ 0, & \text{if } x \neq \lambda_i \end{cases},$$

we have

$$p(\boldsymbol{A}) = \boldsymbol{E}_i \quad \text{and} \quad p(\boldsymbol{A}) = \frac{\prod\limits_{\substack{j=0\\j\neq i}}^{d} (\boldsymbol{A} - \lambda_j I)}{\prod\limits_{\substack{j=0\\j\neq i}}^{d} (\lambda_i - \lambda_j)} = \frac{1}{\phi_i} \prod_{\substack{j=0\\j\neq i}}^{d} (\boldsymbol{A} - \lambda_j I),$$

and the result follows. □

(11.06) Theorem

Principal idempotents of Γ *represents the orthogonal projectors onto* $\mathcal{E}_i = \ker(\boldsymbol{A} - \lambda_i I)$ *(along* $\operatorname{im}(\boldsymbol{A} - \lambda_i I)$*).*

Proof: First recall some basic definitions from Linear algebra. Subspaces $\mathcal{X}$, $\mathcal{Y}$ of a space $\mathcal{V}$ are said to be <u>*complementary*</u> whenever

$$\mathcal{V} = \mathcal{X} + \mathcal{Y} \quad \text{and} \quad \mathcal{X} \cap Y = \{\mathbf{0}\},$$

in which case $\mathcal{V}$ is said to be the <u>*direct sum*</u> of $\mathcal{X}$ and $\mathcal{Y}$, and this is denoted by writing $\mathcal{V} = \mathcal{X} \oplus \mathcal{Y}$. This is equivalent to saying that for each $v \in \mathcal{V}$ there are unique vectors $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ such that $v = x + y$. Vector x is called the <u>*projection*</u> of v onto $\mathcal{X}$ along $\mathcal{Y}$. Vector

y is called the projection of v onto $\mathcal{Y}$ along $\mathcal{X}$. Operator P defined by $Pv = x$ is unique linear operator with property $Pv = x$ ($v = x + y$, $x \in \mathcal{X}$ and $y \in \mathcal{Y}$) and is called the *projector* onto $\mathcal{X}$ along $\mathcal{Y}$. Vector m is called the *orthogonal projection* of v onto $\mathcal{M}$ if and only if $v = m + n$ where $m \in \mathcal{M}$, $n \in \mathcal{M}^{\perp}$ and $\mathcal{M} \subseteq \mathcal{V}$. The projector $P_{\mathcal{M}}$ onto $\mathcal{M}$ along $\mathcal{M}^{\perp}$ is called the *orthogonal projector* onto $\mathcal{M}$.

Pick arbitrary principal idempotent $\boldsymbol{E}_i$ of Γ. The proof that $\boldsymbol{E}_i$ is projector rests on the fact that

$$\boldsymbol{E}_i^2 = \boldsymbol{E}_i \quad \Longrightarrow \quad \mathrm{im}(\boldsymbol{E}_i) \text{ and } \ker(\boldsymbol{E}_i) \text{ are complementary subspaces.} \tag{23}$$

To prove this, observe that $\mathbb{R}^n = \mathrm{im}(\boldsymbol{E}_i) + \ker(\boldsymbol{E}_i)$ because for each $v \in \mathbb{R}^n$,

$$v = \boldsymbol{E}_i v + (I - \boldsymbol{E}_i)v, \text{ where } \boldsymbol{E}_i v \in \mathrm{im}(\boldsymbol{E}_i) \text{ and } (I - \boldsymbol{E}_i)v \in \ker(\boldsymbol{E}_i) \tag{24}$$

(and $(I - \boldsymbol{E}_i)v$ is in $\ker(\boldsymbol{E}_i)$ becouse $\boldsymbol{E}_i((I - \boldsymbol{E}_i)v) = (\boldsymbol{E}_i - \boldsymbol{E}_i^2)v = (\boldsymbol{E}_i - \boldsymbol{E}_i)v = \boldsymbol{0}$). Furthermore, $\mathrm{im}(\boldsymbol{E}_i) \cap \ker(\boldsymbol{E}_i) = \{\boldsymbol{0}\}$ because

$$x \in \mathrm{im}(\boldsymbol{E}_i) \cap \ker(\boldsymbol{E}_i) \quad \Longrightarrow \quad x = \boldsymbol{E}_i v \text{ and } \boldsymbol{E}_i x = \boldsymbol{0} \quad \Longrightarrow \quad x = \boldsymbol{E}_i v = \boldsymbol{E}_i^2 v = \boldsymbol{E}_i x = \boldsymbol{0},$$

and thus (23) is established. Now since we know $\mathrm{im}(\boldsymbol{E}_i)$ and $\ker(\boldsymbol{E}_i)$ are complementary, we can conclude that $\boldsymbol{E}_i$ is a projector because each $v \in V$ can be uniquely written as $v = x + y$, where $x \in \mathrm{im}(\boldsymbol{E}_i)$ and $y \in \ker(\boldsymbol{E}_i)$, and (24) guarantees $\boldsymbol{E}_i v = x$.

With this we had showed that $\boldsymbol{E}_i$ is projector on $\mathrm{im}(\boldsymbol{E}_i)$ and that

$$\mathbb{R}^n = \mathrm{im}(\boldsymbol{E}_i) \oplus \ker(\boldsymbol{E}_i).$$

Now notice that

$$x \in \mathrm{im}(\boldsymbol{E}_i)^{\perp} \;\Leftrightarrow\; \langle \boldsymbol{E}_i y, x \rangle = 0 \;\Leftrightarrow\; y^{\top} \boldsymbol{E}_i^{\top} x = 0 \;\Leftrightarrow\; \langle y, \boldsymbol{E}_i^{\top} x \rangle = 0 \;\overset{\forall y \in \mathbb{R}^n}{\Leftrightarrow}\; \boldsymbol{E}_i^{\top} x = \boldsymbol{0} \;\Leftrightarrow\; x \in \ker(\boldsymbol{E}_i^{\top})$$

and this holds for every y in $\mathbb{R}^n$ that is

$$\mathrm{im}(\boldsymbol{E}_i)^{\perp} = \ker(\boldsymbol{E}_i^{\top})$$

which is equivalent with

$$\mathrm{im}(\boldsymbol{E}_i) = \ker(\boldsymbol{E}_i^{\top})^{\perp}.$$

Since $\boldsymbol{E}_i^{\top} = (U_i U_i^{\top})^{\top} = U_i U_i^{\top} = \boldsymbol{E}_i$ we have that

$$\mathrm{im}(\boldsymbol{E}_i) = \ker(\boldsymbol{E}_i)^{\perp}.$$

But $\boldsymbol{E}_i$ must be an orthogonal projector because last equation allows us to write

$$\boldsymbol{E}_i = \boldsymbol{E}_i^{\top} \iff \mathrm{im}(\boldsymbol{E}_i) = \mathrm{im}(\boldsymbol{E}_i^{\top}) \iff \mathrm{im}(\boldsymbol{E}_i) = \ker(\boldsymbol{E}_i)^{\perp} \iff \mathrm{im}(\boldsymbol{E}_i) \perp \ker(\boldsymbol{E}_i).$$

And we obtained that

$$\mathbb{R}^n = \mathrm{im}(\boldsymbol{E}_i) \oplus \ker(\boldsymbol{E}_i) = \ker(\boldsymbol{E}_i)^{\perp} \oplus \ker(\boldsymbol{E}_i) = \mathrm{im}(\boldsymbol{E}_i) \oplus \mathrm{im}(\boldsymbol{E}_i)^{\perp}.$$

It is only left to show that $\mathrm{im}(\boldsymbol{E}_i) = \ker(\boldsymbol{A} - \lambda_i I)$ and that $\ker(\boldsymbol{E}_i) = \mathrm{im}(\boldsymbol{A} - \lambda_i I)$. To establish that $\mathrm{im}(\boldsymbol{E}_i) = \ker(\boldsymbol{A} - \lambda_i I)$, use $\mathrm{im}(AB) \subseteq \mathrm{im}(A)$ and $U_i^{\top} U_i = I$ to write

$$\mathrm{im}(\boldsymbol{E}_i) = \mathrm{im}(U_i U_i^{\top}) \subseteq \mathrm{im}(U_i) = \mathrm{im}(U_i U_i^{\top} U_i) = \mathrm{im}(\boldsymbol{E}_i U_i) \subseteq \mathrm{im}(\boldsymbol{E}_i).$$

Thus

$$\mathrm{im}(\boldsymbol{E}_i) = \mathrm{im}(U_i) = \ker(\boldsymbol{A} - \lambda_i I).$$

To show $\ker(\boldsymbol{E}_i) = \operatorname{im}(A - \lambda_i I)$, use $\boldsymbol{A} = \sum_{j=1}^{k} \lambda_j \boldsymbol{E}_j$ with the already established properties of the $\boldsymbol{E}_i$'s to conclude

$$\boldsymbol{E}_i(\boldsymbol{A} - \lambda_i I) = \boldsymbol{E}_i \left(\sum_{j=1}^{k} \lambda_j \boldsymbol{E}_j - \lambda_i \sum_{j=1}^{k} \boldsymbol{E}_j \right) = 0 \;\Rightarrow\; \operatorname{im}(\boldsymbol{A} - \lambda_i I) \subseteq \ker(\boldsymbol{E}_i).$$

But we already know that $\ker(\boldsymbol{A} - \lambda_i I) = \operatorname{im}(E_i)$, so

$$\dim \operatorname{im}(\boldsymbol{A} - \lambda_i I) = n - \dim \ker(\boldsymbol{A} - \lambda_i I) = n - \dim \operatorname{im}(\boldsymbol{E}_i) = \dim \ker(\boldsymbol{E}_i),$$

and therefore,

$$\operatorname{im}(\boldsymbol{A} - \lambda_i I) = \ker(E_i).$$

Therefore, $\boldsymbol{E}_i$ is orthogonal projector onto $\mathcal{E}_i$ (along $\operatorname{im}(\boldsymbol{A} - \lambda_i I)$). □

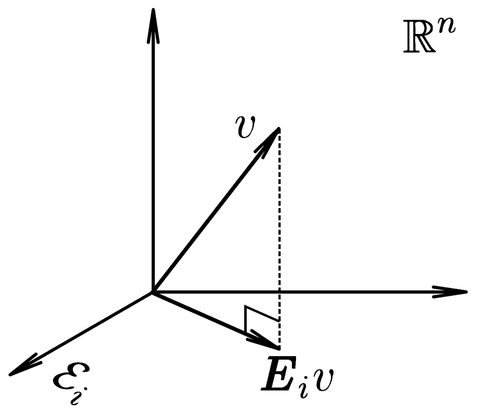


FIGURE 44
$\boldsymbol{E}_i$ projects on the λ_i-eigenspace $\mathcal{E}_i$.

(11.07) Definition (predistance polynomials)

Let $\Gamma = (V, E)$ be a simple connected graph with $|V| = n$ (number of vertices is n). The *predistance polynomials* p_0, p_1, ..., p_d, $\operatorname{dgr} p_i = i$, associated with a given graph Γ with spectrum $\operatorname{spec}(\Gamma) = \operatorname{spec}(\boldsymbol{A}) = \{\lambda_0^{m_0}, \lambda_1^{m_1}, ..., \lambda_d^{m_d}\}$, are orthogonal polynomials with respect to the scalar product

$$\langle p, q \rangle = \frac{1}{n} \operatorname{trace}(p(\boldsymbol{A}) q(\boldsymbol{A})) = \frac{1}{n} \sum_{k=0}^{d} m_k \, p(\lambda_k) q(\lambda_k)$$

on the space of all polynomials with degree at most d, normalized in such a way that $\|p_i\|^2 = p_i(\lambda_0)$. ◇

(11.08) Problem

Prove that polynomials $p_i(x)$ from Definition 11.07 exists for all $i = 0, 1, ..., d$ (so that given definition makes sense). ♢

Solution: Consider linearly independent set $\{1, x, x^2, ..., x^d\}$ of $d+1$ elements. Since we have scalar product $\langle \star, \star \rangle$ we can use Gram-Schmidt orthogonalization procedure and form orthonormal system $\{r_0, r_1, ..., r_d\}$ (because of definition Gram-Schmidt orthogonalization procedure notice that for our system $\{r_0, r_1, ..., r_d\}$ we will have $\operatorname{dgr} r_j = j$ and $\|r_j\| = 1$).

Now for arbitrary numbers $\alpha_0, \alpha_1, ..., \alpha_d$ set $\{\alpha_0 r_0, \alpha_1 r_1, ..., \alpha_d r_r\}$ is orthogonal set (because $\langle \alpha_j r_j, \alpha_i r_i \rangle = \alpha_j \alpha_i \langle r_j, r_i \rangle = 0$ for $i \neq j$). This means that if we for arbitrary r_j define $c := r_j(\lambda_0)$ and $p_j(x) := c r_j(x)$ we have

$$\|p_j\|^2 = \langle c r_j, c r_j \rangle = c^2 \|r_j\| = c \cdot c = c r_j(\lambda_0) = p_j(\lambda_0)$$

that is $\|p_j\|^2 = p_j(\lambda_0)$. Therefore, set $\{p_0, p_1, ..., p_d\}$ where $p_j(x) := r_j(\lambda_0)r_j(x)$ is orthogonal system and $\|p_j\|^2 = p_j(\lambda_0)$ for $j = 0, 1, ..., d$. ♢

(11.09) Comment

We can now observe polinomyal p_0 from Definition 11.07. Notice that dgr $(p_0) = 0$ so we can, for example, say that $p_0 = c$. Since

$$\langle p_0, p_0\rangle = \frac{1}{n}\sum_{k=0}^{d} m_k\, p_0(\lambda_k)p_0(\lambda_k) = \frac{c^2}{n}\sum_{k=0}^{d} m_k = c^2$$

and $\|p_i\|^2 = p_i(\lambda_0)$ we have that $c^2 = c$, and this is possible if and only if $c = 1$. Therefore $p_0 = 1$.

If Γ is δ-regular then

$$\langle 1, x\rangle = \tfrac{1}{n}\sum_{i=0}^{d} m_i\,\lambda_i \overset{by\,Theo.\,4.07}{=\!=\!=\!=} \text{trace}(\boldsymbol{A}) = 0,$$
$$\|1\|^2 = \tfrac{1}{n}\sum_{i=0}^{d} m_i = 1,$$
$$\|x\|^2 = \tfrac{1}{n}\sum_{i=0}^{d} m_i\lambda_i^2 \overset{by\,Theo.\,4.07}{=\!=\!=\!=} \text{trace}(\boldsymbol{A}^2) = \delta = \lambda_0.$$

It is clear from the above three lines, that if $p_1 = x$, then we have that p_1 is orthogonal to p_0 and that $\|p_1\|^2 = p_1(\lambda_0)$. ♢

(11.10) Comment

From Proposition 10.07 we see that distance polynomials of regular graph are orthogonal with respect to the scalar product $\langle p, q\rangle = \frac{1}{n}\text{trace}(p(\boldsymbol{A})q(\boldsymbol{A}))$. Since this polynomials satisfy condition $\|p_i\|^2 = p_i(\lambda_0)$ (see Exercise 10.06), we have that if distance polynomials p_i of regular graph have degree i then they are in fact predistance polynomials. ♢

(11.11) Proposition

Let $\Gamma = (V, E)$ be a simple (connected) regular graph, with spec$(\Gamma) = \{\lambda_0^{m_0}, \lambda_1^{m_1}, ..., \lambda_d^{m_d}\}$, *and let p_0, p_1, ..., p_d, be sequence of predistance polynomials. If $q_i = \sum_{j=0}^{i} p_j$ then*

$$q_d(\boldsymbol{A}) = \boldsymbol{J},$$

or in other words, q_d is the well-known Hoffman-polynomial.

Proof: To prove the claim, we first show that q_i is the (unique) polynomial p of degree i that maximizes $p(\lambda_0)$ subject to the constraint that $\langle p, p\rangle = \langle q_i, q_i\rangle$. To show this property, write a polynomial p of degree i as $p = \sum_{j=0}^{i} \alpha_j p_j$ for certain α_j (for fixed i). Then the problem reduces to maximizing $p(\lambda_0) = \sum_{j=0}^{i} \alpha_j p_j(\lambda_0)$ subject to $\sum_{j=0}^{i} \alpha_j^2 p_j(\lambda_0) = \langle q_i, q_i\rangle$, becouse

$$\langle p, p\rangle = \langle\sum_{j=0}^{i} \alpha_j p_j, \sum_{k=0}^{i} \alpha_j p_k\rangle = \sum_{j=0}^{i} \alpha_j^2\langle p_j, p_j\rangle = \sum_{j=0}^{i} \alpha_j^2 p_j(\lambda_0).$$

Notice that

$$\langle p, q_i\rangle = \langle\sum_{j=0}^{i} \alpha_j p_j, \sum_{k=0}^{i} p_k\rangle = \sum_{j=0}^{i} \alpha_j\langle p_j, p_j\rangle = \sum_{j=0}^{i} \alpha_j p_j(\lambda_0),$$

$$\langle q_i, q_i\rangle = \langle\sum_{j=0}^{i} p_j, \sum_{k=0}^{i} p_k\rangle = \sum_{j=0}^{i}\langle p_j, p_j\rangle = \sum_{j=0}^{i} p_j(\lambda_0).$$

Now problem become: find polynomial p od degree i that maximizes

$$p(\lambda_0) = \sum_{j=0}^{i} \alpha_j p_j(\lambda_0) \tag{25}$$

subject to the constraint that

$$\sum_{j=0}^{i} \alpha_j^2 p_j(\lambda_0) = \sum_{j=0}^{i} p_j(\lambda_0), \tag{26}$$

or in another words

$$\sum_{j=0}^{i} (1-\alpha_j^2) p_j(\lambda_0) = 0.$$

Since $p_j(\lambda_0) > 0$, $j = 0, 1, ..., d$ (Problem 11.08), we have that given constraint become $1 - \alpha_j^2 = 0$ for $j = 0, 1, ..., i$. Now it is not hard to see that for maximal $p(\lambda_0)$ subject to the constraint that $\langle p, p\rangle = \langle q_i, q_i\rangle$ we must have $\alpha_0 = \alpha_1 = ... = \alpha_i = 1$, and therefore q_i is the optimal p.

Same conclusion we will obtain if we consider Cauchy-Schwartz inequality $|\langle p, q_i\rangle| \le \|p\|\|q_i\|$ (with equality iff polynomials p and q_i are linearly dependent), that is $|\langle p, q_i\rangle|^2 \le \|p\|^2\|q_i\|^2$ or in another words

$$p(\lambda_0)^2 \overset{(25)}{=} \left[\sum_{j=0}^{i} \alpha_j p_j(\lambda_0)\right]^2 \le \left[\sum_{j=0}^{i} \alpha_j^2 p_j(\lambda_0)\right]\left[\sum_{j=0}^{i} p_j(\lambda_0)\right] \overset{(26)}{=} \left[\sum_{j=0}^{i} p_j(\lambda_0)\right]\left[\sum_{j=0}^{i} p_j(\lambda_0)\right] = q_i(\lambda_0)^2$$

with equality if and only if all α_j are equal to one. The constraint and the fact that $p_j(\lambda_0) > 0$ for all j guarantees that q_i is the optimal p.

On the other hand, since $\langle p, p\rangle = \frac{1}{n}p(\lambda_0)^2 + \frac{1}{n}\sum_{j=1}^{d} m_j p(\lambda_j)^2$(Definition 11.07), that is

$$\frac{1}{n}p(\lambda_0)^2 = \langle p, p\rangle - \frac{1}{n}\sum_{j=1}^{d} m_j p(\lambda_j)^2,$$

the objective of the optimization problem is clearly equivalent to minimizing $\sum_{j=1}^{d} m_j p(\lambda_j)^2$. For $i = d$, there is a trivial solution for this: take the polynomial that is zero on λ_j for all $j = 1, 2, ..., d$. Hence (since q_d is the optimal p) we may conclude that $q_d(\lambda_j) = 0$ for $j = 1, 2, ..., d$, and from the constraint it futher follows that

$$q_d(\lambda_0) = \sum_{j=1}^{d} p_j(\lambda_0) = \langle q_d, q_d\rangle = \langle p, p\rangle = \frac{1}{n}p(\lambda_0)^2 = \frac{1}{n}q_d(\lambda_0)^2$$

that is

$$q_d(\lambda_0) = n.$$

Recall that in Example 11.02 we had $\boldsymbol{E}_0 = \frac{1}{n}\boldsymbol{J}$, and from Proposition 5.02(iii) $p(\boldsymbol{A}) = \sum_{i=0}^{d} p(\lambda_i)\boldsymbol{E}_i$, so we have

$$q_d(\boldsymbol{A}) = \sum_{i=0}^{d} q_d(\lambda_i)\boldsymbol{E}_i = q_d(\lambda_0)\boldsymbol{E}_0 = \boldsymbol{J}.$$

□

(11.12) Lemma

Let p_0, p_1, ..., p_d, be sequence of predistance polynomials. If $\boldsymbol{A}_d = p_d(\boldsymbol{A})$, then $\boldsymbol{A}_i = p_i(\boldsymbol{A})$ for all $i = 0, 1, ..., d$.

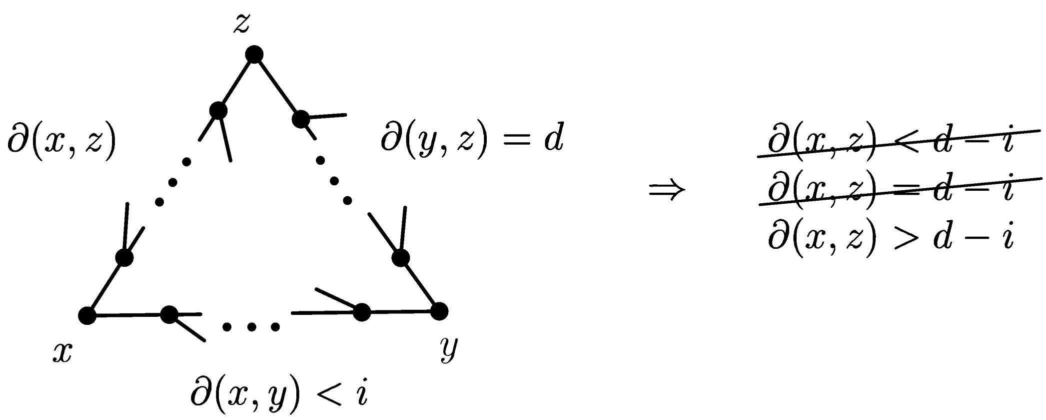


FIGURE 45
If $\partial(x, y) < i$ and $\partial(y, z) = d$ then $\partial(x, z) > d - i$.

Proof: Since p_i is a polynomial of degree i, it follows that if x and y are two vertices at distance larger than i, then $(p_i(\boldsymbol{A}))_{xy} = 0$. Suppose now that $\boldsymbol{A}_d = p_d(\boldsymbol{A})$. In Proposition 13.07, that is obtained independently of this lemma, we will see that for any orthogonal system r_0, r_1, ..., r_d we have that $r_{d-i}(x) = \overline{r}_i(x)r_d(x)$ for some polynomial $\overline{r}_i(x)$ of degree i. Since predistance polynomials form an orthogonal system, it follow $p_i(\boldsymbol{A}) = \overline{p}_{d-i}(\boldsymbol{A})\boldsymbol{A}_d$. If the distance between x and y is smaller than i, then for all vertices z at distance d from y, we have that the distance between z and x is more than $d - i$ (by the triangle inequality), hence $(\overline{p}_{d-i}(\boldsymbol{A}))_{xz} = 0$. Thus

$$(p_i(\boldsymbol{A}))_{xy} = (\overline{p}_{d-1}(\boldsymbol{A})\boldsymbol{A}_d)_{xy} = \sum_z (\overline{p}_{d-i}(\boldsymbol{A}))_{xz}(\boldsymbol{A}_d)_{zy} = 0$$

(if the second factor in sum (that is $(\boldsymbol{A}_d)_{zy}$) is non zero, then (by the previous comments) the first factor (that is $(\overline{p}_{d-i}(\boldsymbol{A}))_{xz}$) is zero). Therefore, for arbitrary x, y we have that $p_i(\boldsymbol{A})_{xy} = 0$ for $\partial(x, y) > i$ and for $\partial(x, y) < i$. Because this holds for all $i = 0, 1, ..., d$ and because $\sum_{i=0}^{d} p_i(\boldsymbol{A}) = q_d(\boldsymbol{A}) = \boldsymbol{J}$, it follows that $p_i(\boldsymbol{A}) = \boldsymbol{A}_i$ for all $i = 0, 1, ..., d$. □

(11.13) Lemma

The algebras $\mathcal{A}$ and $\mathbb{R}[x]/\langle Z\rangle$, with their respective scalar products $\langle R, S\rangle = \frac{1}{n}\mathrm{trace}(RS)$ and $\langle p, q\rangle = \frac{1}{n}\sum_{i=0}^{d} m_i p(\lambda_i)q(\lambda_i)$, are isometric (where $\lambda_0 > \lambda_1 > ... > \lambda_d$ is a mesh of real numbers and $\langle Z\rangle$ is the ideal generated by the polynomial $Z = \prod_{i=0}^{d}(x - \lambda_i)$ - much more about $\mathbb{R}[x]/\langle Z\rangle$ we will say in Section 12).

Proof: Just identify both algebras through the isometry $p = p(\boldsymbol{A})$, that is, for any $R, S \in \mathcal{A}$:

$$\langle R, S\rangle = \langle p(A), q(A)\rangle = \frac{1}{n}\mathrm{trace}(p(A)q(A)) = \frac{1}{n}\sum_{i=0}^{d} m_i p(\lambda_i)q(\lambda_i) = \langle p, q\rangle.$$

□

(11.14) Theorem

Let Γ be a regular graph and let p_0, p_1, ..., p_d be its sequence of predistance polynomials. Let $\delta_d = \|\boldsymbol{A}_d\|^2 = \frac{1}{n}\mathrm{trace}(\boldsymbol{A}_d\boldsymbol{A}_d)$. Then $\delta_d \le p_d(\lambda_0)$, and equality is attained if and only if $\boldsymbol{A}_d = p_d(\boldsymbol{A})$.

Proof: Consider vector space of matrices $\mathcal{T} = \mathcal{A} + \mathcal{D}$ (where $\mathcal{A}$ is Bose-Mesner algebra and $\mathcal{D}$ is distance $\circ$-algebra). In the regular case $I, \boldsymbol{A}$ and $\boldsymbol{J}$ are matrices in $\mathcal{A} \cap \mathcal{D}$, as $\boldsymbol{A}_0 + \boldsymbol{A}_1 + ... + \boldsymbol{A}_d = \boldsymbol{J} = H(\boldsymbol{A}) \in \mathcal{A}$. Thus we have that $\dim(\mathcal{T}) \leq d + D - 1$. Notice that we can define an scalar product into $\mathcal{T}$, in two equivalent forms

$$\langle R, S\rangle = \frac{1}{n}\text{trace}(RS) = \frac{1}{n}\sum_u (RS)_{uu} = \frac{1}{n}\sum_u\sum_v (R)_{uv}(S)_{vu} = \frac{1}{n}\sum_u\sum_v (R)_{uv}(S)_{uv} =$$

$$= \frac{1}{n}\sum_u\sum_v (R \circ S)_{uv} = \frac{1}{n}\sum_{uv} (R \circ S)_{uv}$$

Let $\{p_i\}_{0\leq i\leq d}$ be sequence of predistans polynomials. From Lemma 11.13 space $\mathcal{A}$ is isometric with $\mathbb{R}[x]/\langle Z\rangle$, so since $\{p_i\}_{0\leq i\leq d}$ is orthogonal basis for $\mathbb{R}[x]/\langle Z\rangle$ we have also that $\{R_i = p_i(A)\}_{0\leq i\leq d}$ are orthogonal basis for $\mathcal{A}$. If we use given scalar product, we can expand $\{R_i\}_{0\leq i\leq d}$ to the basis of space $\mathcal{T}$, say to $\{R_i\}_{0\leq i\leq d+D-1}$. Now arbitrary matrix $S \in \mathcal{T}$ we can write in form

$$S = \sum_{i=0}^{d+D-1} \frac{\langle S, R_i\rangle}{\|R_i\|^2} R_i = \underbrace{\sum_{i=0}^{d} \frac{\langle S, p_i(\boldsymbol{A})\rangle}{\|p_i(\boldsymbol{A})\|^2} p_i(\boldsymbol{A})}_{\in\mathcal{A}} + \underbrace{\sum_{i=d+1}^{d+D-1} \frac{\langle S, R_i\rangle}{\|R_i\|^2} R_i}_{\in\mathcal{A}^\perp}$$

Notice that for $0 \leq i \leq d-1$

$$\langle \boldsymbol{A}_d, p_i(\boldsymbol{A})\rangle = 0 \tag{27}$$

becouse p_i is of degree i, $p_i(\boldsymbol{A}) = c_i\boldsymbol{A}^i + ... + c_0 I$ for some constants c_0, ...,c_i and $(\boldsymbol{A}^\ell)_{uv}$ is number of walks of length ℓ from u to v.

Now consider the orthogonal projection

$$\mathcal{T} \longrightarrow \mathcal{A}$$

denoted by

$$S \longrightarrow \widetilde{S}.$$

Using in $\mathcal{A}$ the orthogonal base p_0, p_1, ..., p_d of predistance polynomials, this projection can be expressed as

$$\widetilde{S} = \sum_{i=0}^{d} \frac{\langle S, p_i\rangle}{\|p_i\|^2} p_i = \sum_{i=0}^{d} \frac{\langle S, p_i\rangle}{p_i(\lambda_0)} p_i.$$

Now consider the projection of $\boldsymbol{A}_d$

$$\widetilde{\boldsymbol{A}}_d = \sum_{j=0}^{d} \frac{\langle \boldsymbol{A}_d, p_j\rangle}{\|p_j\|^2} p_j \overset{(27)}{=} \frac{\langle \boldsymbol{A}_d, p_d\rangle}{\|p_d\|^2} p_d = \frac{\langle \boldsymbol{A}_d, p_0 + p_1 + ... + p_d\rangle}{\|p_d\|^2} p_d =$$

$$\overset{\text{Prop. 11.11}}{=\!=\!=} \frac{\langle \boldsymbol{A}_d, H\rangle}{\|p_d\|^2} p_d = \frac{\langle \boldsymbol{A}_d, \boldsymbol{A}_d\rangle}{\|p_d\|^2} p_d = \frac{\delta_d}{p_d(\lambda_0)} p_d,$$

where

$$\delta_d = \frac{1}{n}\text{trace}(\boldsymbol{A}_d\boldsymbol{A}_d) = \frac{1}{n}\sum_u (\boldsymbol{A}_d\boldsymbol{A}_d)_{uu} = \frac{1}{n}\sum_u \underbrace{\sum_v (\boldsymbol{A}_d)_{uv}(\boldsymbol{A}_d)_{vu}}_{|\Gamma_d(u)\cap\Gamma_d(u)|} = \frac{1}{n}\sum_{u\in V} |\Gamma_d(u)|$$

and $\Gamma_d(u)$ is the set of vertices at distance d from u.

Consider the equality $\boldsymbol{A}_d = \widetilde{\boldsymbol{A}}_d + N$, with $N \in \mathcal{A}^{\perp}$. Combining both Pitagoras Theorem and equation $\widetilde{\boldsymbol{A}}_d = \frac{\delta_d}{p_d(\lambda_0)} p_d$ obtained above, we obtain

$$\|N\|^2 = \|\boldsymbol{A}_d\|^2 - \|\widetilde{\boldsymbol{A}}_d\|^2 = \delta_d - \frac{\delta_d^2}{p_d(\lambda_0)} = \delta_d \left(1 - \frac{\delta_d}{p_d(\lambda_0)}\right).$$

This implies the inequality. Moreover, equality is attained if and only if N is zero ($\delta_d = p_d(\lambda_0)$ $\Leftrightarrow N = \{\boldsymbol{0}\} \Leftrightarrow \widetilde{\boldsymbol{A}}_d = \boldsymbol{A}_d \Leftrightarrow \boldsymbol{A}_d \in \mathcal{A} \Leftrightarrow p_d(\boldsymbol{A}) = \boldsymbol{A}_d$). □

We point out that the relation $\delta_d \le p_d(\lambda_0)$ holds for any graph. Now we can prove one characterization that is very similar to one given in Theorem 10.09 from Section 10.

(11.15) Theorem (characterization D')

A graph $\Gamma = (V, E)$ with diameter D and $d+1$ distinct eigenvalues is distance-regular if and only if Γ is regular, has spectrally maximum diameter ($D = d$) and the matrix $\boldsymbol{A}_D$ is polynomial in $\boldsymbol{A}$.

Proof: Let Γ_k be the graph with the same vertex set as Γ and where two vertices are adjacent whenever they are at distance k in Γ. Then, for example $\boldsymbol{A}_d$ is adjacency matrix for Γ_d. For matrix $\boldsymbol{A}$ we know that $\boldsymbol{A}\boldsymbol{j} = \lambda_0 \boldsymbol{j}$ so that

$$\boldsymbol{A}^k \boldsymbol{j} = \lambda_0^k \boldsymbol{j} \quad \text{for any} \quad k \in \mathbb{N}.$$

If $\boldsymbol{A}_d = q(\boldsymbol{A})$ we have $\boldsymbol{A}_d \boldsymbol{j} = q(\boldsymbol{A})\boldsymbol{j} = q(\lambda_0)\boldsymbol{j}$ and this is possible if and only if Γ_d is a regular graph of degree $q(\lambda_0)$. Next, notice that $\delta_d = q(\lambda_0)$ because

$$\delta_d = \|A_d\|^2 = \langle \boldsymbol{A}_d, \boldsymbol{A}_d \rangle = \frac{1}{n} \sum_{u \in V} |\Gamma_d(u)| = \frac{1}{n} \sum_{u \in V} q(\lambda_0) = q(\lambda_0).$$

It is clear that q has degree d, and since

$$\langle q, r \rangle = \frac{1}{n} \sum_{i=0}^{d} m_i q(\lambda_i) r(\lambda_i) = \frac{1}{n} \text{trace}(q(\boldsymbol{A}) r(\boldsymbol{A})) = \langle \boldsymbol{A}_d, r(\boldsymbol{A}) \rangle = 0$$

for every $r \in \mathbb{R}_{d-1}[x]$, we have

$$q = \sum_{i=0}^{d} \frac{\langle q, p_i \rangle}{\|p_i\|^2} p_i = \frac{\langle q, p_d \rangle}{\|p_d\|^2} p_d, \tag{28}$$

that is

$$p_d(x) = c q(x)$$

where $c = \frac{\|p_d\|^2}{\langle q, p_d \rangle}$. Let us prove that $q = p_d$. Indeed,

$$\|q\|^2 = \langle q, q \rangle = \frac{1}{n} \sum_{i=0}^{d} m_i q(\lambda_i) q(\lambda_i) = \frac{1}{n} \text{trace}(q(\boldsymbol{A}) q(\boldsymbol{A})) = \langle \boldsymbol{A}_d, \boldsymbol{A}_d \rangle = \delta_d = q(\lambda_0),$$

and because of equation (28)

$$q(\lambda_0) = \frac{\langle q, p_d \rangle}{\|p_d\|^2} p_d(\lambda_0) = \langle q, p_d \rangle = \langle q, cq \rangle$$

that is

$$\langle q, q \rangle = c \langle q, q \rangle \quad \Rightarrow \quad c = 1.$$

Therefore $q = p_d$. Now result follow from Lemma 11.12 and Theorem 10.08 (characterization D).

Proof in opposite direction is trivial. □

(11.16) Theorem (characterization F)

Let Γ be a graph with diameter D, adjacency matrix $\boldsymbol{A}$ and $d+1$ distinct eigenvalues $\lambda_0 > \lambda_1 > ... > \lambda_d$. Let $\boldsymbol{A}_i$, $i = 0,1,...,D$, be the distance-i matrices of Γ, $\boldsymbol{E}_j$, $j = 0,1,...,d$, be the principal idempotents of Γ, let p_{ji}, $i = 0,1,...,D$, $j = 0,1,...,d$, be constants and p_j, $j = 0,1,...,d$, be the predistance polynomials. Finally, let $\mathcal{A}$ be the adjacency algebra of Γ, and $d = D$. Then

$$\begin{aligned}
\Gamma \text{ distance-regular} &\iff \boldsymbol{A}_i\boldsymbol{E}_j = p_{ji}\boldsymbol{E}_j, \quad i,j = 0,1,...,d(=D),\\
&\iff \boldsymbol{A}_i = \sum_{j=0}^{d} p_{ji}\boldsymbol{E}_j, \quad i = 0,1,...,d(=D),\\
&\iff \boldsymbol{A}_i = \sum_{j=0}^{d} p_i(\lambda_j)\boldsymbol{E}_j, \quad i = 0,1,...,d(=D),\\
&\iff \boldsymbol{A}_i \in \mathcal{A}, \quad i = 0,1,...,d(=D).
\end{aligned}$$

Proof: We will prove that: Γ distance-regular $\Rightarrow \boldsymbol{A}_i\boldsymbol{E}_j = p_{ji}\boldsymbol{E}_j \Rightarrow \boldsymbol{A}_i = \sum_{j=0}^{d} p_{ji}\boldsymbol{E}_j \Rightarrow$ $\boldsymbol{A}_i \in \mathcal{A} \Rightarrow \Gamma$ distance-regular. And, that Γ distance-regular $\Leftrightarrow \boldsymbol{A}_i = \sum_{j=0}^{d} p_i(\lambda_j)\boldsymbol{E}_j$.

(Γ distance-regular $\Rightarrow \boldsymbol{A}_i\boldsymbol{E}_j = p_{ji}\boldsymbol{E}_j$). We will prove this by mathematical induction.

BASIS OF INDUCTION

Pick arbitrary $\boldsymbol{E}_j$ for some $j = 0,1,...,d$. Since $\boldsymbol{A}_0 = I$ we have $\boldsymbol{A}_0\boldsymbol{E}_j = \boldsymbol{E}_j = 1\boldsymbol{E}_j$. Therefore $p_{j0} = 1$ for $j = 0,1,...,d$. If we consider product $\boldsymbol{A}\boldsymbol{E}_j$ we have

$$\boldsymbol{A}\boldsymbol{E}_j = \boldsymbol{A}U_jU_j^\top = \boldsymbol{A}\begin{bmatrix} | & | & & | \\ u_{j1} & u_{j2} & ... & u_{jk_j} \\ | & | & & | \end{bmatrix} U_j^\top = \begin{bmatrix} | & | & & | \\ \boldsymbol{A}u_{j1} & \boldsymbol{A}u_{j2} & ... & \boldsymbol{A}u_{jk_j} \\ | & | & & | \end{bmatrix} U_j^\top =$$

$$= \begin{bmatrix} | & | & & | \\ \lambda_j u_{j1} & \lambda_j u_{j2} & ... & \lambda_j u_{jk_j} \\ | & | & & | \end{bmatrix} U_j^\top = \lambda_j U_j U_j^\top = \lambda_j \boldsymbol{E}_j.$$

Therefore, $p_{j1} = \lambda_j$ for $j = 0,1,...,d$.

INDUCTION STEP

Assume that for any $\boldsymbol{E}_j$ there exist some p_{ji} such that $\boldsymbol{A}_i\boldsymbol{E}_j = p_{ji}\boldsymbol{E}_j$ for $i = 0,1,...,k < D$. We want to use this assumption and to prove that exist some $p_{j,k+1}$ such that $\boldsymbol{A}_{k+1}\boldsymbol{E}_j = p_{j,k+1}\boldsymbol{E}_j$ for $j = 0,1,...,d$.

In Theorem 8.02 we have shown that for arbitrary graph $\Gamma = (V,E)$ which is distance-regular around each of its vertices and with the same intersection array, the distance-i matrices of Γ satisfies

$$\boldsymbol{A}\boldsymbol{A}_i = b_{i-1}\boldsymbol{A}_{i-1} + a_i\boldsymbol{A}_i + c_{i+1}\boldsymbol{A}_{i+1}, \quad 0 \le i \le D$$

for some a_i, b_i and c_i. If we choose k for i, we can multiply this equation from the right side by $\boldsymbol{E}_j$, and get

$$\boldsymbol{A}\boldsymbol{A}_k\boldsymbol{E}_j = b_{k-1}\boldsymbol{A}_{k-1}\boldsymbol{E}_j + a_k\boldsymbol{A}_k\boldsymbol{E}_j + c_{k+1}\boldsymbol{A}_{k+1}\boldsymbol{E}_j.$$

Since, by assumption $\boldsymbol{A}_k\boldsymbol{E}_j = p_{jk}\boldsymbol{E}_j$ and $\boldsymbol{A}_{k-1}\boldsymbol{E}_j = p_{j,k-1}\boldsymbol{E}_j$ we have

$$\boldsymbol{A}p_{jk}\boldsymbol{E}_j = b_{k-1}p_{j,k-1}\boldsymbol{E}_j + a_kp_{jk}\boldsymbol{E}_j + c_{k+1}\boldsymbol{A}_{k+1}\boldsymbol{E}_j,$$

that is

$$p_{jk}p_{j1}\boldsymbol{E}_j = p_{j,k-1}b_{k-1}\boldsymbol{E}_j + p_{jk}a_k\boldsymbol{E}_j + c_{k+1}\boldsymbol{A}_{k+1}\boldsymbol{E}_j.$$

Now it is not hard to see that

$$\boldsymbol{A}_{k+1}\boldsymbol{E}_j = \underbrace{\frac{1}{c_{k+1}}(p_{jk}p_{j1} - p_{j,k-1}b_{k-1} - p_{jk}a_k)}_{p_{j,k+1}}\boldsymbol{E}_j.$$

The result follows.

$(\boldsymbol{A}_i\boldsymbol{E}_j = p_{ji}\boldsymbol{E}_j \Rightarrow \boldsymbol{A}_i = \sum_{j=0}^{d} p_{ji}\boldsymbol{E}_j)$. First recall that $\boldsymbol{E}_0 + \boldsymbol{E}_1 + ... + \boldsymbol{E}_d = I$ (Proposition 5.02(iv)). For any i we have

$$\boldsymbol{A}_i = \boldsymbol{A}_i I = \boldsymbol{A}_i(\boldsymbol{E}_0+\boldsymbol{E}_1+...+\boldsymbol{E}_d) = \boldsymbol{A}_i\boldsymbol{E}_0+\boldsymbol{A}_i\boldsymbol{E}_1+...+\boldsymbol{A}_i\boldsymbol{E}_d = p_{0i}\boldsymbol{E}_0+p_{1i}\boldsymbol{E}_1+...+p_{di}\boldsymbol{E}_d = \sum_{j=0}^{d} p_{ji}\boldsymbol{E}_j.$$

The result follows.

$(\boldsymbol{A}_i = \sum_{j=0}^{d} p_{ji}\boldsymbol{E}_j \Rightarrow \boldsymbol{A}_i \in \mathcal{A})$. Since $\{\boldsymbol{E}_0, \boldsymbol{E}_1, ..., \boldsymbol{E}_d\}$ is an orthogonal basis of adjacency algebra $\mathcal{A}(\Gamma)$ (Proposition 11.04) and since any $\boldsymbol{A}_i$ $(i = 0, 1, ..., d)$ we can write like $\boldsymbol{A}_i = \sum_{j=0}^{d} p_{ji}\boldsymbol{E}_j$, the result follows.

$(\boldsymbol{A}_i \in \mathcal{A} \Rightarrow \Gamma$ distance-regular$)$. Since $\{I, \boldsymbol{A}, \boldsymbol{A}_2, ..., \boldsymbol{A}_D\}$ is linearly independent set, $\dim(\mathcal{A}) = d = D$ and $\boldsymbol{A}_i \in \mathcal{A}$ we have that $\{I, \boldsymbol{A}, \boldsymbol{A}_2, ..., \boldsymbol{A}_D\}$ is basis of the $\mathcal{A}$ and result follow from Theorem 8.22 (characterization C).

$(\Gamma$ distance-regular $\Leftrightarrow \boldsymbol{A}_i = \sum_{j=0}^{d} p_i(\lambda_j)\boldsymbol{E}_j,)$. Assume that Γ is distance-regular graph, and let U_i be matrix which columns are orthonormal basis for $\ker(\boldsymbol{A} - \lambda_i I)$ (see proof of Lemma 4.04). For this direction we will use mathematical induction.

BASIS OF INDUCTION

Let $P = [U_0|U_1|...|U_d]$. Notice that we have

$$\boldsymbol{A}_0 = I = \boldsymbol{E}_1 + \boldsymbol{E}_2 + ... + \boldsymbol{E}_d$$

and from this it follow $p_0(x) = 1$. Also, we have

$$\boldsymbol{A} = PDP^\top = [U_0|U_1|...|U_d]\begin{bmatrix}\lambda_0 I & 0 & ... & 0\\ 0 & \lambda_1 I & ... & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & ... & \lambda_d I\end{bmatrix}\begin{bmatrix}U_0^\top\\ \hline U_1^\top\\ \hline \vdots\\ \hline U_d^\top\end{bmatrix} = \lambda_0\boldsymbol{E}_0 + \lambda_1\boldsymbol{E}_1 + ... + \lambda_d\boldsymbol{E}_d$$

and it follow $p_1(x) = x$.

INDUCTION STEP

Assume that $\boldsymbol{A}_i = \sum_{j=0}^{d} p_i(\lambda_j)\boldsymbol{E}_j$, for $i = 1, 2, .., k$ $(k < D)$ and use this assumption to show that there exist polynomial $p_{k+1}(x)$ of degree $k+1$ such that

$$\boldsymbol{A}_{k+1} = \sum_{j=0}^{d} p_{k+1}(\lambda_j)\boldsymbol{E}_j.$$

We know that distance-i matrices of distance-regular graph satisfies three term recurrence $\boldsymbol{A}\boldsymbol{A}_k = b_{k-1}\boldsymbol{A}_{k-1} + a_k\boldsymbol{A}_k + c_{k+1}\boldsymbol{A}_{k+1}$ for some constants a_k, b_k and c_k (Theorem 8.02). Since

$$c_{k+1}\boldsymbol{A}_{k+1} = \boldsymbol{A}\boldsymbol{A}_k - b_{k-1}\boldsymbol{A}_{k-1} - a_k\boldsymbol{A}_k =$$

$$= \left(\sum_{j=0}^{d} \lambda_j \boldsymbol{E}_j\right)\left(\sum_{j=0}^{d} p_k(\lambda_j)\boldsymbol{E}_j\right) - b_{k-1}\left(\sum_{j=0}^{d} p_{k-1}(\lambda_j)\boldsymbol{E}_j\right) - a_k\left(\sum_{j=0}^{d} p_k(\lambda_j)\boldsymbol{E}_j\right)$$

$$= \left(\sum_{j=0}^{d} \lambda_j p_k(\lambda_j)\boldsymbol{E}_j\right) + \sum_{j=0}^{d}(-b_{k-1}p_{k-1}(\lambda_j) - a_k p_k(\lambda_j))\boldsymbol{E}_j$$

that is

$$\boldsymbol{A}_{k+1} = \frac{1}{c_{k+1}}\sum_{j=0}^{d}\Big(\lambda_j p_k(\lambda_j) - b_{k-1}p_{k-1}(\lambda_j) - a_k p_k(\lambda_j)\Big)\boldsymbol{E}_j,$$

and from this it is not hard to see that for polynomial

$$p_{k+1}(x) = \frac{1}{c_{k+1}}\Big(xp_k(x) - b_{k-1}p_{k-1}(x) - a_k p_k(x)\Big),$$

of degree $k+1$ we have $\boldsymbol{A}_{k+1} = \sum_{j=0}^{d} p_{k+1}(\lambda_j)\boldsymbol{E}_j$ for $j = 0, 1, ..., d$.

But now, polynomials p_0, p_1, ..., p_{k+1} satisfy $xp_k = b_{k-1}p_{k-1} + a_k p_k + c_{k+1}p_{k+1}$ and since Γ is distance-regular (around every vertex) from Proposition 8.04 we have $p_i(\boldsymbol{A}) = \boldsymbol{A}_i$, $i = 0, 1, ..., k+1$. From Exercise 10.06 and Proposition 10.07 it follows that obtained polynomial are predistance polynomial and the result follows.

Conversely, suppose that $\boldsymbol{A}_i = \sum_{j=0}^{d} p_i(\lambda_j)\boldsymbol{E}_j$ for predistance polynomials p_j, $j = 0, 1, ..., d$. Immediately we see that $\boldsymbol{A}_i \in \mathcal{A}$ for $i = 0, 1, ..., D$ and since $d = D$ the result follow. □

(11.17) Proposition (characterization G)

A graph Γ with diameter D and $d+1$ distinct eigenvalues is a distance-regular graph if and only if for every $0 \le i \le d$ and for every pair of vertices u, v of Γ, the (u,v)-entry of $\boldsymbol{E}_i$ depends only on the distance between u and v.

Proof: $(\Rightarrow)$ Suppose that Γ is a distance-regular graph, so that it has spectrally maximum diameter $D = d$ (Theorem 8.22 (characterization C) and Proposition 5.04). We know that

$$p(\boldsymbol{A}) = \sum_{i=0}^{d} p(\lambda_i)\boldsymbol{E}_i,$$

for every polynomial $p \in \mathbb{R}[x]$, where $\lambda_i \in \sigma(\boldsymbol{A})$ (Proposition 5.02). Now, taking p in equation above to be the distance-polynomial p_k, $0 \le k \le d$, we get

$$\boldsymbol{A}_k = \sum_{i=0}^{d} p_k(\lambda_i)\boldsymbol{E}_i \quad (0 \le k \le d)$$

or, in matrix form,

$$\begin{pmatrix} \boldsymbol{A}_0 \\ \boldsymbol{A}_1 \\ \vdots \\ \boldsymbol{A}_d \end{pmatrix} = \underbrace{\begin{pmatrix} p_0(\lambda_0) & p_0(\lambda_1) & ... & p_0(\lambda_d) \\ p_1(\lambda_0) & p_1(\lambda_1) & ... & p_1(\lambda_d) \\ \vdots & \vdots & & \vdots \\ p_d(\lambda_0) & p_d(\lambda_1) & ... & p_d(\lambda_d) \end{pmatrix}}_{=P} \begin{pmatrix} \boldsymbol{E}_0 \\ \boldsymbol{E}_1 \\ \vdots \\ \boldsymbol{E}_d \end{pmatrix}$$

We have alredy considered matrix P in the proofe of Proposition 10.13 where we noticed that

$$P^{-1} = \frac{1}{n}\begin{pmatrix} m(\lambda_0)\frac{p_0(\lambda_0)}{k_0} & m(\lambda_0)\frac{p_1(\lambda_0)}{k_1} & ... & m(\lambda_0)\frac{p_d(\lambda_0)}{k_d} \\ m(\lambda_1)\frac{p_0(\lambda_1)}{k_0} & m(\lambda_1)\frac{p_1(\lambda_1)}{k_1} & ... & m(\lambda_1)\frac{p_d(\lambda_1)}{k_d} \\ \vdots & \vdots & & \vdots \\ m(\lambda_d)\frac{p_0(\lambda_d)}{k_0} & m(\lambda_d)\frac{p_1(\lambda_d)}{k_1} & ... & m(\lambda_d)\frac{p_d(\lambda_d)}{k_d} \end{pmatrix}$$

is the inverse of P. So

$$\begin{pmatrix} \boldsymbol{E}_0 \\ \boldsymbol{E}_1 \\ \vdots \\ \boldsymbol{E}_d \end{pmatrix} = \frac{1}{n} \begin{pmatrix} m(\lambda_0)\frac{p_0(\lambda_0)}{k_0} & m(\lambda_0)\frac{p_1(\lambda_0)}{k_1} & \dots & m(\lambda_0)\frac{p_d(\lambda_0)}{k_d} \\ m(\lambda_1)\frac{p_0(\lambda_1)}{k_0} & m(\lambda_1)\frac{p_1(\lambda_1)}{k_1} & \dots & m(\lambda_1)\frac{p_d(\lambda_1)}{k_d} \\ \vdots & \vdots & & \vdots \\ m(\lambda_d)\frac{p_0(\lambda_d)}{k_0} & m(\lambda_d)\frac{p_1(\lambda_d)}{k_1} & \dots & m(\lambda_d)\frac{p_d(\lambda_d)}{k_d} \end{pmatrix} \begin{pmatrix} \boldsymbol{A}_0 \\ \boldsymbol{A}_1 \\ \vdots \\ \boldsymbol{A}_d \end{pmatrix}.$$

Consequently,

$$\boldsymbol{E}_i = \sum_{j=0}^{d} (P^{-1})_{ij} \boldsymbol{A}_j = \frac{m(\lambda_i)}{n} \sum_{j=0}^{d} \frac{p_j(\lambda_i)}{p_j(\lambda_0)} \boldsymbol{A}_j, \quad (0 \le i \le d),$$

and, equating the corresponding (u, v) entries, it follows that for vertices u, v with $\partial(u, v) = h$, the (u, v)-entry of $\boldsymbol{E}_i$ is equal to $\frac{m(\lambda_i)p_h(\lambda_i)}{np_h(\lambda_0)}$. Therefore, the (u, v)-entry of $\boldsymbol{E}_i$ depends only on the distance between u and v.

($\Leftarrow$) Conversly, assume that for every $0 \le i \le d$ and for every pair of vertices u, v of Γ, the (u, v)-entry of $\boldsymbol{E}_i$ depends only on the distance between u and v. Then

$$\boldsymbol{E}_\ell = \sum_{j=0}^{D} q_{j\ell} \boldsymbol{A}_j \quad (0 \le \ell \le d)$$

for some constants $q_{j\ell}$. Notice that the set $\{\boldsymbol{A}_0, \boldsymbol{A}_1, ..., \boldsymbol{A}_D\}$ is linearly independent because no two vertices u, v can have two different distances from each other, so for any position (u, v) in the set of distance matrices, there is only one matrix with a one entry in that position, and all the other matrices have zero. So this set is a linearly independent set of $D + 1$ elements.

The fact that $\{\boldsymbol{E}_0, \boldsymbol{E}_1, ..., \boldsymbol{E}_d\}$ is a basis of adjacency algebra $\mathcal{A}(\Gamma)$ (Proposition 11.04), (any element of the $\mathcal{A}$ can be writen like linear combination of $\boldsymbol{E}_i$'s), since $\{I, \boldsymbol{A}, ..., \boldsymbol{A}_D\}$ is linearly independent set and since the above equation imply that every $\boldsymbol{E}_i$'s can be writen as linear combination of $\boldsymbol{A}_i$'s we have that $\{I, \boldsymbol{A}, ..., \boldsymbol{A}_D\}$ is also a basis of $\mathcal{A}$ and the result follows. □

(11.18) Theorem (characterization H)

Let $\Gamma = (V, E)$ be a graph with diameter D, $|V| = n$, adjacency matrix $\boldsymbol{A}$ and $d + 1$ distinct eigenvalues $\lambda_0 > \lambda_1 > ... > \lambda_d$. Let $\boldsymbol{A}_i$, $i = 0, 1, ..., D$, be the distance-i matrices of Γ, $\boldsymbol{E}_j$, $j = 0, 1, ..., d$, be the principal idempotents of Γ, let q_{ij}, $i = 0, 1, ..., D$, $j = 0, 1, ..., d$, be constants and p_j, $j = 0, 1, ..., d$, be the predistance polynomials. Finally, let q_j, $j = 0, 1, ..., d$ be polynomials defined by $q_i(\lambda_j) = m_j \frac{p_i(\lambda_j)}{p_i(\lambda_0)}$, $i, j = 0, 1, ..., d$, let $\mathcal{A}$ be the adjacency algebra of Γ and $\mathcal{D}$ be distance $\circ$-algebra. Then

$$\begin{aligned} \Gamma \textit{ distance-regular } &\iff \boldsymbol{E}_j \circ \boldsymbol{A}_i = q_{ij} \boldsymbol{A}_i, \quad i, j = 0, 1, ..., d(= D), \\ &\iff \boldsymbol{E}_j = \sum_{i=0}^{D} q_{ij} \boldsymbol{A}_i, \quad j = 0, 1, ..., d(= D), \\ &\iff \boldsymbol{E}_j = \frac{1}{n} \sum_{i=0}^{d} q_i(\lambda_j) \boldsymbol{A}_i, \quad j = 0, 1, ..., d(= D), \\ &\iff \boldsymbol{E}_j \in \mathcal{D}, \quad j = 0, 1, ..., d(= D). \end{aligned}$$

Proof: We will show that: Γ distance-regular $\Rightarrow \boldsymbol{E}_j \circ \boldsymbol{A}_i = q_{ij} \boldsymbol{A}_i \Rightarrow \boldsymbol{E}_j = \sum_{i=0}^{D} q_{ij} \boldsymbol{A}_i \Rightarrow \boldsymbol{E}_j \in \mathcal{D} \Rightarrow \Gamma$ distance-regular and that Γ distance-regular $\Leftrightarrow \boldsymbol{E}_j = \frac{1}{n} \sum_{i=0}^{d} q_i(\lambda_j) \boldsymbol{A}_i$.

(Γ distance-regular $\Rightarrow \boldsymbol{E}_j \circ \boldsymbol{A}_i = q_{ij} \boldsymbol{A}_i$). If graph Γ is distance-regular the by Theorem

8.22 (characterization C) we have that set $\{I, \boldsymbol{A}, ..., \boldsymbol{A}_D\}$ is basis for $\mathcal{A} = \text{span}\{I, \boldsymbol{A}, ..., \boldsymbol{A}^d\} = \text{span}\{\boldsymbol{E}_0, \boldsymbol{E}_1, ..., \boldsymbol{E}_d\}$, and therefore for evry $\boldsymbol{E}_j$ there are unique constants c_{0j}, c_{1j}, ..., c_{Dj} such that

$$\boldsymbol{E}_j = \sum_{i=0}^{D} c_{ij}\boldsymbol{A}_i.$$

So we have

$$\boldsymbol{E}_j \circ \boldsymbol{A}_i = \left(\sum_{k=0}^{D} c_{kj}\boldsymbol{A}_k\right) \circ \boldsymbol{A}_i = c_{ij}\boldsymbol{A}_i$$

Now if we define $q_{ij} := c_{ij}$ it follow $\boldsymbol{E}_j \circ \boldsymbol{A}_i = q_{ij}\boldsymbol{A}_i$.

$(\boldsymbol{E}_j \circ \boldsymbol{A}_i = q_{ij}\boldsymbol{A}_i \Rightarrow \boldsymbol{E}_j = \sum_{i=0}^{D} q_{ij}\boldsymbol{A}_i)$. For arbitrary $\boldsymbol{E}_j$ we have

$$\boldsymbol{E}_j \circ \boldsymbol{A}_0 + \boldsymbol{E}_j \circ \boldsymbol{A}_2 + ... + \boldsymbol{E}_j \circ \boldsymbol{A}_D = \boldsymbol{E}_j \circ (\boldsymbol{A}_0 + \boldsymbol{A}_1 + ... + \boldsymbol{A}_D) = \boldsymbol{E}_j \circ \boldsymbol{J} = \boldsymbol{E}_j.$$

On the other hand $\sum_{i=0}^{D} \boldsymbol{E}_j \circ \boldsymbol{A}_i = \sum_{i=0}^{D} q_{ij}\boldsymbol{A}_i$ and the result follow.

$(\boldsymbol{E}_j = \sum_{i=0}^{D} q_{ij}\boldsymbol{A}_i \Rightarrow \boldsymbol{E}_j \in \mathcal{D})$. This is trivial.

$(\boldsymbol{E}_j \in \mathcal{D} \Rightarrow \Gamma$ distance-regular$)$. Since $\{\boldsymbol{E}_0, \boldsymbol{E}_1, ..., \boldsymbol{E}_d\}$ is orthogonal basis for $\mathcal{A}$ and $\boldsymbol{E}_j = \sum_{i=0}^{D} q_{ij}\boldsymbol{A}_i$ it follow that $\mathcal{A} \subseteq \mathcal{D}$. Next, since $\{I, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^d\}$ is basis of $\mathcal{A}$, and $\{I, \boldsymbol{A}, \boldsymbol{A}^2, ..., \boldsymbol{A}^D\}$ is linearly independent set we have $\dim\mathcal{A} = d + 1 \leq D + 1 = \dim\mathcal{D}$, which imply $\mathcal{A} = \mathcal{D}$, and the result follow.

$(\Gamma$ distance-regular $\Leftrightarrow \boldsymbol{E}_j = \frac{1}{n}\sum_{i=0}^{d} q_i(\lambda_j)\boldsymbol{A}_i)$. Assume that Γ is distance-regular. From Theorem 8.22 (characterization C) we have that $D = d$, and because of Proposition 5.02(iii) we have $p(\boldsymbol{A}) = \sum_{i=0}^{d} p(\lambda_i)\boldsymbol{E}_i$. Distance polinomials of distance-regular graph are equal to predistance polynomials (see Comment 11.10) and if we for p in the above equation set distance polynomials $\{p_k\}_{0\leq k\leq d}$ we have $\boldsymbol{A}_k = \sum_{i=0}^{d} p_k(\lambda_i)\boldsymbol{E}_i$. Now we can continue like in the proof of Theorem 11.17 and obtain

$$\boldsymbol{E}_j = \frac{1}{n}\sum_{i=0}^{d} m(\lambda_j)\frac{p_i(\lambda_j)}{p_i(\lambda_0)}\boldsymbol{A}_i.$$

If we define polynomials $\{q_i\}_{0\leq i\leq d}$ by $q_i(\lambda_j) = m(\lambda_j)\frac{p_i(\lambda_j)}{p_i(\lambda_0)}$, the result follows.

Converse is trivial. □

(11.19) Theorem (characterization I)

Let Γ be a graph with diameter D, adjacency matrix $\boldsymbol{A}$ and $d+1$ distinct eigenvalues $\lambda_0 > \lambda_1 > ... > \lambda_d$. Let $\boldsymbol{A}_i$, $i = 0, 1, ..., D$, be the distance-i matrix of Γ, $\boldsymbol{E}_j$, $j = 0, 1, ..., d$, be the principal idempotents of Γ, and let $a_i^{(j)}$ and q_{ij}, $i = 0, 1, ..., D$, $j = 0, 1, ..., d$, be constants. Finally, let $\mathcal{A}$ be the adjacency algebra of Γ, $\mathcal{D}$ be distance $\circ$-algebra and $d = D$. Then

$$\begin{aligned}
\Gamma \textit{ distance-regular} &\iff \boldsymbol{A}^j \circ \boldsymbol{A}_i = a_i^{(j)}\boldsymbol{A}_i, \quad i, j = 0, 1, ..., d(= D),\\
&\iff \boldsymbol{A}^j = \sum_{i=0}^{d} a_i^{(j)}\boldsymbol{A}_i, \quad i, j = 0, 1, ..., d(= D),\\
&\iff \boldsymbol{A}^j = \sum_{i=0}^{d}\sum_{\ell=0}^{d} q_{i\ell}\lambda_\ell^j\boldsymbol{A}_i, \quad j = 0, 1, ..., d(= D),\\
&\iff \boldsymbol{A}^j \in \mathcal{D}, \quad j = 0, 1, ..., d.
\end{aligned}$$

Proof: We will show that: Γ distance-regular $\Rightarrow \boldsymbol{A}^j \circ \boldsymbol{A}_i = a_i^{(j)} \boldsymbol{A}_i \Rightarrow \boldsymbol{A}^j = \sum_{i=0}^{D} a_i^{(j)} \boldsymbol{A}_i \Rightarrow$ $\boldsymbol{A}^j \in \mathcal{D} \Rightarrow \Gamma$ distance-regular and that Γ distance-regular $\Leftrightarrow \boldsymbol{A}^j = \sum_{i=0}^{D} \sum_{\ell=0}^{d} q_{i\ell} \lambda_\ell^j \boldsymbol{A}_i$.

(Γ distance-regular $\Rightarrow \boldsymbol{A}^j \circ \boldsymbol{A}_i = a_i^{(j)} \boldsymbol{A}_i$). Since Γ is distance-regular, by Theorem 8.22 (characterization C) we have that basis for the adjacency algebra $\mathcal{A} = \text{span}\{I, \boldsymbol{A}, ..., \boldsymbol{A}^d\}$ is $\{I, \boldsymbol{A}, ..., \boldsymbol{A}_D\}$, so $d = D$. Now we have that for arbitrary $\boldsymbol{A}^j \in \mathcal{A}$ there exist unique constants c_{j0}, c_{j1}, ..., c_{jD} such that

$$\boldsymbol{A}^j = c_{j0} I + c_{j1} \boldsymbol{A} + ... + c_{jD} \boldsymbol{A}_D = \sum_{k=0}^{D} c_{jk} \boldsymbol{A}_k.$$

So

$$(\boldsymbol{A}^j \circ \boldsymbol{A}_i) = (\sum_{k=0}^{D} c_{jk} \boldsymbol{A}_k) \circ \boldsymbol{A}_i = c_{ji} \boldsymbol{A}_i$$

Therefore, if we define $a_i^{(j)} := c_{ji}$, it follow

$$\boldsymbol{A}^j \circ \boldsymbol{A}_i = a_i^{(j)} \boldsymbol{A}_i, \qquad i, j = 0, 1, ..., d(= D).$$

($\boldsymbol{A}^j \circ \boldsymbol{A}_i = a_i^{(j)} \boldsymbol{A}_i \Rightarrow \boldsymbol{A}^j = \sum_{j=0}^{d} a_i^{(j)} \boldsymbol{A}_i$). For arbitrary $\boldsymbol{A}^j$ we have

$$\sum_{i=0}^{D} (\boldsymbol{A}^j \circ \boldsymbol{A}_i) = \boldsymbol{A}^j \circ \sum_{i=0}^{D} \boldsymbol{A}_i = \boldsymbol{A}^j \circ \boldsymbol{J} = \boldsymbol{A}^j$$

and since $\boldsymbol{A}^j \circ \boldsymbol{A}_i = a_i^{(j)} \boldsymbol{A}_i$ it follow

$$\sum_{i=0}^{D} (\boldsymbol{A}^j \circ \boldsymbol{A}_i) = \sum_{i=0}^{D} a_i^{(j)} \boldsymbol{A}_i.$$

Therefore

$$\boldsymbol{A}^j = \sum_{i=0}^{D} a_i^{(j)} \boldsymbol{A}_i.$$

($\boldsymbol{A}^j = \sum_{i=0}^{d} a_i^{(j)} \boldsymbol{A}_i \Rightarrow \boldsymbol{A}^j \in \mathcal{D}$). This is trivial.

($\boldsymbol{A}^j \in \mathcal{D} \Rightarrow \Gamma$ distance-regular). We known that $\mathcal{A} = \text{span}\{I, \boldsymbol{A}, .., \boldsymbol{A}^d\}$. Since $\boldsymbol{A}^j \in \mathcal{D}$ for any j it is not hard to see that $\{I, \boldsymbol{A}, ..., \boldsymbol{A}_D\}$ is basis of the $\mathcal{A}$, and the result follow from Theorem 8.22 (characterization C).

(Γ distance-regular $\Leftrightarrow \boldsymbol{A}^j = \sum_{i=0}^{D} \sum_{\ell=0}^{d} q_{i\ell} \lambda_\ell^j \boldsymbol{A}_i$). First notice that from Proposition 4.05

$$\boldsymbol{A}^j = \sum_{\ell=0}^{d} \lambda_\ell^j \boldsymbol{E}_\ell.$$

If we denote the (u, v)-entry of $\boldsymbol{A}^j$ by $a_{uv}^{(j)}$, and (u, v)-entry of $\boldsymbol{E}_\ell$ denote by $m_{uv}(\lambda_\ell)$, previous equation imply

$$a_{uv}^{(j)} = (\boldsymbol{A}^j)_{uv} = \sum_{\ell=0}^{d} m_{uv}(\lambda_\ell) \lambda_\ell^j.$$

If Γ is distance-regular, by Theorem 11.17 (characterization G) it follows that (u,v)-entry of $\boldsymbol{E}_\ell$ depends only on the distance between u and v. Therefore, for arbitrary vertices u,v at distance $\partial(u,v)=i$, we have $m_{uv}(\lambda_\ell)=q_{i\ell}$ and

$$a_{uv}^{(j)} = (\boldsymbol{A}^j)_{uv} = \sum_{\ell=0}^{d} q_{i\ell}\lambda_\ell^j.$$

Since $\boldsymbol{A}^j = \sum_{i=0}^{D} a_i^{(j)}\boldsymbol{A}_i$ (see second equivalence above), it is not hard to see that $a_{uv}^{(j)} = a_i^{(j)}$ for vertices u,v at distance $\partial(u,v)=i$. Therefore

$$\boldsymbol{A}^j = \sum_{i=0}^{D} a_{uv}^{(j)}\boldsymbol{A}_i = \sum_{i=0}^{D}\sum_{\ell=0}^{d} q_{i\ell}\lambda_\ell^j\boldsymbol{A}_i$$

Conversely, suppose that $\boldsymbol{A}^j = \sum_{i=0}^{D}\sum_{\ell=0}^{d} q_{i\ell}\lambda_\ell^j\boldsymbol{A}_i$, that is $\boldsymbol{A}^j \in \mathcal{D}$. Now, from this, it is not hard to see that the result follows. □

(11.20) Proposition (Folklore)
The following statements are equivalent:
(i) Γ *is distance-regular,*
(ii) $\mathcal{D}$ *is an algebra with the ordinary product,*
(iii) $\mathcal{A}$ *is an algebra with the Hadamard product,*
(iv) $\mathcal{A}=\mathcal{D}$.

Proof: This proposition is now a corollary of the above characterizations. □

Chapter III

Characterization of DRG which involve the spectrum

In this section we begin by surveying some known results about orthogonal polynomials of a discrete variable. We will describe one interesting family of orthogonal polynomials: the canonical orthogonal system. We begin by presenting some notation and basic facts. In Definition 12.04 we will define the scalar product associated to $(\mathcal{M}, g)$, and in Definition 14.21 we will define canonical orthogonal system. Let we here say that for set of finite many real numbers $\mathcal{M} = \{\lambda_0, \lambda_1, ..., \lambda_d\}$ and for $g_\ell := g(\lambda_\ell)$, we define the scalar product $\langle p, q\rangle := \sum_{\ell=0}^{d} g_\ell p(\lambda_\ell)q(\lambda_\ell)$, where $p, q \in \mathbb{R}_d[x]$, and for this product we say that is associated to $(\mathcal{M}, g)$. Let us also say that sequence of polynomials $(p_k)_{0\le k\le d}$, defined with $p_0 := q_0 = 1$, $p_1 := q_1 - q_0$, $p_2 := q_2 - q_1$, ... , $p_{d-1} := q_{d-1} - q_{d-2}$, $p_d := q_d - q_{d-1} = H_0 - q_{d-1}$ will be called the canonical orthogonal system associated to $(\mathcal{M}, g)$, where q_k denote the orthogonal projection of $H_0 := \frac{1}{g_0\pi_0}\prod_{i=1}^{d}(x-\lambda_i)$ (where $\pi_0 = \prod_{i=1}^{d}(\lambda_0 - \lambda_i)$) onto $\mathbb{R}_k[x]$. Main results from this chapter are the following:

(14.22) Let r_0, r_1, ..., r_{d-1}, r_d be an orthogonal system with respect to the scalar product associated to $(\mathcal{M}, g)$. Then the following assertions are all equivalent:

(a) $(r_k)_{0\le k\le d}$ is the canonical orthogonal system associated to $(\mathcal{M}, g)$;

(b) $r_0 = 1$ and the entries of the recurrence matrix $\boldsymbol{R}$ associated to $(r_k)_{0\le k\le d}$, satisfy $a_k + b_k + c_k = \lambda_0$, for any $k = 0, 1, ..., d$;

(c) $r_0 + r_1 + ... + r_d = H_0$;

(d) $\|r_k\|^2 = r_k(\lambda_0)$ for any $k = 0, 1, ..., d$.

(K) A graph $\Gamma = (V, E)$ with predistance polynomials $\{p_k\}_{0\le k\le d}$ is distance-regular if and only if the number of vertices at distance k from every vertex $u \in V$ is

$$p_k(\lambda_0) = |\Gamma_k(u)| \quad (0 \le k \le d).$$

(J) A regular graph Γ with n vertices and predistance polynomials $\{p_k\}_{0\le k\le d}$ is distance-regular if and only if

$$q_k(\lambda_0) = \frac{n}{\sum_{u\in V}\frac{1}{s_k(u)}} \quad (0 \le k \le d),$$

where $q_k = p_0 + ... + p_k$, $s_k(u) = |N_k(u)| = |\Gamma_0(u)| + |\Gamma_1(u)| + ... + |\Gamma_k(u)|$.

(J') A regular graph Γ with n vertices and spectrum $\text{spec}(\Gamma) = \{\lambda_0^{m(\lambda_0)}, \lambda_1^{m(\lambda_1)}, ..., \lambda_d^{m(\lambda_d)}\}$ is distance-regular if and only if

$$\frac{\sum_{u\in V} n/(n-k_d(u))}{\sum_{u\in V} k_d(u)/(n-k_d(u))} = \sum_{i=0}^{d} \frac{\pi_0^2}{m(\lambda_i)\pi_i^2}.$$

where $\pi_h = \prod_{\substack{i=0\\ i\neq h}}^{d} (\lambda_h - \lambda_i)$ and $k_d(u) = |\Gamma_d(u)|$.

12 Basic facts on orthogonal polynomials of a discrete variable

Set of finite many distance real numbers $\{\alpha_1, \alpha_2, ..., \alpha_m\}$ are called a *mesh of real numbers*.

Let $\mathcal{M} := \{\lambda_0, \lambda_1, ..., \lambda_d\}$, $\lambda_0 > \lambda_1 > ... > \lambda_d$, be a mesh of real numbers and let $\langle Z\rangle = \{Z(x)q(x) : q \in \mathbb{R}[x]\}$ be ideal[1] generated by tha polynomial $Z(x) := \prod_{\ell=0}^{d}(x-\lambda_\ell)$. From Abstract algebra we know that $\mathbb{R}[x]/\langle Z\rangle$ forms a ring (known as quotient ring), where for any $\langle Z\rangle + a$, $\langle Z\rangle + b \in \mathbb{R}[x]/\langle Z\rangle$ operations addition and multiplication are defined on following way

$$(\langle Z\rangle + a) + (\langle Z\rangle + b) = \langle Z\rangle + (a+b)$$

and

$$(\langle Z\rangle + a)(\langle Z\rangle + b) = \langle Z\rangle + ab.$$

(12.01) Example

For example consider quotient ring $\mathbb{R}[x]/I$ where $I = \langle (x-1)(x+4)\rangle$ and two elements $I + x, I + x^2 + 1 \in \mathbb{R}[x]/I$. Then

(*a*)

$$(I+x) + (I+x^2+1) = I + x^2 + x + 1 = I + 1\cdot \underbrace{(x^2+3x-4)}_{(x-1)(x+4)} + (-2x+5) = I - 2x + 5$$

(notice that $I + x^2 + 1 = I + \underbrace{(x^2+3x-4)}_{(x-1)(x+4)} + (-3x+5) = I - 3x + 5$).

(*b*)

$$(I+x)\cdot(I-3x+5) = I - 3x^2 + 5x = I - 3\cdot \underbrace{(x^2+3x-4)}_{(x-1)(x+4)} + (14x-12) = I + 14x - 12.$$

Notice that we have

$$(I+x)\cdot(I+x^2+1) = I + x^3 + x = I + x\cdot \underbrace{(x^2+3x-4)}_{(x-1)(x+4)} + (-3x^2+5x) =$$

$$= I - 3x^2 + 5x = I - 3\cdot \underbrace{(x^2+3x-4)}_{(x-1)(x+4)} + 14x - 12 = I + 14x - 12.$$

◊

[1]Recall: A nonempty subset I of a ring R is said to be a (two-sided) *ideal* of R if I is a subgroup of R under addition, and if $ar \in I$ and $ra \in I$ for all $a \in I$ and all $r \in R$. With another words $rI = \{ra \,|\, a \in I\} \subseteq I$ and $Ir = \{ar \,|\, a \in I\} \subseteq I$ for all $r \in R$.

It is not hard to show that in this ring, that is in $\mathbb{R}[x]/\langle Z\rangle$, also holds

$$\alpha(pq) = (\alpha p)q = p(\alpha q)$$

for all $p, q \in \mathbb{R}[x]/\langle Z\rangle$ and all scalars α. Therefore $\mathbb{R}[x]/\langle Z\rangle$ forms quotient algebra.

Next, notice that $Z(x) = \prod_{\ell=0}^{d}(x - \lambda_\ell)$ is polynomial of degree $d + 1$. From Abstract algebra we also know that every element of $\mathbb{R}[x]/\langle Z\rangle$ can be uniquely expressed in the form

$$\langle Z\rangle + (b_0 + b_1x + ... + b_dx^d)$$

where $b_0, ..., b_d \in \mathbb{R}$, and if we set $r(x) = b_0 + b_1x + ... + b_dx^d$ then $r(x)$ is polynomial such that either $r(x) = 0$ or $\operatorname{dgr} r < \operatorname{dgr} Z$. This imply that $\mathbb{R}[x]/\langle Z\rangle$ we can identify as $\mathbb{R}_d[x]$. With this in mind it is not hard to prove following lemma.

(12.02) Lemma ($\mathbb{R}[x]/\langle Z\rangle \longleftrightarrow \mathbb{R}_d[x]$)

For any monic polynomial $Z(x) = \prod_{\ell=0}^{d}(x - \lambda_\ell)$ of degree $d + 1$ over the field $\mathbb{R}$, quotient algebra $\mathbb{R}[x]/\langle Z(x)\rangle$ is isomorphic with the algebra of polynomials modulo $Z(x)$ (the set of all polynomials with a degree smaller than that of $Z(x)$), together with polynomial addition and polynomial multiplication modulo $Z(x)$. This algebra we will conventionally denoted by $\mathbb{R}_d[x]$.

By $\mathcal{F}_\mathcal{M}$ we will denote set of all real functions defined on the mesh $\mathcal{M} = \{\lambda_0, \lambda_1, ..., \lambda_d\}$

$$\mathcal{F}_\mathcal{M} := \{f : \mathcal{M} \to \mathbb{R} \,|\, \mathcal{M} = \{\lambda_0, \lambda_1, ..., \lambda_d\},\ \lambda_0 > \lambda_1 > ... > \lambda_d\}.$$

(12.03) Lemma ($\mathcal{F}_\mathcal{M}$)

Let $\mathcal{M} = \{\lambda_0, \lambda_1, ..., \lambda_d\}$, be a mesh of real numbers and let $\mathcal{F}_\mathcal{M}$ denote a set of all real functions defined on $\mathcal{M}$. Then $\mathcal{F}_\mathcal{M}$ is a vector space of dimension $d + 1$ and basis $\{e_0, e_1, ..., e_d\}$ where e_i are functions defined on following way

$$e_i(\lambda_j) = \begin{cases} 1 & \textit{if } i = j \\ 0 & \textit{otherwise.} \end{cases}$$

Proof: We will left like interesting exercise for reader to show that $\mathcal{F}_\mathcal{M}$ satisfy all axioms from definition of vector space[2].

Let $\mathcal{M} = \{\lambda_0, \lambda_1, ..., \lambda_d\}$ be a mesh of real numbers and consider functions e_i, defined on following way

$$e_i(\lambda_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Is the set of functions $\{e_0, e_1, ..., e_d\}$ linearly independent? That is, are there scalars α_0, α_1, ..., α_d, not all zero, such that

$$(\alpha_0e_0 + \alpha_1e_1 + ... + \alpha_de_d)(\lambda_i) = 0, \quad \text{for all } i = 1, 2, ..., n?$$

[2] A nonempty set $\mathcal{V}$ is said to be a <u>*vector space*</u> over a field $\mathbb{F}$ if: (*i*) there exists an operation called addition that associates to each pair $x, y \in V$ a new vector $x + y \in \mathcal{V}$ called the sum of x and y; (*ii*) there exists an operation called scalar multiplication that associates to each $\alpha \in \mathbb{F}$ and $x \in \mathcal{V}$ a new vector $\alpha x \in \mathcal{V}$ called the product of α and x; (*iii*) these operations satisfy the following axioms:

(V1)-(V5). $\mathcal{V}$ is an additive Abelian group (with neutral element 0).

(V6). $1v = v$, for all $v \in \mathcal{V}$ where 1 is the (multiplicative) identity in $\mathbb{F}$

(V7). $\alpha(\beta v) = (\alpha\beta)v$ for all $v \in \mathcal{V}$ and all $\alpha, \beta \in \mathbb{F}$.

(V8)-(V9). There worth two law of distribution:

(*a*) $\alpha(u + v) = \alpha u + \alpha v$ for all $u, v \in \mathcal{V}$ and all $\alpha \in \mathbb{F}$;

(*b*) $(\alpha + \beta)v = \alpha v + \beta v$ for all $v \in \mathcal{V}$ and all $\alpha, \beta \in \mathbb{F}$.

The members of $\mathcal{V}$ are called vectors, and the members of $\mathbb{F}$ are called scalars. The vector $0 \in \mathcal{V}$ is called the zero vector, and the vector $-x$ is called the negative of the vector x. We mention only in passing that if we replace the field $\mathbb{F}$ by an arbitrary ring R, then we obtain what is called an <u>*R-module*</u> (or simply a <u>*module*</u> over R).

From this, for arbitrary i, we have

$$\alpha_0 e_0(\lambda_i) + \alpha_1 e_1(\lambda_i) + ... + \alpha_d e_d(\lambda_i) = 0,$$
$$\alpha_i e_i(\lambda_i) = 0,$$
$$\alpha_i = 0.$$

Set of functions $\{e_0, e_1, ..., e_d\}$ is linearly independent.

Now, pick arbitrary function $f \in \mathcal{F}_{\mathcal{M}}$. We want to show that $f \in \text{span}\{e_0, e_1, ..., e_d\}$. Define numbers c_0, c_1, ..., c_d as $c_0 = f(\lambda_0)$, $c_1 = f(\lambda_1)$, ..., $c_d = f(\lambda_d)$. Now it is not hard to see that function f we can write in form

$$f = c_0 e_0 + c_1 e_1 + ... + c_d e_d.$$

Therefore, set $\{e_0, e_1, ..., e_d\}$ is linearly independent set that span vector space $\mathcal{F}_{\mathcal{M}}$. Dimension of $\mathcal{F}_{\mathcal{M}}$ is $d + 1$. □

For arbitrary polynomials $a_0 + a_1 x + ... + a^d x^d, b_0 + b_1 x + ... + b^d x^d \in R_d[x]$ there are unique function $f, g \in \mathcal{F}_{\mathcal{M}}$, respectly, such that $f(\lambda_0) = a_0$, $f(\lambda_1) = a_1$, ..., $f(\lambda_d) = a_d$ and $g(\lambda_0) = b_0$, $g(\lambda_1) = b_1$, ..., $g(\lambda_d) = b_d$. If we define mapping $F : \mathcal{F}_{\mathcal{M}} \longrightarrow \mathbb{R}_d[x]$ with

$$F(f) = f(\lambda_d)x^d + ... + f(\lambda_1)x + f(\lambda_0) = a_0 + a_1 x + ... + a_d x^d,$$

$$F(g) = g(\lambda_d)x^d + ... + g(\lambda_1)x + g(\lambda_0) = b_0 + b_1 x + ... + b_d x^d,$$

since $(f + g)(\lambda_i) = f(\lambda_i) + g(\lambda_i) = a_i + b_i$, we have

$$F(f + g) = (a_0 + b_0) + (a_1 + b_1)x + ... + (a_d + b_d)x^d =$$

$$= (a_0 + a_1 x + ... + a^d x^d) + (b_0 + b_1 x + ... + b_d x^d) = F(f) + F(g).$$

Interesting question, which we will not consider here, is: Is it possible to define multiplication of elements in $\mathcal{F}_{\mathcal{M}}$ such that $\mathcal{F}_{\mathcal{M}}$ form an algebra that is isomorphic with $\mathbb{R}_d[x]$?

From now on, we are interest in sets $\mathbb{R}[x]/\langle Z(x)\rangle$, $\mathbb{R}_d[x]$ and $\mathcal{F}_{\mathcal{M}}$ just as vector spaces, and we invite reader to show that these sets are isomorphic as vector spaces, that is, that we have following natural identifications

$$\mathcal{F}_{\mathcal{M}} \longleftrightarrow \mathbb{R}[x]/\langle Z(x)\rangle \longleftrightarrow \mathbb{R}_d[x] \tag{29}$$

For simplicity, we represent by the same symbol, say p, any of the three mathematical objects identified in (29). When we need to specify one of the above three sets, we will be explicit.

(12.04) Definition (the scalar product associated to $(\mathcal{M}, g)$)

Let $g : \mathcal{M} \longrightarrow \mathbb{R}$ be positive function defined on mesh $\mathcal{M} = \{\lambda_0, \lambda_1, ..., \lambda_d\}$. We shall write, for short, $g_\ell := g(\lambda_\ell)$. From the pair $(\mathcal{M}, g)$ we can define an inner product in $\mathbb{R}_d[x]$ (indistinctly in $\mathcal{F}_{\mathcal{M}}$ or in $\mathbb{R}[x]/(Z)$) as

$$\langle p, q\rangle := \sum_{\ell=0}^{d} g_\ell p(\lambda_\ell) q(\lambda_\ell), \quad p, q \in \mathbb{R}_d[x],$$

with corresponding norm $\| \star \|$. From now on, this will be referred to as *the scalar product associated to $(\mathcal{M}, g)$*. Function g will be called a *weight function* on $\mathcal{M}$. We say that it is normalized when $g_0 + g_1 + ... + g_d = 1$. Note that $\langle 1, 1\rangle = 1$ is the condition concerning the normalized character of the weight function g, which will be hereafter assumed. ◇

For sake of next definition, it is interesting to consider mesh $\mathcal{M} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$, and real numbers π_0, π_1, π_2 and π_3 defined as follows:

$$\pi_0 = |\lambda_0 - \lambda_1||\lambda_0 - \lambda_2||\lambda_0 - \lambda_3| = \prod_{\ell=0\,(\ell\neq 0)}^{3} |\lambda_0 - \lambda_\ell|,$$

$$\pi_1 = |\lambda_1 - \lambda_0||\lambda_1 - \lambda_2||\lambda_1 - \lambda_3| = \prod_{\ell=0\,(\ell\neq 1)}^{3} |\lambda_1 - \lambda_\ell|,$$

$$\pi_2 = |\lambda_2 - \lambda_0||\lambda_2 - \lambda_1||\lambda_2 - \lambda_3| = \prod_{\ell=0\,(\ell\neq 2)}^{3} |\lambda_2 - \lambda_\ell|,$$

$$\pi_3 = |\lambda_3 - \lambda_0||\lambda_3 - \lambda_1||\lambda_3 - \lambda_2| = \prod_{\ell=0\,(\ell\neq 3)}^{3} |\lambda_3 - \lambda_\ell|.$$

Since $\lambda_0 > \lambda_1 > \lambda_2 > \lambda_3$ we have

$$\pi_0 = (\lambda_0 - \lambda_1)(\lambda_0 - \lambda_2)(\lambda_0 - \lambda_3) = (-1)^0 \prod_{\ell=0\,(\ell\neq 0)}^{3} (\lambda_0 - \lambda_\ell),$$

$$\pi_1 = (-1)(\lambda_1 - \lambda_0)(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) = (-1)^1 \prod_{\ell=0\,(\ell\neq 1)}^{3} (\lambda_1 - \lambda_\ell),$$

$$\pi_2 = (-1)(\lambda_2 - \lambda_0)(-1)(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3) = (-1)^2 \prod_{\ell=0\,(\ell\neq 2)}^{3} (\lambda_2 - \lambda_\ell),$$

$$\pi_3 = (-1)(\lambda_3 - \lambda_0)(-1)(\lambda_3 - \lambda_1)(-1)(\lambda_3 - \lambda_2) = (-1)^3 \prod_{\ell=0\,(\ell\neq 3)}^{3} (\lambda_3 - \lambda_\ell).$$

In order to simplify some expressions, it is useful to introduce the following momentlike parameters, computed from the points of the mesh $\mathcal{M}$, and the family of interpolating polynomials (with degree d).

(12.05) Definition (π_k, Z_k)
Let $\mathcal{M} = \{\lambda_0, \lambda_1, ..., \lambda_d\}$, $\lambda_0 > \lambda_1 > ... > \lambda_d$ be mesh of real numbers. We define parameters π_k $(0 \le k \le d)$ and polynomials Z_k $(0 \le k \le d)$ on the following way

$$\pi_k := \prod_{\ell=0\,(\ell\neq k)}^{d} |\lambda_k - \lambda_\ell| = (-1)^k \prod_{\ell=0\,(\ell\neq k)}^{d} (\lambda_k - \lambda_\ell) \qquad (0 \le k \le d);$$

$$Z_k := \frac{(-1)^k}{\pi_k} \prod_{\ell=0\,(\ell\neq k)}^{d} (x - \lambda_\ell) \qquad (0 \le k \le d).$$

◇

(12.06) Proposition
Interpolating polynomials Z_k $(0 \le k \le d)$ satisfy

$$Z_k(\lambda_h) = \delta_{hk}, \qquad \langle Z_h, Z_k \rangle = \delta_{hk} g_k.$$

Proof:

$$Z_k(\lambda_h) = \frac{(-1)^k}{\pi_k} \prod_{\ell=0\,(\ell\neq k)}^{d} (\lambda_h - \lambda_\ell) = \frac{1}{\pi_k} \underbrace{(-1)^k \prod_{\ell=0\,(\ell\neq k)}^{d} (\lambda_h - \lambda_\ell)}_{\begin{cases} 0, & \text{if } h \neq k \\ \pi_k, & \text{if } h = k \end{cases}} = \delta_{hk}.$$

$$\langle Z_h, Z_k\rangle = \sum_{\ell=0}^{d} g_\ell Z_h(\lambda_\ell) Z_k(\lambda_\ell) = g_h Z_h(\lambda_h) \cdot \underbrace{Z_k(\lambda_h)}_{\begin{cases} 1, & \text{if } k = h \\ 0, & \text{if } k \neq h \end{cases}} = g_h \delta_{hk} = \delta_{hk} g_k.$$

□

For a given set of m points $\mathcal{S} = \{(x_1, y_1), (x_2, y_2), ..., (x_m, y_m)\}$ in which the x_i's are distinct, we know that there are unique polynomial

$$p(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + ... + \alpha_{m-1} t^{m-1}$$

of degree $m-1$ that passes through each point in $\mathcal{S}$. In fact this polynomial must be given by

$$p(t) = \sum_{i=1}^{m} \left(y_i \frac{\prod_{j=1\,(j\neq i)}^{m} (t - x_j)}{\prod_{j=1\,(j\neq i)}^{m} (x_i - x_j)} \right)$$

and is known as the *Lagrange interpolation polynomial* of degree $m-1$.

Consider arbitrary polynomial $p(x) = ax^3 + bx^2 + cx + d \in \mathbb{R}_3[x]$ and consider set of points $\mathcal{S} = \{(\lambda_0, p(\lambda_0)), (\lambda_1, p(\lambda_1)), (\lambda_2, p(\lambda_2)), (\lambda_3, p(\lambda_3))\}$. Lagrange interpolation polynomial for this set is

$$p(t) = \sum_{i=1}^{3} \left(p(\lambda_i) \frac{\prod_{j=1\,(j\neq i)}^{3} (t - \lambda_j)}{\prod_{j=1\,(j\neq i)}^{3} (\lambda_i - \lambda_j)} \right) = \sum_{i=1}^{3} \left(p(\lambda_i) \frac{(-1)^i}{(-1)^i \prod_{j=1\,(j\neq i)}^{3} (\lambda_i - \lambda_j)} \prod_{j=1\,(j\neq i)}^{3} (t - \lambda_j) \right)$$

$$= \sum_{i=1}^{3} \left(p(\lambda_i) \frac{(-1)^i}{\pi_i} \prod_{j=1\,(j\neq i)}^{3} (t - \lambda_j) \right) = \sum_{i=1}^{3} p(\lambda_i) Z_i(t).$$

(12.07) Proposition

Let $\mathcal{M} = \{\lambda_0, \lambda_1, ..., \lambda_d\}$, $\lambda_0 > \lambda_1 > ... > \lambda_d$ be mesh of real numbers. For arbitrary polynomial $p \in \mathbb{R}_d[x]$ we have

$$p = \sum_{k=0}^{d} p(\lambda_k) Z_k$$

where $\{Z_0, Z_1, ..., Z_d\}$ is the family of interpolating polynomials from Definition 12.05.

Proof: Let $p(x) = a_d x^d + ... + a_1 x + a_0$ be arbitrary polynomial of degree d, and consider set of points $\mathcal{S} = \{(\lambda_0, p(\lambda_0)), (\lambda_1, p(\lambda_1)), ..., (\lambda_d, p(\lambda_d))\}$. Lagrange interpolation polynomial for $\mathcal{S}$ is unique polynomial of degree d that passes through each point in $\mathcal{S}$, and is given by

$$p(t) = \sum_{i=0}^{d} \left(p(\lambda_i) \frac{\prod_{j=0\,(j\neq i)}^{d} (t - \lambda_j)}{\prod_{j=0\,(j\neq i)}^{d} (\lambda_i - \lambda_j)} \right) = \sum_{i=0}^{d} \left(p(\lambda_i) \frac{(-1)^i}{(-1)^i \prod_{j=0\,(j\neq i)}^{d} (\lambda_i - \lambda_j)} \prod_{j=0\,(j\neq i)}^{d} (t - \lambda_j) \right)$$

$$= \sum_{i=0}^{d} \left(p(\lambda_i) \frac{(-1)^i}{\pi_i} \prod_{j=0\,(j\neq i)}^{d} (t-\lambda_j) \right) = \sum_{i=0}^{d} p(\lambda_i) Z_i(t).$$

□

For mesh $\mathcal{M} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$ consider family of interpolating polynomials $\{Z_0, Z_1, Z_2, Z_3\}$. If we set $p(x) = 1$, from Proposition 12.07 we have

$$1 = \sum_{k=0}^{d} 1 \cdot Z_k = Z_0 + Z_1 + Z_2 + Z_3 = \sum_{k=0}^{3} \frac{(-1)^k}{\pi_k} \prod_{i=0\,(i\neq k)}^{3} (x-\lambda_i)$$

If we look at coefficients of order 3 of above equation we have

$$\sum_{k=0}^{3} \frac{(-1)^k}{\pi_k} = 0.$$

Now let p be $p(x) = x$. From Proposition 12.07

$$x = \sum_{k=0}^{d} \lambda_k \cdot Z_k = \sum_{k=0}^{3} \lambda_k \frac{(-1)^k}{\pi_k} \prod_{i=0\,(i\neq k)}^{3} (x-\lambda_i)$$

and if we look at coefficients of order 3 we have

$$\sum_{k=0}^{3} \frac{(-1)^k}{\pi_k} \lambda_k = 0.$$

Next, let p be $p(x) = x^2$. From Proposition 12.07

$$x^2 = \sum_{k=0}^{d} \lambda_k^2 \cdot Z_k = \sum_{k=0}^{3} \lambda_k^2 \frac{(-1)^k}{\pi_k} \prod_{i=0\,(i\neq k)}^{3} (x-\lambda_i)$$

and if we look at coefficients of order 3 we have

$$\sum_{k=0}^{3} \frac{(-1)^k}{\pi_k} \lambda_k^2 = 0.$$

Finaly, if we for p set $p(x) = x^3$, from Proposition 12.07

$$x^3 = \sum_{k=0}^{d} \lambda_k^3 \cdot Z_k = \sum_{k=0}^{3} \lambda_k^3 \frac{(-1)^k}{\pi_k} \prod_{i=0\,(i\neq k)}^{3} (x-\lambda_i)$$

we have

$$\sum_{k=0}^{3} \frac{(-1)^k}{\pi_k} \lambda_k^3 = 1.$$

(12.08) Corollary

Momentlike parameters $\pi_k := \prod_{\ell=0\,(\ell\neq k)}^{d} |\lambda_k - \lambda_\ell|$ *satisfy*

$$\sum_{k=0}^{d} \frac{(-1)^k}{\pi_k} \lambda_k^i = 0 \quad (0 \le i \le d-1), \qquad \sum_{k=0}^{d} \frac{(-1)^k}{\pi_k} \lambda_k^d = 1.$$

Proof: For function p from Proposition 12.07 if we use $p(x)=1$, $p(x)=x$, ..., $p(x)=x^d$ we have

$$x^i=\sum_{k=0}^{d}\lambda_k^i Z_k=\sum_{k=0}^{d}\lambda_k^i\frac{(-1)^k}{\pi_k}\prod_{j=0\,(j\neq k)}^{d}(x-\lambda_j),\qquad (0\leq i\leq d).$$

From this it is not hard to see that

$$\sum_{k=0}^{d}\frac{(-1)^k}{\pi_k}\lambda_k^i=0\qquad (0\leq i\leq d-1),\qquad\qquad \sum_{k=0}^{d}\frac{(-1)^k}{\pi_k}\lambda_k^d=1.$$

□

(12.09) Proposition

Suppose that $\mathcal{V}$ is a finite-dimensional real inner product space. If $\{v_1,...,v_n\}$ is an orthogonal basis of $\mathcal{V}$, then for every vector $u\in\mathcal{V}$, we have

$$u=\frac{\langle u,v_1\rangle}{\|v_1\|^2}v_1+...+\frac{\langle u,v_n\rangle}{\|v_n\|^2}v_n.$$

Furthermore, if $\{v_1,...,v_n\}$ is an orthonormal basis of $\mathcal{V}$, then for every vector $u\in\mathcal{V}$, we have

$$u=\langle u,v_1\rangle v_1+...+\langle u,v_n\rangle v_n.$$

Proof: Since $\{v_1,...,v_n\}$ is a basis of $\mathcal{V}$, there exist unique $\alpha_1,...,\alpha_n\in\mathbb{R}$ such that

$$u=\alpha_1v_1+...+\alpha_nv_n.$$

For every $i=1,...,n$, we have

$$\langle u,v_i\rangle=\langle\alpha_1v_1+...+\alpha_nv_n,v_i\rangle=\alpha_1\langle v_1,v_i\rangle+...+\alpha_n\langle v_n,v_i\rangle=\alpha_i\langle v_i,v_i\rangle$$

since $\langle v_i,v_j\rangle=0$ if $j\neq i$. Clearly $v_i\neq 0$, so that $\langle v_i,v_i\rangle\neq 0$, and so

$$\alpha_i=\frac{\langle u,v_i\rangle}{\langle v_i,v_i\rangle}$$

for every $i=1,...,n$. The first assertion follows immediately. For the second assertion, note that $\|v_i\|^2=\langle v_i,v_i\rangle=1$ for every $i=1,...,n$. □

(12.10) Definition (Fourier expansions, Fourier coefficients)

If $\mathcal{B}=\{u_1,u_2,...,u_n\}$ is an orthonormal basis for an inner-product space $\mathcal{V}$, then each $x\in\mathcal{V}$ can be expressed as

$$x=\langle x,u_1\rangle u_1+...+\langle x,u_n\rangle u_n.$$

This is called the *Fourier expansion* of x. The scalars $\alpha_i=\langle x,u_i\rangle$ are the coordinates of x with respect to $\mathcal{B}$, and they are called the *Fourier coefficients*. Geometrically, the Fourier expansion resolves x into n mutually orthogonal vectors $\langle x,u_i\rangle u_i$, each of which represents the orthogonal projection of x onto the space (line) spanned by u_i. ◇

(12.11) Theorem (Gram-Schmidt orthogonalization procedure)

If $\mathcal{B}=\{x_1,x_2,...,x_n\}$ is a basis for a general inner-product space $\mathcal{S}$, then the Gram-Schmidt sequence defined by

$$u_1=\frac{x_1}{\|x_1\|}\quad\text{and}\quad u_k=\frac{x_k-\sum_{i=1}^{k-1}\langle u_i,x_k\rangle u_i}{\|x_k-\sum_{i=1}^{k-1}\langle u_i,x_k\rangle u_i\|}\quad\text{for } k=2,...,n$$

is an orthonormal basis for $\mathcal{S}$.

Proof: Proof can be find in any book of linear algebra (for example see [37], page 309). □

(12.12) Problem

Let $p(x) = (x - \lambda_0)(x - \lambda_1)(x - \lambda_2) \in \mathbb{R}[x]$ *be irreducible polynomial and let* $\mathbb{R}^3 := \{(\alpha_1, \alpha_2, \alpha_3)^\top \,|\, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\}$. *Consider vector space* $\mathbb{R}[x]/I$ *where* $I = \langle p \rangle := \{pq \,|\, q \in \mathbb{R}[x]\}$. *Recall that elements of* $\mathbb{R}[x]/I$ *are cossets of form* $I + r$ *where for any* $I + r_1, I + r_2 \in \mathbb{R}[x]/I$, $\alpha \in \mathbb{R}$ *we define*

vector addition: $(I + r_1) + (I + r_2) = I + (r_1 + r_2)$ *and*
scalar multiplication: $\alpha(I + r_1) = I + (\alpha r_1)$.

Show that vector spaces $\mathbb{R}[x]/I$ *and* $\mathbb{R}^3$ *are isomorphic.*

Solution: Roots of polynomial $p(x)$ are λ_0, λ_1 and λ_2. From Abstract algebra we know[3] that arbitrary element $I + r \in \mathbb{R}[x]/I$ can be uniquely expressed in the form

$$I + (ax^2 + bx + c)$$

where $a, b, c \in \mathbb{R}$. Define function Φ on following way

$$\Phi : \mathbb{R}[x]/I \longrightarrow \mathbb{R}^3,$$

$$\Phi(I + ax^2 + bx + c) = (a, b, c)^\top.$$

We first want to show that Φ is homomorphism of vector spaces:

$$\Phi((I+ax^2+bx+c)+(I+a_1x^2+b_1x+c_1)) = \Phi(I+(a+a_1)x^2+(b+b_1)x+(c+c_1)) = (a+a_1, b+b_1, c+c_1)^\top =$$

$$= (a, b, c)^\top + (a_1, b_1, c_1)^\top = \Phi(I + ax^2 + bx + c) + \Phi(I + a_1x^2 + b_1x + c_1),$$

$$\Phi(\alpha(I + ax^2 + bx + c)) = \Phi(I + \alpha(ax^2 + bx + c)) = \Phi(I + (\alpha ax^2 + \alpha bx + \alpha c)) =$$

$$= (\alpha a, \alpha b, \alpha c)^\top = \alpha(a, b, c)^\top = \alpha\Phi(I + ax^2 + bx + c)$$

Is Φ well defined? Assume that $I + ax^2 + bx + c = I + a_1x^2 + b_1x + c_1$. Then we have

$$I + (a - a_1)x^2 + (b - b_1)x + (c - c_1) = I$$

that is

$$(a - a_1)x^2 + (b - b_1)x + (c - c_1) \in I.$$

With another words

$$(a - a_1)x^2 + (b - b_1)x + (c - c_1) = p(x)q(x)$$

for some $q(x) \in \mathbb{R}[x]$. If $q \neq 0$ then degree of left side of above equation is 2, but degree of right side is at last 3, a contradiction. So $q = 0$, which imply that

$$(a - a_1)x^2 + (b - b_1)x + (c - c_1) = 0$$

that is

$$a = a_1, \;\; b = b_1, \;\; c = c_1, \;\; \Rightarrow \;\; (a, b, c)^\top = (a_1, b_1, c_1)^\top$$

Therefore

$$\Phi(I + ax^2 + bx + c) = \Phi(I + a_1x^2 + b_1x + c_1).$$

[3] **Theorem** Suppose $p = a_0 + a_1x + ... + a_nx^n \in R[x]$ (R is a ring), $a_n \neq 0$, and let $I = \langle p \rangle = p\,R[x] = \{pf : f \in R[x]\}$. Then every element of $R[x]/I$ can be uniquely expressed in the form $I + (b_0 + b_1x + ... + b_{n-1}x^{n-1})$ where $b_0, ..., b_{n-1} \in \mathbb{F}$.

Finally, we want to show that Φ is injective and sirjective. Pick arbitrary $(a,b,c)^\top \in \mathbb{R}^3$. Then there exists at last one element from $\mathbb{R}[x]/I$ which are mapping in $(a,b,c)^\top$ (for example $I+(ax^2+bx+c)$). Therefore Φ is sirjective. Now assume that

$$\Phi(I+(ax^2+bx+c)) = \Phi(I+(a_1x^2+b_1x+c_1))$$

for some $a,b,c,a_1,b_1,c_1 \in \mathbb{R}[x]$. With another words $(a,b,c)^\top = (a_1,b_1,c_1)^\top$ which imply $a=a_1,\ b=b_1,\ c=c_1$ so

$$I+(ax^2+bx+c) = I+(a_1x^2+b_1x+c_1).$$

Therefore, Φ is isomorphism of vector spaces. $\Diamond$

13 Orthogonal systems

A family of polynomials $r_0,\ r_1,\ ...,\ r_d$ is said to be an <u>*orthogonal system*</u> when each polynomial r_k is of degree k and $\langle r_h, r_k\rangle = 0$ for any $h \neq k$.

(13.01) Lemma

Let $r_0,\ r_1,\ ...,\ r_d$ be orthogonal system. Then every of $r_k(x)$, $k=0,1,...,d$, is orthogonal on arbitrary polynomial of lower degree.

Proof: From Proposition 5.03 we know that $\{r_0,r_1,...,r_d\}$ is linearly independent set. So for any $k-1$ where $0 \le k-1 < d$ the set $\{r_0,r_1,...,r_{k-1}\}$ is basis for $\mathbb{R}_{k-1}[x]$ ($\mathbb{R}_{k-1}[x]$ is vector space of all polynomials of degree at most $k-1$). Now notice that arbitrary $q \in \mathbb{R}_{k-1}[x]$ can be write like linear combination of $\{r_0,r_1,...,r_{k-1}\}$. So we have

$$\langle q, r_k\rangle = \langle \alpha_0 r_0 + \alpha_1 r_1 + ... + \alpha_{k-1} r_{k-1}, r_k\rangle = 0.$$

$\square$

Example 13.02 will help us to easier understand Proposition 13.03.

(13.02) Example

In this example we work in space $\mathbb{R}[x]/\langle Z\rangle$ where $Z=(x-\lambda_0)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$, $\lambda_0=3$, $\lambda_1=1$, $\lambda_2=-1$, $\lambda_3=-3$, and inner product is defined by $\langle p,q\rangle = \sum_{i=0}^{3} g_i p(\lambda_i)q(\lambda_i)$, $g_0=g_1=g_2=g_3=1/4$.

It is not hard to check that family of polynomials $\{r_0,r_1,r_2,r_3\} = \{1, x, x^2-5, 5x^3-41x\}$ is orthogonal system. From Lemma 13.01 every r_k is orthogonal on arbitrary polynomial of lower degree. Easy computation gives

$$x\begin{pmatrix} r_0\\ r_1\\ r_2\\ r_3\end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0\\ 5 & 0 & 1 & 0\\ 0 & 16/5 & 0 & 144/720\\ 0 & 0 & 144/16 & 0\end{pmatrix}\begin{pmatrix} r_0\\ r_1\\ r_2\\ r_3\end{pmatrix}.$$

Notice that $xr_3 = \frac{144}{16}r_2 = 9x^2-45$ and $xr_3 = 5x^4-41x^2 = 5Z+9x^2-45$ where $Z=(x-3)(x-1)(x+1)(x+3) = x^4-10x^2+9$.

Given any $k=0,1,2,3$ let $Z_k^* = \prod_{\ell=0,\ell\neq k}^{3}(x-\lambda_\ell)$. For any k we have $Z_k^* = \prod_{\ell=0,\ell\neq k}^{3}(x-\lambda_\ell) = x^3+... = \xi_0 r_3 + \xi_1 r_2 + \xi_2 r_1 + \xi_3 r_0$, for some ξ's, where ξ_0 does not depend on k (because $\{r_0,r_1,r_2,r_3\}$ is basis for $\mathbb{R}_3[x]$ and only r_3 has degree 3). In our case

$$Z_0^* = (x-1)(x+1)(x+3) = x^3+3x^2-x-3, \quad Z_1^* = (x-3)(x+1)(x+3) = x^3+x^2-9x-9,$$

$Z_2^* = (x-3)(x-1)(x+3) = x^3 - x^2 - 9x + 9, \quad Z_3^* = (x-3)(x-1)(x+1) = x^3 - 3x^2 - x + 3,$

$$Z_0^* = (1/5)r_3 + 3r_2 + (36/5)r_1 + 18r_0, \qquad Z_1^* = (1/5)r_3 + r_2 - (4/5)r_1 - 4r_0,$$
$$Z_2^* = (1/5)r_3 - r_2 - (4/5)r_1 + 4r_0, \qquad Z_3^* = (1/5)r_3 - 3r_2 + (36/5)r_1 - 12r_0.$$

Next, we want to compute $\langle r_3, Z_0^*\rangle$, $\langle r_3, Z_1^*\rangle$, $\langle r_3, Z_2^*\rangle$ and $\langle r_3, Z_3^*\rangle$

$$\langle r_3, Z_0^*\rangle = 144, \qquad \langle r_3, Z_1^*\rangle = 144, \qquad \langle r_3, Z_2^*\rangle = 144, \qquad \langle r_3, Z_3^*\rangle = 144.$$

Thus, for $\pi_k = (-1)^k \prod_{\ell=0\,(\ell\neq k)}^{d} (\lambda_k - \lambda_\ell)$,

$$\langle r_3, Z_k^*\rangle = \sum_{i=0}^{3} g_i\, r_3(\lambda_i) Z_k^*(\lambda_i) = g_k r_3(\lambda_k) Z_k^*(\lambda_k) = g_k r_3(\lambda_k)(-1)^k\,\pi_k =$$

$$= \langle r_3, \underbrace{\xi_0 r_3 + \xi_1 r_2 + \xi_2 r_1 + \xi_3 r_0}_{Z_k^*}\rangle = \xi_0 \|r_3\|^2 = const.,$$

that is

$$\langle r_3, Z_k^*\rangle = g_k r_3(\lambda_k)(-1)^k\,\pi_k = \xi_0\|r_3\|^2 = const.$$

so if we, for example, in above equality set $k = 0$, we have

$$\langle r_3, Z_0^*\rangle = g_0 r_3(\lambda_0)\,\pi_0 = \xi_0\|r_3\|^2 \neq 0,$$

and since

$$\xi_0\|r_3\|^2 = \langle r_3, Z_k^*\rangle$$

we have

$$(-1)^k g_k \pi_k r_3(\lambda_k) = g_0 \pi_0 r_3(\lambda_0),$$
$$\frac{r_3(\lambda_k)}{r_3(\lambda_0)} = (-1)^k \frac{g_0\pi_0}{g_k\pi_k}.$$

We have already seen that

$$Z_k^* = \prod_{\ell=0,\ell\neq k}^{3} (x - \lambda_\ell) = \xi_0 r_3 + \xi_1 r_2 + \xi_2 r_1 + \xi_3 r_0,$$

for some ξ_i's, where ξ_0 does not depend on k, and since

$$Z_k^* = \frac{\langle Z_k^*, r_3\rangle}{\|r_3\|^2} r_3 + \frac{\langle Z_k^*, r_2\rangle}{\|r_2\|^2} r_2 + \frac{\langle Z_k^*, r_1\rangle}{\|r_1\|^2} r_1 + \frac{\langle Z_k^*, r_0\rangle}{\|r_0\|^2} r_0$$

(Fourier expansion) we have

$$\xi_0 = \frac{\langle Z_k^*, r_3\rangle}{\|r_3\|^2} = \frac{g_0 r_3(\lambda_0)\,\pi_0}{\|r_3\|^2}.$$

From $Z_k^* = \xi_0 r_3 + \xi_1 r_2 + \xi_2 r_1 + \xi_3 r_0$, we see that

$$\xi_0 r_3 = Z_k^* - (\xi_1 r_2 + \xi_2 r_1 + \xi_3 r_0),$$

$$r_3 - \frac{1}{\xi_0} Z_k^* = -\frac{1}{\xi_0}(\xi_1 r_2 + \xi_2 r_1 + \xi_3 r_0),$$

so for any $k = 0, 1, 2, 3$ we get

$$r_3 - \frac{\|r_3\|^2}{r_3(\lambda_0)} \frac{1}{g_0\pi_0} Z_k^* \in \mathbb{R}_2[x].$$

Then, the equality $xr_3 = \alpha r_2 + \beta r_3$ (for some $\alpha, \beta \in \mathbb{R}$, because $\langle xr_3, r_0\rangle = \langle r_3, xr_0\rangle = 0$, $\langle xr_3, r_1\rangle = \langle r_3, xr_1\rangle = 0$), holding in $\mathbb{R}[x]/\langle Z\rangle$, and the comparison of the degrees allows us to establish the existence of $\psi \in \mathbb{R}$ such that $xr_3 = \alpha r_2 + \beta r_3 + \psi Z$ in $\mathbb{R}[x]$. In our example

$$x\begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 5 & 0 & 1 & 0 \\ 0 & 16/5 & 0 & 144/720 \\ 0 & 0 & 144/16 & 0 \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 5Z \end{pmatrix}.$$

Notice that ψ is the first coefficient of r_3, we get, from fact that $r_3 - \frac{\|r_3\|^2}{r_3(\lambda_0)}\frac{1}{g_0\pi_0}Z_k^* \in \mathbb{R}_2[x]$

$$\psi = \frac{\|r_3\|^2}{r_3(\lambda_0)}\frac{1}{g_0\pi_0}.$$

$\Diamond$

(13.03) Proposition

Let $Z := \prod_{\ell=0}^{d}(x-\lambda_\ell)$ *where* $\mathcal{M} = \{\lambda_0, \lambda_1, ..., \lambda_d\}$, $\lambda_0 > \lambda_1 > ... > \lambda_d$, *is a mesh of real numbers. Every orthogonal system* r_0, r_1, ..., r_d *satisfies the following properties:*

(a) There exists a tridiagonal matrix $\boldsymbol{R}$ *(called the recurrence matrix of the system) such that, in* $\mathbb{R}[x]/\langle Z\rangle$*):*

$$x\boldsymbol{r} := x\begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \\ r_{d-2} \\ r_{d-1} \\ r_d \end{pmatrix} = \begin{pmatrix} a_0 & c_1 & 0 & & & & \\ b_0 & a_1 & c_2 & 0 & & & \\ 0 & b_1 & a_2 & \dots & \dots & & \\ & 0 & \vdots & \vdots & \dots & 0 & \\ & & \vdots & \dots & a_{d-2} & c_{d-1} & 0 \\ & & & 0 & b_{d-2} & a_{d-1} & c_d \\ & & & & 0 & b_{d-1} & a_d \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \\ r_{d-2} \\ r_{d-1} \\ r_d \end{pmatrix} = \boldsymbol{R}\boldsymbol{r},$$

and this equality, in $\mathbb{R}[x]$*, reads:*

$$x\boldsymbol{r} = \boldsymbol{R}\boldsymbol{r} + \begin{pmatrix} 0 & 0 & \dots & 0 & \frac{\|r_d\|^2}{r_d(\lambda_0)}\frac{1}{g_0\pi_0}Z \end{pmatrix}^\top.$$

(b) All the entries b_k, c_k, *of matrix* $\boldsymbol{R}$ *are nonzero and satisfy* $b_k c_{k+1} > 0$.

(c) The matrix $\boldsymbol{R}$ *diagonalizes with eigenvalues the elements of* $\mathcal{M}$*. An eigenvector associated to* λ_k *is* $(r_0(\lambda_k), r_1(\lambda_k), ..., r_{d-1}(\lambda_k), r_d(\lambda_k))^\top$.

(d) For every $k = 1, ..., d$ *the polynomial* r_k *has real simple roots. If* $\mathcal{M}_k$ *denotes the mesh of the ordered roots of* r_k*, then (the points of) the mesh* $\mathcal{M}_d$ *interlaces* $\mathcal{M}$ *and, for each* $k = 1, 2, ..., d-1$, $\mathcal{M}_k$ *interlaces* $\mathcal{M}_{k+1}$.

Proof: (*a*) Working in $\mathbb{R}[x]/\langle Z\rangle$, we have $\langle xr_k, r_h\rangle = 0$ provided that $k < h-1$ (because r_h is orthogonal to arbitrary polynomial of lower degree, and r_k is of degree k) and, by symmetry ($\langle xr_k, r_h\rangle = \langle r_k, xr_h\rangle$), the result is also zero when $h < k-1$. Therefore, for $k = 0$, we can write,

$$xr_0 = \sum_{h=0}^{d}\frac{\langle x\,r_0, r_h\rangle}{\|r_h\|^2}r_h = \frac{\langle x\,r_0, r_0\rangle}{\|r_0\|^2}r_0 + \frac{\langle x\,r_0, r_1\rangle}{\|r_1\|^2}r_1 + \frac{\langle x\,r_0, r_2\rangle}{\|r_2\|^2}r_2 + ... + \frac{\langle x\,r_0, r_d\rangle}{\|r_d\|^2}r_d =$$

$$= \frac{\langle x\,r_0, r_0\rangle}{\|r_0\|^2}r_0 + \frac{\langle x\,r_0, r_1\rangle}{\|r_1\|^2}r_1,$$

and for any $k = 1, ..., d-1$,

$$xr_k = \sum_{h=0}^{d} \frac{\langle x\, r_k, r_h\rangle}{\|r_h\|^2} r_h = \sum_{h=k-1}^{k+1} \frac{\langle x\, r_k, r_h\rangle}{\|r_h\|^2} r_h = \frac{\langle x\, r_k, r_{k-1}\rangle}{\|r_{k-1}\|^2} r_{k-1} + \frac{\langle x\, r_k, r_k\rangle}{\|r_k\|^2} r_k + \frac{\langle x\, r_k, r_{k+1}\rangle}{\|r_{k+1}\|^2} r_{k+1}.$$

One question that immediately jump up is: What for xr_d? Since we work in $\mathbb{R}[x]/\langle Z\rangle$ we have that xr_d is degree $< d+1$ (in space $\mathbb{R}[x]$ polynomial xr_d is of degree $d+1$). Notice that $xr_d = \psi Z + p(x)$ for some $\psi \in \mathbb{R}$ and some $p(x) \in \mathbb{R}_d[x]$ (of what degree is $p(x)$ - is this matter?). Next, we want to consider $\langle xr_d, r_1\rangle$, ...,$\langle xr_d, r_{d-1}\rangle$:

$$\langle xr_d, r_1\rangle = \langle r_d, xr_1\rangle = 0,$$

$$\langle xr_d, r_2\rangle = \langle r_d, xr_2\rangle = 0,$$

$$\vdots$$

$$\langle xr_d, r_{d-2}\rangle = \langle r_d, xr_{d-2}\rangle = 0,$$

$$\langle xr_d, r_{d-1}\rangle = \langle r_d, xr_{d-1}\rangle \neq 0,$$

so

$$xr_d = \sum_{h=0}^{d} \frac{\langle x\, r_d, r_h\rangle}{\|r_h\|^2} r_h = 0 + ... + 0 + \frac{\langle x\, r_d, r_{d-1}\rangle}{\|r_{d-1}\|^2} r_{d-1} + \frac{\langle x\, r_d, r_d\rangle}{\|r_d\|^2} r_d =$$

$$= \frac{\langle x\, r_d, r_{d-1}\rangle}{\|r_{d-1}\|^2} r_{d-1} + \frac{\langle x\, r_d, r_d\rangle}{\|r_d\|^2} r_d.$$

Then, for any $k = 0, 1, ..., d$ the parameters b_k, a_k, c_k are defined by:

$$b_k = \frac{\langle x\, r_{k+1}, r_k\rangle}{\|r_k\|^2} \quad (0 \le k \le d-1), \qquad b_d = 0,$$

$$a_k = \frac{\langle x\, r_k, r_k\rangle}{\|r_k\|^2} \quad (0 \le k \le d),$$

$$c_0 = 0, \qquad c_k = \frac{\langle x\, r_{k-1}, r_k\rangle}{\|r_k\|^2} \quad (1 \le k \le d),$$

from which we get

$$xr_0 = a_0 r_0 + c_1 r_1,$$
$$xr_k = b_{k-1} r_{k-1} + a_k r_k + c_{k+1} r_{k+1}, \quad k = 1, 2, ..., d-1,$$
$$xr_d = b_{d-1} r_{d-1} + a_d r_d.$$

Given any $k = 0, 1, ..., d$ let $\pi_k = (-1)^k \prod_{\ell=0\,(\ell\neq k)}^{d} (\lambda_k - \lambda_\ell)$, $\mathbb{Z}_k^* := \prod_{\ell=0,\ell\neq k}^{d} (x - \lambda_\ell) =$ $= \xi_0 r_d + \xi_1 r_{d-1} + ...$, and notice that ξ_0 does not depend on k. We have

$$\langle r_d, \mathbb{Z}_k^*\rangle = \sum_{i=0}^{d} g_i r_d(\lambda_i) \mathbb{Z}_k^*(\lambda_i) = g_k r_d(\lambda_k)(-1)^k \pi_k,$$

$$\langle r_d, \mathbb{Z}_k^*\rangle = \langle r_d, \xi_0 r_d + \xi_1 r_{d-1} + ...\rangle = \langle r_d, \xi_0 r_d\rangle = \xi_0 \|r_d\|^2 = const.$$

thus,

$$g_k r_d(\lambda_k)(-1)^k \pi_k = \xi_0 \|r_d\|^2 = const. = g_0 r_d(\lambda_0) \pi_0 \neq 0,$$

$$g_k r_d(\lambda_k)(-1)^k \pi_k = g_0 r_d(\lambda_0)\pi_0 \quad \Longrightarrow \quad \frac{r_d(\lambda_k)}{r_d(\lambda_0)} = (-1)^k \frac{g_0\pi_0}{g_k\pi_k}.$$

Moreover, since

$$Z_k^* = \frac{\langle Z_k^*, r_d\rangle}{\|r_d\|^2} r_d + ... + \frac{\langle Z_k^*, r_1\rangle}{\|r_1\|^2} r_1 + \frac{\langle Z_k^*, r_0\rangle}{\|r_0\|^2} r_0$$

(Fourier expansion) we have

$$\xi_0 = \frac{\langle Z_k^*, r_d\rangle}{\|r_d\|^2} = \frac{g_0 r_d(\lambda_0)\,\pi_0}{\|r_d\|^2} = \frac{r_d(\lambda_0)}{\|r_d\|^2} g_0\pi_0$$

and for any $k = 0, 1, ..., d$ we get

$$r_d - \frac{\|r_d\|^2}{r_d(\lambda_0)} \frac{1}{g_0\pi_0} Z_k^* \in \mathbb{R}_{d-1}[x], \qquad \frac{r_d(\lambda_k)}{r_d(\lambda_0)} = (-1)^k \frac{g_0\pi_0}{g_k\pi_k}.$$

Then the first coefficient of r_d is

$$\psi = \frac{\|r_d\|^2}{r_d(\lambda_0)} \frac{1}{g_0\pi_0}.$$

Equality $xr_d = b_{d-1}r_{d-1} + a_d r_d$, holding in $\mathbb{R}[x]/\langle Z\rangle$, and the comparison of the degrees allows us to establish the existence of $\alpha \in \mathbb{R}$ such that $xr_d = b_{d-1}r_{d-1} + a_d r_d + \alpha Z$ in $\mathbb{R}[x]$. Indeed,

$$xr_d = \alpha Z + q(x)$$

for some $\alpha \in \mathbb{R}$ and some $q(x)$ such that $\operatorname{dgr} q(x) \le d$ (notice that $\operatorname{dgr} Z = d+1$ and that $\operatorname{dgr}(xr_d) = d+1$). Since remainder $q(x)$ is unique, and we had $xr_d = b_{d-1}r_{d-1} + a_d r_d$ in $\mathbb{R}[x]/\langle Z\rangle$, we can conclude that

$$q(x) = b_{d-1}r_{d-1} + a_d r_d.$$

Finaly, since $r_d - \frac{\|r_d\|^2}{r_d(\lambda_0)} \frac{1}{g_0\pi_0} Z_k^* \in \mathbb{R}_{d-1}[x]$ we have that

$$\alpha = \frac{\|r_d\|^2}{r_d(\lambda_0)} \frac{1}{g_0\pi_0} = \psi.$$

(b) By looking again to the degrees of xr_k $(0 \le k \le d-1)$

$$\begin{aligned} xr_0 &= a_0 r_0 + c_1 r_1, \\ xr_k &= b_{k-1}r_{k-1} + a_k r_k + c_{k+1}r_{k+1}, \quad k = 1, ..., d-1 \end{aligned}$$

we realize that c_1, c_2, ..., c_d are nonzero. For $k = 0, 1, ..., d-1$, from the equality

$$xr_k = \sum_{h=0}^{d} \frac{\langle x\,r_k, r_h\rangle}{\|r_h\|^2} r_h = \underbrace{\frac{\langle x\,r_k, r_{k-1}\rangle}{\|r_{k-1}\|^2}}_{b_{k-1}} r_{k-1} + \underbrace{\frac{\langle x\,r_k, r_k\rangle}{\|r_k\|^2}}_{a_k} r_k + \underbrace{\frac{\langle x\,r_k, r_{k+1}\rangle}{\|r_{k+1}\|^2}}_{c_{k+1}} r_{k+1}$$

we have

$$b_k = \frac{\langle x\,r_{k+1}, r_k\rangle}{\|r_k\|^2} = \frac{\langle x\,r_{k+1}, r_k\rangle}{\|r_{k+1}\|^2} \frac{\|r_{k+1}\|^2}{\|r_k\|^2} \overset{\text{by definition of } \langle\cdot,\cdot\rangle}{=\!=\!=\!=} \frac{\|r_{k+1}\|^2}{\|r_k\|^2} \frac{\langle x\,r_k, r_{k+1}\rangle}{\|r_{k+1}\|^2} = \frac{\|r_{k+1}\|^2}{\|r_k\|^2} c_{k+1},$$

that the parameters b_0, b_1, ..., b_{d-1} are also nonnull and, moreover, $b_k c_{k+1} > 0$ for any $k = 0, 1, ..., d-1$.

(c) Pick arbitrary λ_h for some $h = 0, 1, ..., d$. In proof of (a) we have seen that

$$xr_0 = a_0 r_0 + c_1 r_1,$$

$$xr_k = b_{k-1} r_{k-1} + a_k r_k + c_{k+1} r_{k+1}, \quad k = 1, ..., d-1,$$

$$xr_d = b_{d-1} r_{d-1} + a_d r_d.$$

On both sides we have polynomials, so

$$(xr_0)(\lambda_h) = (a_0 r_0 + c_1 r_1)(\lambda_h)$$

$$(xr_k)(\lambda_h) = (b_{k-1} r_{k-1} + a_k r_k + c_{k+1} r_{k+1})(\lambda_h), \quad k = 1, ..., d-1,$$

$$(xr_d)(\lambda_h) = (b_{d-1} r_{d-1} + a_d r_d)(\lambda_h).$$

and this is equivalent with

$$\lambda_h r_0(\lambda_h) = a_0 r_0(\lambda_h) + c_1 r_1(\lambda_h)$$

$$\lambda_h r_k(\lambda_h) = b_{k-1} r_{k-1}(\lambda_h) + a_k r_k(\lambda_h) + c_{k+1} r_{k+1}(\lambda_h), \quad k = 1, ..., d-1,$$

$$\lambda_h r_d(\lambda_h) = b_{d-1} r_{d-1}(\lambda_h) + a_d r_d(\lambda_h).$$

If this, we write in matrix form we have

$$\lambda_h \begin{pmatrix} r_0(\lambda_h) \\ r_1(\lambda_h) \\ r_2(\lambda_h) \\ \vdots \\ r_{d-2}(\lambda_h) \\ r_{d-1}(\lambda_h) \\ r_d(\lambda_h) \end{pmatrix} = \underbrace{\begin{pmatrix} a_0 & c_1 & 0 & & & & \\ b_0 & a_1 & c_2 & 0 & & & \\ 0 & b_1 & a_2 & \dots & \dots & & \\ & 0 & \vdots & \vdots & \dots & 0 & \\ & & \vdots & \dots & a_{d-2} & c_{d-1} & 0 \\ & & & 0 & b_{d-2} & a_{d-1} & c_d \\ & & & & 0 & b_{d-1} & a_d \end{pmatrix}}_{=R} \begin{pmatrix} r_0(\lambda_h) \\ r_1(\lambda_h) \\ r_2(\lambda_h) \\ \vdots \\ r_{d-2}(\lambda_h) \\ r_{d-1}(\lambda_h) \\ r_d(\lambda_h) \end{pmatrix}.$$

Therefore, an eigenvector associated to λ_h is $(r_0(\lambda_h), r_1(\lambda_h), ..., r_{d-1}(\lambda_h), r_d(\lambda_h))^\top$.

Now we want to show that the matrix $\boldsymbol{R}$ diagonalizes with eigenvalues the elements of $\mathcal{M}$. From above we have that $\{(\lambda_0, \boldsymbol{r}_0), (\lambda_1, \boldsymbol{r}_1), ..., (\lambda_d, \boldsymbol{r}_d)\}$ is set of eigenpairs for $\boldsymbol{R}$ where $\boldsymbol{r}_0 = (r_0(\lambda_0), r_1(\lambda_0), ..., r_{d-1}(\lambda_0), r_d(\lambda_0))^\top$, $\boldsymbol{r}_1 = (r_0(\lambda_1), r_1(\lambda_1), ..., r_{d-1}(\lambda_1), r_d(\lambda_1))^\top$, ..., $\boldsymbol{r}_d = (r_0(\lambda_d), r_1(\lambda_d), ..., r_{d-1}(\lambda_d), r_d(\lambda_d))^\top$. If we use Proposition 2.11 we have that $\{\boldsymbol{r}_0, \boldsymbol{r}_1, ..., \boldsymbol{r}_d\}$ is a linearly independent set. Therefore, by Proposition 2.07, $\boldsymbol{R}$ is diagonalizes with eigenvalues the elements of $\mathcal{M}$.

(d) In proof of (a) we have obtained that

$$\frac{r_d(\lambda_k)}{r_d(\lambda_0)} = (-1)^k \frac{g_0 \pi_0}{g_k \pi_k}$$

for any $k = 0, 1, ..., d$. Since $\prod_{\ell=0\,(\ell\neq k)}^{d} |\lambda_k - \lambda_\ell| = \pi_k > 0$, $g_k > 0$ for $k = 0, 1, ..., d$ and $r_d(\lambda_k) = (-1)^k r_d(\lambda_0) \frac{g_0 \pi_0}{g_k \pi_k}$, we observe that r_d takes alternating signs on the points of $\mathcal{M} = \{\lambda_0, \lambda_1, ..., \lambda_d\}$. Hence, this polynomial has d simple roots θ_i whose mesh $\mathcal{M}_d = \{\theta_0, \theta_1, ..., \theta_{d-1}\}$ interlaces $\mathcal{M}$. Noticing that

$$Z = \prod_{k=0}^{d} (x - \lambda_k)$$

and $\lambda_0 > \theta_0 > \lambda_1 > \theta_1 > ... > \lambda_{d-1} > \theta_{d-1} > \lambda_d$, so Z takes alternating signs over the elements of $\mathcal{M}_d$. From the equality $b_{d-1} r_{d-1} = (x - a_d) r_d - \psi Z$, since $r_d(\theta_i) = 0$ for $i = 0, 1, ..., d-1$, it turns out that r_{d-1} takes alternating signs on the elements of $\mathcal{M}_d$; whence $\mathcal{M}_{d-1} = \{\gamma_0, \gamma_1, ..., \gamma_{d-2}\}$ interlaces $\mathcal{M}_d$ and r_d has alternating signs on $\mathcal{M}_{d-1}$ (because $r_d(\theta_i) = 0$ and $\theta_0 > \gamma_0 > \theta_1 > \gamma_1 > ... > \theta_{d-2} > \gamma_{d-2} > \theta_{d-1}$). Recursively, suppose that, for $k = 1, ..., d-2$, the polynomials r_{k+1} and r_{k+2} have simple real roots α_i and β_i, respectly, and that $\mathcal{M}_{k+1} = \{\alpha_0, \alpha_1, ..., \alpha_k\}$ interlaces $\mathcal{M}_{k+2} = \{\beta_0, \beta_1, ..., \beta_{k+1}\}$, so that r_{k+2} takes alternating signs on $\mathcal{M}_{k+1}$. Then, the result follows by just evaluating the equality $b_k r_k = (x - a_{k+1}) r_{k+1} - c_{k+2} r_{k+2}$ at the points of $\mathcal{M}_{k+1}$. □

(13.04) Example

Consider space $\mathbb{R}[x]/\langle Z\rangle$ where $Z = (x-3)(x-1)(x+1)(x+3)$, $\lambda_0 = 3$, $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = -3$ i.e. $\mathcal{M} = \{3, 1, -1, -3\}$, and $g_0 = g_1 = g_2 = g_3 = 1/4$.

In Example 13.02 we have shown that $r_0 = 1$, $r_1 = x$, $r_2 = x^2 - 5$, $r_3 = 5x^3 - 41x$,

$$\frac{r_3(\lambda_k)}{r_3(\lambda_0)} = (-1)^k \frac{g_0 \pi_0}{g_k \pi_k},$$

and it is not hard to compute that $\pi_0 = 48$, $\pi_1 = 16$, $\pi_2 = 16$ and $\pi_3 = 48$. From last equation we observe that r_3 takes alternating signs on the points of $\mathcal{M}$. Easy computation will give that roots of r_3 are

$$\sqrt{\frac{41}{5}}, \quad 0, \quad -\sqrt{\frac{41}{5}}.$$

Since $3 > \sqrt{\frac{41}{5}} > 1 > 0 > -1 > -\sqrt{\frac{41}{5}} > -3$ mesh $\mathcal{M}_3$ interlaces $\mathcal{M}$. Next notice that in same example we had $xr_3 = \frac{144}{16} r_2 + 5Z$. From the equality $\frac{144}{16} r_2 = xr_3 - 5Z$ it turns out that r_2 takes alternating signs on the elements of $\mathcal{M}_3$ (since Z takes alternating signs on the elements of $\mathcal{M}_3$); whence $\mathcal{M}_2$ interlaces $\mathcal{M}_3$ and r_3 has alternating signs on $\mathcal{M}_2$. Indeed, roots of r_2 are

$$\sqrt{5} \quad \text{and} \quad -\sqrt{5},$$

and notice that $\sqrt{\frac{41}{5}} > \sqrt{5} > 0 > -\sqrt{5} > -\sqrt{\frac{41}{5}}$. ◊

(13.05) Problem

Let $Z := \prod_{\ell=0}^{d}(x - \lambda_\ell)$ where $\mathcal{M} = \{\lambda_0, \lambda_1, ..., \lambda_d\}$, $\lambda_0 > \lambda_1 > ... > \lambda_d$, is a mesh of real numbers. If r_0, r_1, ..., r_d is orthogonal system associated to $(\mathcal{M}, g)$ in space $\mathbb{R}[x]/\langle Z\rangle$, prove or disprove that $r_i(\lambda_0) > 0$, $i = 0, 1, ..., d$.

Hint: From Proposition 13.03(d) we see that every orthogonal system r_0, r_1, ..., r_d satisfies the following property: For every $i = 1, ..., d$ the polynomial r_i has real simple roots, and if $\mathcal{M}_i$ denotes the mesh (set of finite many distance real numbers) of the ordered roots of r_i, then (the points of) the mesh $\mathcal{M}_d$ interlaces $\mathcal{M} = \{\lambda_0, \lambda_1, ..., \lambda_d\}$ and, for each $i = 1, 2, ..., d-1$, $\mathcal{M}_i$ interlaces $\mathcal{M}_{i+1}$. If elements of set $\mathcal{M}_i$ we denote by $\mathcal{M}_i = \{\theta_{i1}, \theta_{i2}, ..., \theta_{ii},\}$ this mean that every r_i is of the form $r_i(x) = c_i \prod_{j=1}^{i}(x - \theta_{ij})$ for some $c_i \in \mathbb{R}$.

(13.06) Proposition

Let r_0, r_1, ..., r_{d-1}, r_d be an orthogonal system with respect to the scalar product associated to $(\mathcal{M}, g)$, let $Z = \prod_{k=0}^{d}(x - \lambda_k)$, $H_0 = \frac{1}{g_0\pi_0}\prod_{i=1}^{d}(x - \lambda_i)$ and $\pi_0 = \prod_{i=1}^{d}(\lambda_0 - \lambda_i)$. Then the following assertions are all equivalent:

(a) $r_0 = 1$ and the entries of the tridiagonal matrix $\boldsymbol{R}$ associated to $(r_k)_{0\le k\le d}$, satisfy $a_k + b_k + c_k = \lambda_0$, for any $k = 0, 1, ..., d$;

(b) $r_0 + r_1 + ... + r_d = H_0$;

(c) $\|r_k\|^2 = r_k(\lambda_0)$ for any $k = 0, 1, ..., d$.

Proof: We will show that $(a) \Rightarrow (b)$, $(b) \Leftrightarrow (c)$ and that $(c) \Rightarrow (a)$. Let $\boldsymbol{j} := (1, 1, ..., 1)^\top$.

$(a) \Rightarrow (b)$: Consider the tridiagonal matrix $\boldsymbol{R}$ (Proposition 13.03) associated to the orthogonal system $(r_k)_{0\le k\le d}$

$$x\boldsymbol{r} := x\begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \\ r_{d-2} \\ r_{d-1} \\ r_d \end{pmatrix} = \begin{pmatrix} a_0 & c_1 & 0 & & & & & \\ b_0 & a_1 & c_2 & 0 & & & & \\ 0 & b_1 & a_2 & \dots & \dots & & & \\ & 0 & \vdots & \vdots & \dots & 0 & & \\ & & \vdots & \dots & a_{d-2} & c_{d-1} & 0 \\ & & & 0 & b_{d-2} & a_{d-1} & c_d \\ & & & & 0 & b_{d-1} & a_d \end{pmatrix}\begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \\ r_{d-2} \\ r_{d-1} \\ r_d \end{pmatrix} = \boldsymbol{R}\boldsymbol{r}.$$

Then working in $\mathbb{R}[x]/\langle Z\rangle$, since $x\boldsymbol{r} = \boldsymbol{R}\boldsymbol{r}$, we have:

$$0 = \boldsymbol{j}^\top(x\boldsymbol{r} - \boldsymbol{R}\boldsymbol{r}) = x(\boldsymbol{j}^\top\boldsymbol{r}) - \boldsymbol{j}^\top\boldsymbol{R}\boldsymbol{r} \overset{a_k+b_k+c_k=\lambda_0}{=\!=\!=} x(\boldsymbol{j}^\top\boldsymbol{r}) - \lambda_0\boldsymbol{j}^\top\boldsymbol{r} = (x-\lambda_0)\boldsymbol{j}^\top\boldsymbol{r} = (x-\lambda_0)\sum_{k=0}^{d} r_k,$$

that is $(x-\lambda_0)\sum_{k=0}^{d} r_k = 0$, so $(x-\lambda_0)\sum_{k=0}^{d} r_k = \alpha Z$ for some $\alpha \in \mathbb{R}$. Notice that $H_0(\lambda_0) = \frac{1}{g_0\pi_0}\underbrace{\prod_{i=1}^{d}(\lambda_0-\lambda_i)}_{\pi_0} = \frac{1}{g_0}$. Since we known that $(x-\lambda_0)H_0 = \frac{1}{g_0\pi_0}Z$ we can conclude that in $\mathbb{R}[x]/\langle Z\rangle$ this mean

$$(x-\lambda_0)H_0 = 0,$$

so $H_0, \sum_{k=0}^{d} r_k \in \mathbb{R}_d[x]$ and there exists some ξ such that $\sum_{k=0}^{d} r_k = \xi H_0$. Since, also,

$$\langle r_0, \sum_{k=0}^{d} r_k\rangle = \langle r_0, r_0\rangle = 1$$

we have $\langle r_0, \underbrace{\frac{1}{\xi}\sum_{k=0}^{d} r_k}_{=H_0}\rangle = \frac{1}{\xi}$ and

$$\langle r_0, H_0\rangle = \sum_{i=0}^{d} g_i r_0(\lambda_i)H_0(\lambda_i) = g_0 r_0(\lambda_0)H_0(\lambda_0) = r_0(\lambda_0) = 1$$

it turns out that $\xi = 1$. Consequently, $\sum_{k=0}^{d} r_k = H_0$.

$(b) \Leftrightarrow (c)$: Assume that $r_0 + r_1 + ... + r_d = H_0$. Then

$$\|r_k\|^2 = \langle r_k, r_k\rangle = \langle r_k, r_0 + r_1 + ... + r_d\rangle = \langle r_k, H_0\rangle = \sum_{i=0}^{d} g_i r_k(\lambda_i)H_0(\lambda_i) = r_k(\lambda_0).$$

Conversely, assume that $\|r_k\|^2 = r_k(\lambda_0)$ for any $k = 0, 1, ..., d$. By Fourier expansion (Proposition 12.09) we have

$$H_0 = \frac{\langle H_0, r_0\rangle}{\|r_0\|^2}r_0 + \frac{\langle H_0, r_1\rangle}{\|r_1\|^2}r_1 + ... + \frac{\langle H_0, r_d\rangle}{\|r_d\|^2}r_d.$$

Notice that $r_k(\lambda_0) = \|r_k\|^2$ imply

$$\langle H_0, r_k\rangle = \sum_{i=0}^{d} g_i r_k(\lambda_i) H_0(\lambda_i) = \|r_k\|^2$$

and result follow.

$(c) \Rightarrow (a)$: From $\|r_k\|^2 = r_k(\lambda_0)$ we have that $\langle r_k, r_k\rangle = r_k(\lambda_0)$, and since degre of r_0 is 0 we can write $r_0 = c$, for some $c \in \mathbb{R}$. But $\langle c, c\rangle = c$ imply $c = 1$, and therefore $r_0 = 1$.
In second part of proof we had seen that $\|r_k\|^2 = r_k(\lambda_0)$ imply $r_0 + r_1 + ... + r_d = H_0$. Then, computing xH_0 in $\mathbb{R}[x]/\langle Z\rangle$ in two different ways we get:

$$xH_0 = x\sum_{k=0}^{d} r_k = x\boldsymbol{j}^\top \boldsymbol{r} = \boldsymbol{j}^\top \boldsymbol{R}\boldsymbol{r} = (a_o + b_0, c_1 + a_1 + b_1, ..., c_d + a_d)\boldsymbol{r} = \sum_{k=0}^{d} (a_k + b_k + c_k) r_k;$$

$$xH_0 = \lambda_0 H_0 = \sum_{k=0}^{d} \lambda_0 r_k,$$

(because $(x - \lambda_0)H_0 = \frac{1}{g_0\pi_0} Z$ and in $\mathbb{R}[x]/\langle Z\rangle$ this mean that $(x - \lambda_0)H_0 = 0$) and, from the linear independence of the polynomials r_k, we get $a_k + b_k + c_k = \lambda_0$. □

(13.07) Proposition (the conjugate polynomials)

Let $\{p_0, p_1, ..., p_d\}$ *be some orthogonal system of polynomials with respect to some inner product* $\langle\star, \star\rangle$ *in space* $\mathbb{R}[x]/\langle Z\rangle$*. Then there exist the so-called conjugate polynomials* $\overline{p}_i$ *of degree* i*, for* $i = 0, 1, ..., d$ *with the property that*

$$p_{d-i}(x) = \overline{p}_i(x) p_d(x) \ \text{ for } \ i = 0, 1, ..., d.$$

Proof: From Proposition 13.03 we have

$$\begin{aligned} xp_0 &= a_0 p_0 + c_1 p_1, \\ xp_i &= b_{i-1} p_{i-1} + a_i p_i + c_{i+1} p_{i+1}, \quad i = 1, 2, ..., d-1, \\ xp_d &= b_{d-1} p_{d-1} + a_d p_d. \end{aligned}$$

Notice that

$$\begin{aligned} xp_d &= b_{d-1} p_{d-1} + a_d p_d, \\ xp_{d-j} &= b_{d-j-1} p_{d-j-1} + a_{d-j} p_{d-j} + c_{d-j+1} p_{d-j+1}, \quad j = 1, 2, ..., d-1, \\ xp_0 &= a_0 p_0 + c_1 p_1, \end{aligned}$$

and from this it is not hard to see that

$$\begin{aligned} p_{d-1} &= \tfrac{1}{b_{d-1}} (x - a_d) p_d, \\ p_{d-j-1} &= \tfrac{1}{b_{d-j-1}} \big((x - a_{d-j}) p_{d-j} - c_{d-j+1} p_{d-j+1}\big), \quad j = 1, 2, ..., d-1. \end{aligned}$$

Now, we shall prove this proposition by induction.

BASIS OF INDUCTION

For $i = 0$ we have $p_d(x) = 1 \cdot p_d(x)$ so if we set $\overline{p}_0(x) = 1$ the result follow. For $i = 1$, since $p_{d-1} = \frac{1}{b_{d-1}}(x - a_d)p_d$, if we set $\overline{p}_1(x) = \frac{1}{b_{d-1}}(x - a_d)$, the result follow.

INDUCTION STEP

Assume that for any $i = 1, 2, ..., k$ we have that there exist some polynomial $\overline{p}_i$ of degree i such that $p_{d-i}(x) = \overline{p}_i(x)p_d(x)$ (this assumption include that $p_{d-k}(x) = \overline{p}_k(x)p_d(x)$ and $p_{d-(k-1)}(x) = \overline{p}_{k-1}(x)p_d(x)$ for some $\overline{p}_k$ and $\overline{p}_{k-1}$). From equation $p_{d-j-1} = \frac{1}{b_{d-j-1}}\big((x - a_{d-j})p_{d-j} - c_{d-j+1}p_{d-j+1}\big)$ that we had above, we have

$$p_{d-(k+1)}(x) = p_{d-k-1}(x) = \frac{1}{b_{d-k-1}}\big((x - a_{d-k})p_{d-k}(x) - c_{d-k+1}p_{d-k+1}(x)\big) =$$

$$= \frac{1}{b_{d-k-1}}\big((x - a_{d-k})\overline{p}_k(x)p_d(x) - c_{d-k+1}\overline{p}_{k-1}(x)p_d(x)\big),$$

which provides the induction step. □

14 The canonical orthogonal system

(14.01) Observation (induced linear functional)

Each real number λ induces a linear functional on $\mathbb{R}_d[x]$, defined by $[\lambda](p) := p(\lambda)$. To see this, notice that $[\lambda] : \mathbb{R}_d[x] \to \mathbb{R}$ and for arbitrary polynomials $p(x), q(x) \in \mathbb{R}_d[x]$ and scalar $\alpha \in \mathbb{R}$ we have

$$[\lambda](p + q) = (p + q)(\lambda) = p(\lambda) + q(\lambda) = [\lambda](p) + [\lambda](q)$$

and

$$[\lambda](\alpha p) = (\alpha p)(\lambda) = \alpha p(\lambda) = \alpha[\lambda](p).$$

◇

(14.02) Observation (basis for dual space $\mathbb{R}_d^*[x]$)

Let $\mathcal{M} = \{\lambda_0, \lambda_1, ..., \lambda_d\}$, $\lambda_0 > \lambda_1 > ... > \lambda_d$, be a mesh of real numbers, and let $g : \mathcal{M} \to \mathbb{R}$ be weight function associated to $\langle p, q\rangle$ ($\langle p, q\rangle := \sum_{\ell=0}^{d} g_\ell p(\lambda_\ell)q(\lambda_\ell)$, where for short we write $g_\ell = g(\lambda_\ell)$) in inner product space $\mathbb{R}_d[x]$. From Proposition 12.06 ($\langle Z_h, Z_k\rangle = \delta_{hk}g_k$) we know that family $\{Z_0, Z_1, ..., Z_d\}$ of interpolating polynomials (with degree d)

$$Z_k := \frac{(-1)^k}{\pi_k} \prod_{\ell=0\,(\ell\neq k)}^{d} (x - \lambda_\ell), \quad (0 \le k \le d),$$

are orthogonal basis.

Now notice that linear functionals $[\lambda_0]$, $[\lambda_1]$,..., $[\lambda_d]$ are the dual basis of the polynomials Z_0, Z_1,...,Z_d. Indeed, suppose that α_0,...,α_d are scalars so that

$$\alpha_0[\lambda_0] + ... + \alpha_d[\lambda_d] = \mathbf{0}$$

(where **0** on the right denotes the zero functional, i.e. the functional which sends everything in $\mathbb{R}_d[x]$ to $0 \in \mathbb{R}$). The above equality above is an equality of maps, which should hold for any $p \in \mathbb{R}_d[x]$, we evaluate either side on. In particular, evaluating both sides on Z_i, we have

$$(\alpha_0[\lambda_0] + ... + \alpha_d[\lambda_d])(Z_i) = \alpha_0[\lambda_0](Z_i) + ... + \alpha_d[\lambda_d](Z_i) = \alpha_i Z_i(\lambda_i) = \alpha_i$$

on the left (by Proposition 12.06 ($Z_k(\lambda_h) = \delta_{hk}$)) and 0 on the right. Thus we see that $\alpha_i = 0$ for each i, so $\{[\lambda_0], ..., [\lambda_d]\}$ is linearly independent.

Now we show that $\{[\lambda_0], ..., [\lambda_d]\}$ spans $\mathbb{R}_d^*[x]$. Let $[\lambda] \in \mathbb{R}_d^*$ be arbitrary. For each i, let β_i denote the scalar $[\lambda](Z_i)$. We claim that

$$[\lambda] = \beta_0[\lambda_0] + ... + \beta_d[\lambda_d].$$

Again, this means that both sides should give the same result when evaluating on any $p \in \mathbb{R}_d$. By linearity, it suffices to check that this is true on the basis $\{Z_0, ..., Z_d\}$. Indeed, for each i we have

$$(\beta_0[\lambda_0] + ... + \beta_d[\lambda_d])(Z_i) = \beta_0[\lambda_0](Z_i) + ... + \beta_d[\lambda_d](Z_d) = \beta_i = [\lambda](Z_i),$$

again by the Proposition 12.06 ($Z_k(\lambda_h) = \delta_{hk}$) and definition the β_i. Thus, $[\lambda]$ and $\beta_0[\lambda_0] + ... + \beta_d[\lambda_d]$ agree on the basis, so we conclude that they are equal as elements of $\mathbb{R}_d^*$. Hence $\{[\lambda_0], ..., [\lambda_d]\}$ spans $\mathbb{R}_d^*$ and therefore forms a basis of $\mathbb{R}_d^*$. ⋄

Proof of next theorem can be find in almost any book of linear algebra (for example see [2], page 117).

(14.03) Theorem

Suppose φ is a linear functional on $\mathcal{V}$. Then there is a unique vector $v \in \mathcal{V}$ such that $\varphi(u) = \langle u, v\rangle$ for every $u \in \mathcal{V}$.

(14.04) Observation (induced isomorphism between the space $\mathbb{R}_d[x]$ and its dual)

The scalar product associated to $(\mathcal{M}, g)$ induces an isomorphism between the space $\mathbb{R}_d[x]$ and its dual, where each polynomial p corresponds to the functional ω_p, defined as $\omega_p(q) := \langle p, q\rangle$ and, conversely, each form ω is associated to a polynomial p_ω through $\langle q, p_\omega\rangle = \omega(q)$.

Indeed, consider mapping $\omega : \mathbb{R}_d[x] \to \mathbb{R}_d^*[x]$ defined by $\omega(p) = \omega_p$ where $\omega_p(\star) = \langle \star, p\rangle$. To show that ω is bijection, pick arbitrary $\varphi \in \mathbb{R}_d^*[x]$. From Theorem 14.03 we know that there is a unique polynomial $p \in \mathbb{R}_d[x]$ such that $\varphi(q) = \langle q, p\rangle$ for every $q \in \mathbb{R}_d[x]$. Since $\omega_p(\star) = \langle \star, p\rangle$ we have that

$$\varphi(\star) = \langle \star, p\rangle = \omega_p(\star)$$

that is there is polynomial $p \in \mathbb{R}_d[x]$ such that $\varphi = \omega_p$. Linearity of ω is obvious.

From Theorem 14.03 we know that for arbitrary $\omega \in \mathbb{R}_d^*[x]$ there exist unique polynomial $p \in \mathbb{R}_d[x]$ such that $\omega(q) = \langle q, p\rangle$ for every $q \in \mathbb{R}_d[x]$. In different notation $\omega(\star) = \langle \star, p\rangle$. Now we can define mapping $P : \mathbb{R}_d^*[x] \to \mathbb{R}_d[x]$ with $P(\omega) = p_\omega$ where p_ω is unique polynomial, from Theorem 14.03, such that $\omega(q) = \langle q, p_\omega\rangle$ for every $q \in \mathbb{R}_d[x]$. Now it is not hard to see that P is isomorphism. ⋄

(14.05) Observation (expressions for ω_p and p_ω)

By observing how the isomorphism acts on the bases $\{[\lambda_\ell]\}_{0\le\ell\le d}$, $\{Z_\ell\}_{0\le\ell\le d}$, we get the expressions:

$$\omega_p = \sum_{\ell=0}^{d} g(\lambda_\ell)p(\lambda_\ell)[\lambda_\ell], \qquad p_\omega = \sum_{\ell=0}^{d} \frac{1}{g(\lambda_\ell)}\omega(Z_\ell)Z_\ell.$$

Indeed, consider isomorphism $\omega : \mathbb{R}_d[x] \to \mathbb{R}_d^*[x]$ defined by $\omega(p) = \omega_p$ where $\omega_p(\star) = \langle \star, p\rangle$. Pick arbitrary $p \in \mathbb{R}_d[x]$. Since $\{[\lambda_\ell]\}_{0\le\ell\le d}$, is basis for $\mathbb{R}_d^*[x]$ there exist unique scalars β_0, ..., β_d such that

$$\omega(p) = \omega_p = \beta_0[\lambda_0] + ... + \beta_d[\lambda_d].$$

For short we set $g_\ell = g(\lambda_\ell)$. For every $q \in \mathbb{R}_d[x]$ we have

$$\omega_p(q) = \langle p, q\rangle = \sum_{\ell=0}^{d} g_\ell p(\lambda_\ell)g(\lambda_\ell) = \sum_{\ell=0}^{d} g_\ell p(\lambda_\ell)[\lambda_\ell](g)$$

From last two equations we can conclude that $\beta_0 = g_0 p(\lambda_0)$, ..., $\beta_d = g_d p(\lambda_d)$, that is

$$\omega_p = \sum_{\ell=0}^{d} g_\ell p(\lambda_\ell)[\lambda_\ell].$$

Now consider isomorphism $P : \mathbb{R}_d^*[x] \to \mathbb{R}_d[x]$ defined with $P(\omega) = p_\omega$ where p_ω is unique polynomial (see Theorem 14.03) such that $\omega(q) = \langle q, p_\omega \rangle$ for every $q \in \mathbb{R}_d[x]$. Pick arbitrary $\omega \in \mathbb{R}_d^*[x]$. Since $\{Z_\ell\}_{0 \le \ell \le d}$ is basis for $\mathbb{R}_d[x]$ there exist unique scalars γ_0, ..., γ_d such that

$$P(\omega) = p_\omega = \gamma_0 Z_0 + ... + \gamma_d Z_d.$$

We know that $\omega(q) = \langle q, p_\omega \rangle$ for every $q \in \mathbb{R}_d[x]$. If, for q we pick up Z_ℓ we have $\omega(Z_\ell) = \langle Z_\ell, p_\omega \rangle = \langle Z_\ell, \gamma_0 Z_0 + ... + \gamma_d Z_d \rangle$, and since $\langle Z_h, Z_k \rangle = \delta_{hk} g_k$ we obtain $\omega(Z_\ell) = g_\ell \gamma_\ell$. We see that $\gamma_\ell = \frac{1}{g_\ell}\omega(Z_\ell)$ and therefore

$$p_\omega = \sum_{\ell=0}^{d} \frac{1}{g_\ell} \omega(Z_\ell) Z_\ell.$$

(14.06) Observation (polynomial corresponding to $[\lambda_k]$ and their scalar products) In particular, for short we set $g_\ell = g(\lambda_\ell)$, the polynomial corresponding to $[\lambda_k]$ is

$$H_k := p_{[\lambda_k]} = \sum_{\ell=0}^{d} \frac{1}{g_\ell} [\lambda_k](Z_\ell) Z_\ell = \sum_{\ell=0}^{d} \frac{1}{g_\ell} \delta_{\ell k} Z_\ell =$$

$$= \frac{1}{g_k} Z_k = \frac{(-1)^k}{g_k \pi_k} (x - \lambda_0) ... \widehat{(x - \lambda_k)} ... (x - \lambda_d),$$

(where $\widehat{(x - \lambda_k)}$ denotes that this factor is not present in the product) and their scalar products are

$$\langle H_h, H_k \rangle = \sum_{\ell=0}^{d} g_\ell H_h(\lambda_\ell) H_k(\lambda_\ell) = g_h \frac{1}{g_h} Z_h(\lambda_h) H_k(\lambda_h) = H_k(\lambda_h) = \frac{1}{g_h} Z_k(\lambda_h) = \frac{1}{g_h} \delta_{hk}.$$

Moreover, property

$$\sum_{k=0}^{d} \frac{(-1)^k}{\pi_k} \lambda_k^i = 0 \;\; (0 \le i \le d-1),$$

(see Proposition 12.08) is equivalent to stating that the form $\sum_{k=0}^{d} \frac{(-1)^k}{\pi_k} [\lambda_k]$ annihilates on the space $\mathbb{R}_{d-1}[x]$. Indeed, for arbitrary polynomial $p(x) = a_{d-1} x^{d-1} + ... + a_1 x + a_0$ of degree $d - 1$ we have

$$\sum_{k=0}^{d} \frac{(-1)^k}{\pi_k} [\lambda_k](p(x)) = \sum_{k=0}^{d} \frac{(-1)^k}{\pi_k} [\lambda_k](a_{d-1} x^{d-1} + ... + a_1 x + a_0) =$$

$$= a_{d-1} \sum_{k=0}^{d} \frac{(-1)^k}{\pi_k} [\lambda_k] x^{d-1} + ... + a_1 \sum_{k=0}^{d} \frac{(-1)^k}{\pi_k} [\lambda_k] x + a_0 \sum_{k=0}^{d} \frac{(-1)^k}{\pi_k} [\lambda_k] x^0 = 0.$$

◇

(14.07) Comment (some notation from linear algebra)

The sum of two subspaces $\mathcal{X}$ and $\mathcal{Y}$ of a vector space $\mathcal{V}$ is defined to be the set $\mathcal{X}+\mathcal{Y}=\{x+y \,|\, x\in\mathcal{X} \text{ and } y\in\mathcal{Y}\}$, and it is not hard to establish that $\mathcal{X}+\mathcal{Y}$ is another subspace of $\mathcal{V}$. Subspaces $\mathcal{X}$, $\mathcal{Y}$ of a space $\mathcal{V}$ are said to be *complementary* whenever

$$\mathcal{V}=\mathcal{X}+\mathcal{Y} \quad \text{and} \quad \mathcal{X}\cap\mathcal{Y}=\mathbf{0},$$

in which case $\mathcal{V}$ is said to be the direct sum of $\mathcal{X}$ and $\mathcal{Y}$, and this is denoted by writing $\mathcal{V}=\mathcal{X}\oplus\mathcal{Y}$.

For a subset $\mathcal{L}$ of an inner-product space $\mathcal{V}$, the *orthogonal complement* $\mathcal{L}^{\perp}$ (pronounced "$\mathcal{L}$ perp") of $\mathcal{L}$ is defined to be the set of all vectors in $\mathcal{V}$ that are orthogonal to every vector in $\mathcal{L}$. That is,

$$\mathcal{L}^{\perp}=\{x\in\mathcal{V} \,|\, \langle m,x\rangle=0 \text{ for all } m\in\mathcal{L}\}.$$

If $\mathcal{L}$ is a subspace of a finite-dimensional inner-product space $\mathcal{V}$, then

$$\mathcal{V}=\mathcal{L}\oplus\mathcal{L}^{\perp}.$$

For $v\in\mathcal{V}$, let $v=m+n$, where $m\in\mathcal{L}$ and $n\in\mathcal{L}^{\perp}$. Vector m is called the *orthogonal projection* of v onto $\mathcal{L}$.

If $\mathcal{L}$ is a subspace of an n-dimensional inner-product space, then it is not hard to show that $\dim\mathcal{L}^{\perp}=n-\dim\mathcal{L}$ and $\mathcal{L}^{\perp^{\perp}}=\mathcal{L}$ (proof see in [37], page 404). If $\mathcal{L}_1$ and $\mathcal{L}_2$ are subspaces of an n-dimensional inner-product space, then the following statements are true:

(*i*) $\mathcal{L}_1\subseteq\mathcal{L}_2 \Longrightarrow \mathcal{L}_2^{\perp}\subseteq\mathcal{L}_1^{\perp}$.
(*ii*) $(\mathcal{L}_1+\mathcal{L}_2)^{\perp}=\mathcal{L}_1^{\perp}\cap\mathcal{L}_2^{\perp}$.
(*iii*) $(\mathcal{L}_1\cap\mathcal{L}_2)^{\perp}=\mathcal{L}_1^{\perp}+\mathcal{L}_2^{\perp}$. ◊

(14.08) Observation (the functional $[\lambda_0]$ is represented by the polynomial H_0)

Consider the space $\mathbb{R}_d[x]$ with the scalar product associated to $(\mathcal{M},g)$ ($\langle p,q\rangle:=\sum_{\ell=0}^{d} g_\ell p(\lambda_\ell)q(\lambda_\ell)$). From the identification of such space with its dual (see Observation 14.04) by contraction of the scalar product, the functional $[\lambda_0]: p\to p(\lambda_0)$ is represented by the polynomial $H_0=\frac{1}{g_0\pi_0}(x-\lambda_1)...(x-\lambda_d)$ through $\langle H_0,p\rangle=p(\lambda_0)$.

Indeed, from Theorem 14.03 we know that for arbitrary $\omega\in\mathbb{R}_d^*[x]$ there exist unique polynomial $p_\omega\in\mathbb{R}_d[x]$ such that $\omega(q)=\langle q,p_\omega\rangle$ for every $q\in\mathbb{R}_d[x]$. In different notation $\omega(\star)=\langle\star,p_\omega\rangle$.

For functional $[\lambda_0]: p\to p(\lambda_0)$ on $\mathbb{R}_d$ there exist unique polynomial $p_{[\lambda_0]}$ such that $[\lambda_0](q)=\langle q,p_{[\lambda_0]}\rangle$ for every $q\in\mathbb{R}_d[x]$. We want to evaluate $p_{[\lambda_0]}$. From Observation 14.05 and Proposition 12.06 we have

$$p_{[\lambda_0]}=\sum_{\ell=0}^{d}\frac{1}{g(\lambda_\ell)}[\lambda_0](Z_\ell)Z_\ell=\sum_{\ell=0}^{d}\frac{1}{g(\lambda_\ell)}Z_\ell(\lambda_0)Z_\ell=$$

$$\frac{1}{g_0}Z_0(\lambda_0)Z_0+\frac{1}{g_1}Z_1(\lambda_0)Z_1+...+\frac{1}{g_d}Z_d(\lambda_0)Z_d=\frac{1}{g_0}Z_0(\lambda_0)Z_0=\frac{1}{g_0}Z_0=H_0.$$

◊

(14.09) Definition (orthogonal projection of H_0 onto $\mathbb{R}_k[x]$))

For any given $0\le k\le d-1$, let $q_k\in\mathbb{R}_k[x]$ denote the orthogonal projection of H_0 onto $\mathbb{R}_k[x]$. With another words

$$\mathbb{R}_d[x]=\mathbb{R}_0[x]\oplus\mathbb{R}_0^{\perp}[x],\;\; H_0=q_0+t_0 \text{ where } q_0\in\mathbb{R}_0[x],\, t_0\in\mathbb{R}_0^{\perp}[x],$$

$$\mathbb{R}_d[x]=\mathbb{R}_1[x]\oplus\mathbb{R}_1^{\perp}[x],\;\; H_0=q_1+t_1 \text{ where } q_1\in\mathbb{R}_1[x],\, t_1\in\mathbb{R}_1^{\perp}[x],$$

$$\vdots$$

$$\mathbb{R}_d[x] = \mathbb{R}_{d-1}[x] \oplus \mathbb{R}_{d-1}^{\perp}[x], \;\; H_0 = q_{d-1} + t_{d-1} \text{ where } q_{d-1} \in \mathbb{R}_{d-1}[x], \, t_{d-1} \in \mathbb{R}_{d-1}^{\perp}[x]$$

(see Figure 46). ◇

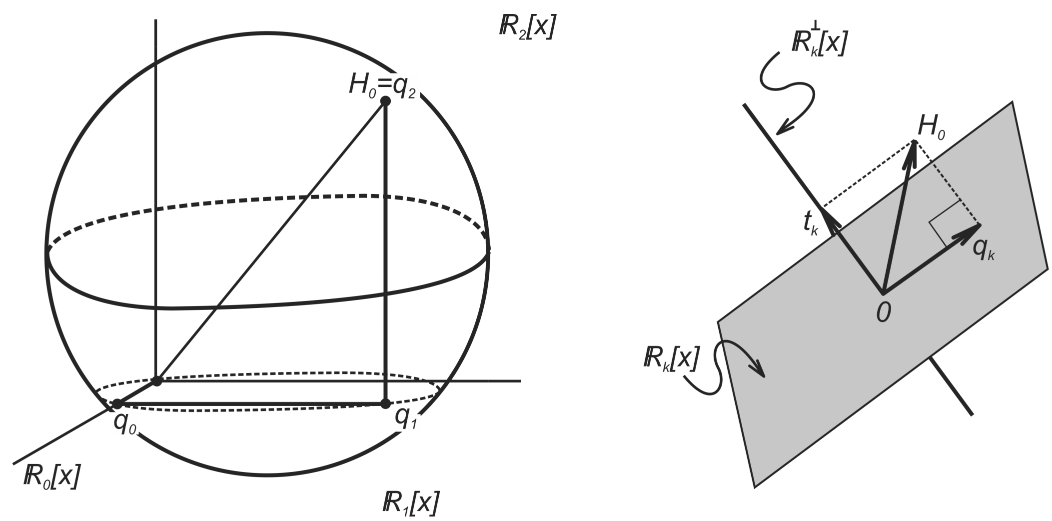


FIGURE 46
Obtaining the q_k by projecting H_0 onto $\mathbb{R}_k[x]$.

We know that we can think of flat surfaces passing through the origin whenever we encounter the term "subspace" in higher dimensions. Alternatively, the polynomial q_k can be defined on following way.

(14.10) Theorem (closest point theorem)

The unique vector in $\mathbb{R}_k[x]$ that is closest to H_0 is q_k, the orthogonal projection of H_0 onto $\mathbb{R}_k[x]$. In other words,

$$\|H_0 - q_k\| = \min_{q \in \mathbb{R}_k[x]} \|H_0 - q\|$$

(see Figure 47).

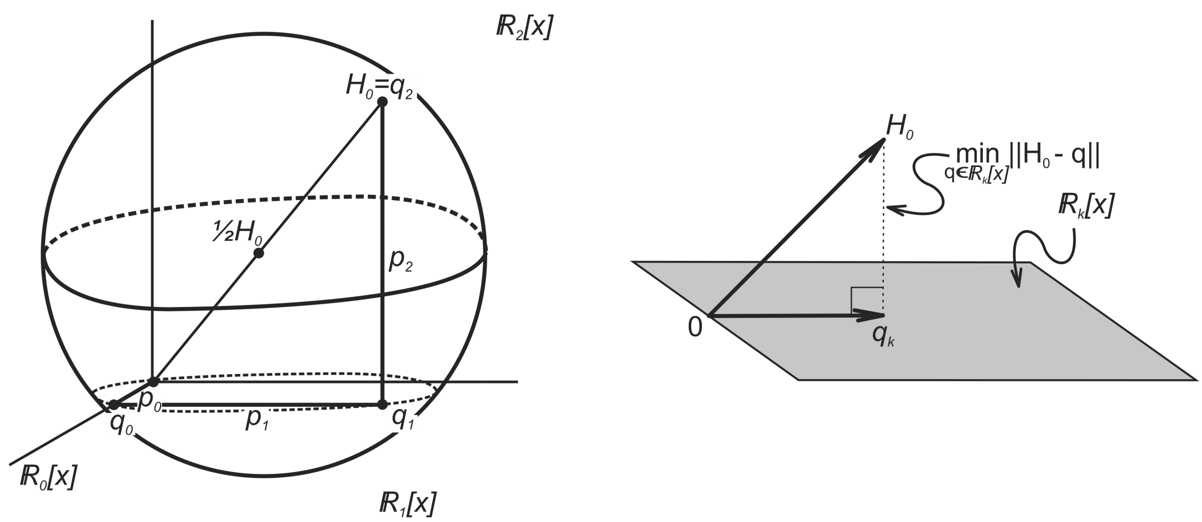


FIGURE 47
Obtaining the q's as closes points to H_0.

Proof: If q_k is orthogonal projection of H_0 onto $\mathbb{R}_k[x]$, then $q_k - m \in \mathbb{R}_k[x]$ for all $m \in \mathbb{R}_k[x]$, and $H_0 = q_k + t_k$ (for unique $q_k \in \mathbb{R}_k[x]$ and $t_k \in \mathbb{R}_k^{\perp}[x]$) so

$$H_0 - q_k \in \mathbb{R}_k^{\perp}[x],$$

and $(q_k - m) \perp (H_0 - q_k)$. The Pythagorean theorem says $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ whenever $x \perp y$, and hence

$$\|H_0 - m\|^2 = \|H_0 - q_k + q_k - m\|^2 = \|H_0 - q_k\|^2 + \|q_k - m\|^2 \geq \|H_0 - q_k\|^2.$$

In other words, $\min\limits_{m \in \mathbb{R}_k[x]} \|H_0 - m\| = \|H_0 - q_k\|$.

Now argue that there is not another point in $\mathbb{R}_k[x]$ that is as close to H_0 as q_k is. If $\widehat{m} \in \mathbb{R}_k[x]$ such that $\|H_0 - \widehat{m}\| = \|H_0 - q_k\|$, then by using the Pythagorean theorem again we see

$$\|H_0 - \widehat{m}\|^2 = \|H_0 - q_k + q_k - \widehat{m}\|^2 = \|H_0 - q_k\|^2 + \|q_k - \widehat{m}\|^2 \quad \Longrightarrow \quad \|q_k - \widehat{m}\| = 0,$$

and thus $\widehat{m} = q_k$. □

Let $\mathcal{S}$ denote the sphere in $\mathbb{R}_d[x]$ such that 0 and H_0 are antipodal points on it; that is, the sphere with center $\frac{1}{2}H_0$ and radius $\frac{1}{2}\|H_0\|$

$$\mathcal{S} = \left\{ p \in \mathbb{R}_d[x] : \|p - \frac{1}{2}H_0\|^2 = \left(\frac{1}{2}\|H_0\|\right)^2 \right\}$$

(if x and y are points on sphere and distance between them is equal to the diameter of sphere, then y is called an *antipodal point* of x, x is called an antipodal point of y, and the points x and y are said to be antipodal to each other).

(14.11) Lemma

Sphere $\mathcal{S}$ in $\mathbb{R}_d[x]$ such that 0 and H_0 are antipodal points on it can also be written as

$$\mathcal{S} = \left\{ p \in \mathbb{R}_d[x] : \|p\|^2 = p(\lambda_0) \right\} = \{ p \in \mathbb{R}_d[x] : \langle H_0 - p, p \rangle = 0 \}.$$

Proof: We have

$$\|p - \frac{1}{2}H_0\|^2 = \left(\frac{1}{2}\|H_0\|\right)^2,$$

$$\langle p - \frac{1}{2}H_0, p - \frac{1}{2}H_0 \rangle = \frac{1}{4}\|H_0\|^2,$$

$$\langle p, p \rangle - \frac{1}{2}\langle p, H_0 \rangle + \langle \frac{1}{2}H_0, p \rangle + \frac{1}{4}\langle H_0, H_0 \rangle = \frac{1}{4}\|H_0\|^2,$$

$$\langle p, p \rangle - \langle H_0, p \rangle + \frac{1}{4}\|H_0\|^2 = \frac{1}{4}\|H_0\|^2,$$

$$\|p\|^2 = \langle H_0, p \rangle,$$

and in Observation 14.08 we had $\langle H_0, p \rangle = p(\lambda_0)$, so

$$\|p\|^2 = p(\lambda_0).$$

Since $\langle H_0, p \rangle - \langle p, p \rangle = 0$ we have

$$\langle H_0 - p, p \rangle = 0.$$

□

(14.12) Problem

Prove that the projection q_k is on the sphere $\mathcal{S}_k := \mathcal{S} \cap \mathbb{R}_k[x]$.

Solution: From Lemma 14.11

$$\mathcal{S} = \{p \in \mathbb{R}_d[x] : \langle H_0 - p, p\rangle = 0\}.$$

Since $H_0 - q_k \in \mathbb{R}_k^{\perp}[x]$ and $q_k \in \mathbb{R}_k[x]$ we have

$$\langle H_0 - q_k, q_k\rangle = 0.$$

$\diamondsuit$

(14.13) Problem

Prove that $\mathcal{S}_0 = \{0, 1\}$.

Solution: We have $\mathcal{S} = \{p \in \mathbb{R}_d[x] : \|p\| = p(\lambda_0)\}$ and $\mathcal{S}_0 := \mathcal{S} \cap \mathbb{R}_0[x]$. Notice that $\mathbb{R}_0[x] = \{\alpha : \alpha \in \mathbb{R}\}$, and pick arbitrary $p \in \mathcal{S}_0$. Then we have $p = \alpha$ for some $\alpha \in \mathbb{R}$. Equation $\|p\| = p(\lambda_0)$ imply $\langle p, p\rangle = p(\lambda_0)$, so

$$\sum_{\ell=0}^{d} g_\ell\, p(\lambda_\ell)\, p(\lambda_\ell) = p(\lambda_0),$$

$$g_0\alpha^2 + g_1\alpha^2 + ... + g_d\alpha^2 = \alpha,$$

$$\alpha^2 = \alpha,$$

$$\alpha^2 - \alpha = 0,$$

$$\alpha(\alpha - 1) = 0 \quad \Rightarrow \quad \alpha = 0 \quad \text{or} \quad \alpha = 1.$$

Therefore $\mathcal{S}_0 = \{0, 1\}$. $\diamondsuit$

(14.14) Problem

Let $\mathbb{R}_n[x]$ represent the vector space of polynomials (with coefficients in $\mathbb{R}$) whose degree is at most n. For every $a \in \mathbb{R}$ let $\mathcal{U}_a = \{p \in \mathbb{R}[x] : p(a) = 0\}$.

(a) Find a basis of $\mathcal{M} = \mathcal{U}_a \cap \mathbb{R}_n[x]$ for all $a \in \mathbb{R}$;

(b) Show that $(\mathcal{U}_3 + \mathcal{U}_4) \cap \mathbb{R}_n[x] = \mathbb{R}_n[x]$.

Solution: (a) Since $\mathcal{U}_a$ is space of all polynimials whose root is a, and $\mathbb{R}_n[x]$ is space of all polynomials with degree at most n, we have that $\mathcal{M} = \mathcal{U}_a \cap \mathbb{R}_n[x]$ is space of all polynomials with degree at most n, whose root is a. Elements from $\mathcal{M}$ are in form $(x - a)(\alpha_{n-1}x^{n-1} + ... + \alpha_1 x + \alpha_0)$ for some $\alpha_{n-1}, ..., \alpha_1, \alpha_0 \in \mathbb{R}$. Now it is not hard to prove that $\{x - a, (x - a)x, (x - a)x^2, ..., (x - a)x^{n-1}\}$ is basis of $\mathcal{M}$ (this basis we have obtained by multiplying the standard basis $\{1, x, x^2, ..., x^{n-1}\}$ of $\mathbb{R}_{n-1}[x]$ by $x - a$).

(b) Notice that $\mathcal{U}_3$ is space of all polynomials whose root is 3, and $\mathcal{U}_4$ is space of all polynomials whose root is 4, and elements of $\mathcal{U}_3 + \mathcal{U}_4$ are in the form $(x - 3)q_1(x) + (x - 4)q_2(x)$ where $q_1(x), q_2(x)$ are some polynomials (of arbitrary degree). For arbitrary polynomial $p \in \mathbb{R}_n[x]$ we have

$$p(x) = 1 \cdot p(x) = ((x - 3) - (x - 4))p(x) = (x - 3)p(x) + (x - 4)(-p(x)) \in \mathcal{U}_3 + \mathcal{U}_4.$$

The result follow. $\diamondsuit$

(14.15) Problem

Let $p \in \mathbb{R}_{n-1}[x]$ be a polynomial of degree $n-1$ $(n-1 \geqslant 0)$.

(a) Let $\mathbb{R}_{n-1}[x]$ be the vector space of polynomials with degree $\leqslant n-1$ over $\mathbb{R}$. Show that $\{p(x), p(x+1), \ldots, p(x+n-1)\}$ is a basis of $\mathbb{R}_{n-1}[x]$.

(b) Let $M_n = \begin{pmatrix} p(x) & p(x+1) & p(x+2) & \ldots & p(x+n) \\ p(x+1) & p(x+2) & p(x+3) & \ldots & p(x+n+1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p(x+n) & p(x+n+1) & p(x+n+2) & \ldots & p(x+2n) \end{pmatrix}$. *Show that* $\det M_n = 0$ *for every* $x \in \mathbb{R}$.

Solution: (a) We will prove this using mathematical induction.

BASIS OF INDUCTION

Consider case when $n = 2$. We then have that $p \in \mathbb{R}_1[x]$ and degree of p is 1, for example let $p(x) = ax + b$ for some $a, b \in \mathbb{R}$ $(a \neq 0)$. We want to show that $\{p(x), p(x+1)\}$ is basis of $\mathbb{R}_1[x]$.

First notice that $p(x+1) = a(x+1) + b = p(x) + a$. Consider equation $\alpha p(x) + \beta p(x+1) = 0$. From this we have $(\alpha + \beta)p(x) + \beta a = 0$, from which it follow that $\alpha = \beta = 0$. That is $\{p(x), p(x+1)\}$ is linear independent set. Now pick arbitrary $r(x) \in \mathbb{R}_1[x]$ (for example let $r(x) = cx + d$ for some $c, d \in \mathbb{R}$). We want to find α and β such that $r(x) = \alpha p(x) + \beta p(x+1)$. This imply

$$cx + d = (\alpha + \beta)(ax + b) + \beta a,$$

$$\alpha = \frac{c}{a} - \beta, \qquad \beta = \frac{d}{a} - \frac{bc}{a^2},$$

that is $r(x) \in \operatorname{span}\{p(x), p(x+1)\}$. Therefore, $\{p(x), p(x+1)\}$ is basis of $\mathbb{R}_1[x]$.

INDUCTION STEP

Assume that the result holds for $n-2 \geq 1$ that is assume that for arbitrary polynomial $p \in \mathbb{R}_{n-2}[x]$ of degree $n-2$ set $\{p(x), p(x+1), ..., p(x+n-2)\}$ is a basis of $\mathbb{R}_{n-2}[x]$, and use this assumption to show that $\{p(x), p(x+1), \ldots, p(x+n-1)\}$ is a basis of $\mathbb{R}_{n-1}[x]$.

Let $q(x) = p(x+1) - p(x)$. Then $\operatorname{dgr} q = \operatorname{dgr} p - 1$. Indeed, if $p(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0$, then

$$p(x+1) - p(x) = (a_{n-1}(x+1)^{n-1} + a_{n-2}(x+1)^{n-2} + ... + a_1(x+1) + a_0) -$$

$$-(a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0) = (n-1)a_{n-1}x^{n-2} + ...$$

By induction hypothesis $\{q(x), ..., q(x+n-2)\}$ generate $\mathbb{R}_{n-2}[x]$. Thus for any polynomial $r(x) = b_{n-1}x^{n-1} + ... + b_1 x + b_0 \in \mathbb{R}_{n-1}[x]$, we can express $r[x] - \frac{b_{n-1}}{a_{n-1}}p(x)$ as linear combination of $q(x)$,...,$q(x+n-2)$, hence of $p(x)$, ..., $p(x+n-1)$ and finally also $r(x)$ as such linear combination.

(b) An easy approach will be to use the idea of Method of Finite Differences for a polynomial. For $p(x)$ these polynomials are $p(x)$, $p_1(x) = p(x+1) - p(x)$, $p_2(x) = p_1(x+1) - p_1(x) = p(x+2) - 2p(x+1) + p(x)$,... (each of them have a different degree). The rows (and columns) satisfy the condition that their nth method of difference is equal to 0. This gives as coefficients, which shows that the columns (rows) are not linearly independent, so the matrix has determinant 0.

For clarity, the nth difference tells us that for any j,

$$0 = \binom{n}{0} p(x+j+n) - \binom{n}{1} p(x+j+n-1) + \binom{n}{1} p(x+j+n-2) - ...+$$

$$+(-1)^{n-1}\binom{n}{n-1}p(x+j+1)+(-1)^n p(x+1).$$

So we will take $\binom{n}{0}$ as the coefficients for the linear combination that is 0. ◊

(14.16) Problem

Prove that space $\mathbb{R}_k[x]\cap\mathbb{R}_{k-1}[x]^\perp$ has dimension one.

Solution: The dimensions of the orthogonal complement of $\mathbb{R}_k[x]$ is $d-k$ ($\dim(\mathbb{R}_k^\perp[x])=d-k$), and the dimensions of the orthogonal complement of $\mathbb{R}_{k-1}[x]^\perp$ is $k-1$ ($\dim(\mathbb{R}_{k-1}[x])=k-1$). Notice that $\dim(\mathbb{R}_k^\perp[x]\cap\mathbb{R}_{k-1}[x])=\emptyset$ (for illustration see Figure 48). The linear subspace generated by the orthogonal complements has dimension exactly $d-k+k-1=d-1$ ($\dim(\mathbb{R}_k^\perp[x]+\mathbb{R}_{k-1}[x])=d-1$). On the end, notice that $(\mathbb{R}_k[x]\cap\mathbb{R}_{k-1}[x]^\perp)^\perp=\mathbb{R}_k^\perp[x]+\mathbb{R}_{k-1}[x]$ (see Comment 14.07)). The result follow.

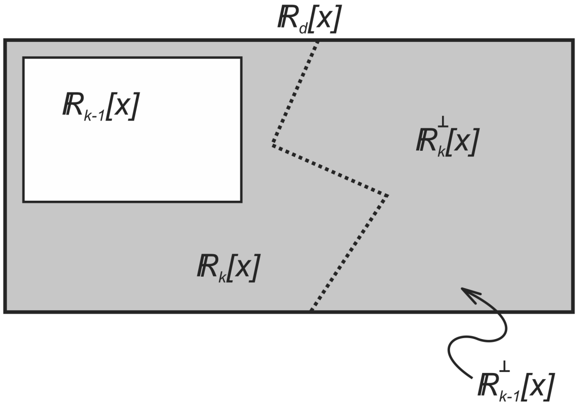


FIGURE 48
Vector space $R_d[x]=R_k[x]\oplus R_k^\perp[x]=R_{k-1}[x]\oplus R_{k-1}^\perp[x]$. ◊

(14.17) Proposition

The polynomial q_k, which is the orthogonal projection of H_0 on $\mathbb{R}_k[x]$, can be defined as the unique polynomial of $\mathbb{R}_k[x]$ satisfying

$$\langle H_0,q_k\rangle=q_k(\lambda_0)=\max\{q(\lambda_0):\ \textit{for all } q\in\mathcal{S}_k\},$$

where $\mathcal{S}_k$ is the sphere $\{q\in\mathbb{R}_k[x]:\|q\|^2=q(\lambda_0)\}$. Equivalently, q_k is the antipodal point of the origin in $\mathcal{S}_k$.

Proof: First notice that
$\mathcal{S}_k:=\mathcal{S}\cap\mathbb{R}_k[x]=\{q\in\mathbb{R}_d[x]:\|q\|^2=q(\lambda_0)\}\cap\mathbb{R}_k[x]=\{q\in\mathbb{R}_k[x]:\|q\|^2=q(\lambda_0)\}$ and since $q_k\in\mathcal{S}_k$ from Theorem 14.10 we have

$$\|H_0-q_k\|^2=\min_{q\in\mathcal{S}_k}\|H_0-q\|.$$

Since q_k is orthogonal to H_0-q_k, we have $\langle H_0-q_k,q_k\rangle=0$ ($\Rightarrow\langle H_0,q_k\rangle=\|q_k\|^2=q_k(\lambda_0)$) and $\|q_k\|^2+\|H_0-q_k\|^2=\|H_0\|^2=\langle H_0,H_0\rangle=\langle\frac{1}{g_0}Z_0,\frac{1}{g_0}Z_0\rangle=\frac{1}{g_0}$. Then, as $q_k\in\mathcal{S}_k$ we get

$$q_k(\lambda_0)=\|q_k\|^2=\frac{1}{g_0}-\|H_0-q_k\|^2=\frac{1}{g_0}-\min_{q\in\mathcal{S}_k}\|H_0-q\|^2.$$

Next, since $\|H_0\|^2=\frac{1}{g_0}$

$$\frac{1}{g_0}-\min_{q\in\mathcal{S}_k}\|H_0-q\|^2=\|H_0\|^2-\min_{q\in\mathcal{S}_k}\|H_0-q\|^2=\max_{q\in\mathcal{S}_k}\|q\|^2=\max_{q\in\mathcal{S}_k}q(\lambda_0)$$

(for illustration see Figure 49).

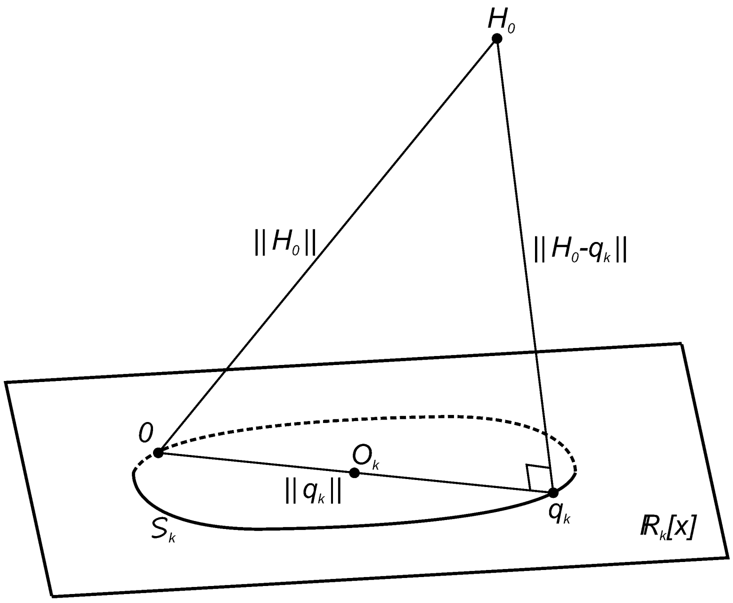


FIGURE 49
Orthogonal projection of H_0 on $\mathbb{R}_k[x]$, sphere $\mathcal{S}_k$ and illustration for $\|H_0\|$, $\|q_k\|$ and $\|H_0 - q_k\|$.

From the above equations we have

$$\|q_k\| = \max_{q \in \mathcal{S}_k} \|q\|,$$

and the proof is complete. □

With the notation $q_d := H_0$, we obtain the family of polynomials q_0, q_1, ..., q_{d-1}, q_d. Let us remark some of their properties.

(14.18) Corollary

The polynomials q_0, q_1, ..., q_{d-1}, q_d, satisfy the following:

(a) Each q_k has degree exactly k.

(b) $1 = q_0(\lambda_0) < q_0(\lambda_1) < ... < q_{d-1}(\lambda_0) < q_d(\lambda_0) = \frac{1}{g_0}$.

(c) The polynomials q_0, q_1, ..., q_{d-1} constitute an orthogonal system with respect to the scalar product associated to the mesh $\{\lambda_1 > \lambda_2 > ... > \lambda_d\}$ and the weight function $\lambda_k \to (\lambda_0 - \lambda_k)g_k$, $k = 1, ..., d$.

Proof: (a) Notice that $\mathcal{S}_0 = \{0, 1\}$ (see Problem 14.13). Consequently, $q_0 = 1$. Assume that q_{k-1} has degree $k-1$, but q_k has degree lesser than k. Because of the uniqueness of the projection and since

$$\|H_0 - q_{k-1}\| = \min_{q \in \mathbb{R}_{k-1}[x]} \|H_0 - q\|, \qquad \|H_0 - q_k\| = \min_{q \in \mathbb{R}_k[x]} \|H_0 - q\|,$$

we would have $q_k = q_{k-1}$, and this imply that $H_0 - q_{k-1}$ would be orthogonal to $\mathbb{R}_k[x]$. In particular,

$$0 = \langle H_0 - q_{k-1}, (x - \lambda_0)q_{k-1} \rangle = \langle (x - \lambda_0)H_0 - (x - \lambda_0)q_{k-1}, q_{k-1} \rangle =$$

$$= \langle (x - \lambda_0)H_0, q_{k-1} \rangle - \langle (x - \lambda_0)q_{k-1}, q_{k-1} \rangle.$$

By definition of inner product

$$\langle (x-\lambda_0)H_0, q_{k-1}\rangle = \sum_{\ell=0}^{d} g_\ell(\lambda_\ell-\lambda_0)H_0(\lambda_\ell)q_{k-1}(\lambda_\ell) = 0.$$

(because $H_0(\lambda_\ell)=0$ for $\ell = 1,2,...,d$). So

$$0 = \langle H_0 - q_{k-1}, (x-\lambda_0)q_{k-1}\rangle = \langle (x-\lambda_0)q_{k-1}, q_{k-1}\rangle = \sum_{\ell=0}^{d} g_\ell(\lambda_\ell-\lambda_0)q_{k-1}^2(\lambda_\ell).$$

Hence, $q_{k-1}(\lambda_\ell)=0$ for any $1\le \ell \le d$ and q_{k-1} would be null (because polynomial q_{k-1} of degree $k-1$ have d roots), a contradiction. The result follows.

(b) Each q_k has degree exactly k and since $q_k(\lambda_0)=\|q_k\|^2$ we have $q_{k-1}(\lambda_0)\le q_k(\lambda_0)$. If $q_{k-1}(\lambda_0)=q_k(\lambda_0)$, from Proposition 14.17 we would get $q_{k-1}=q_k$, which is not possible because of (a). The result follows.

(c) Let $0\le h<k\le d-1$. Since H_0-q_k is orthogonal to $\mathbb{R}_k[x]$ we have, in particular, that

$$0 = \langle H_0-q_k, (x-\lambda_0)q_h\rangle = \langle (x-\lambda_0)H_0-(x-\lambda_0)q_k, q_h\rangle = \langle (\lambda_0-x)q_k, q_h\rangle =$$

$$\sum_{\ell=0}^{d} g_\ell(\lambda_0-\lambda_\ell)q_k(\lambda_\ell)q_h(\lambda_\ell) = \sum_{\ell=0}^{d}(\lambda_0-\lambda_\ell)g_\ell q_k(\lambda_\ell)q_h(\lambda_\ell),$$

establishing the claimed orthogonality. □

The polynomial q_k, as the orthogonal projection of H_0 onto $\mathbb{R}_k[x]$, can also be seen as the orthogonal projection of q_{k+1} onto $R_k[x]$, as $q_{k+1}-q_k = H_0-q_k-(H_0-q_{k+1})$ is orthogonal to $\mathbb{R}_k[x]$ (with another words for arbitrary q_{k+1} there exist unique $q_k\in\mathbb{R}_k[x]$ and $t_k\in\mathbb{R}_k^{\perp}[x]$ such that $q_{k+1}=g_k+t_k$). Consider the family of polynomials defined as

$$\begin{aligned}
&p_0 := q_0 = 1,\\
&p_1 := q_1 - q_0,\\
&p_2 := q_2 - q_1,\\
&\vdots\\
&p_{d-1} := q_{d-1} - q_{d-2},\\
&p_d := q_d - q_{d-1} = H_0 - q_{d-1}.
\end{aligned}$$

Note that, then, $q_k = p_0+p_1+...+p_k$ $(0\le k\le d)$, and, in particular, $p_0+p_1+...+p_d = H_0$. Let us now begin the study of the polynomials $(p_k)_{0\le k\le d}$.

(14.19) Proposition

The polynomials p_0, p_1, ..., p_{d-1}, p_d constitute an orthogonal system with respect to the scalar product associated to $(\mathcal{M}, g)$.

Proof: From $p_k = q_k - q_{k-1}$ we see that p_k has degree k. Moreover, for arbitrary $u\in\mathbb{R}_{k-1}[x]$

$$\langle p_k, u\rangle = \langle q_k-q_{k-1}, u\rangle = \langle H_0-q_{k-1}-(H_0-q_k), u\rangle = \langle H_0-q_{k-1}, u\rangle - \langle H_0-q_k, u\rangle = 0-0=0$$

so $p_k = q_k - q_{k-1}$ is orthogonal to $R_{k-1}[x]$, whence the polynomials p_k form an orthogonal system. □

(14.20) Example

Consider space $\mathbb{R}_3[x]$, let $\lambda_0 = 3$, $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = -3$, $g_0 = g_1 = g_2 = g_3 = 1/4$, and let $\langle p, q\rangle = \sum_{i=0}^{3} g_i p(\lambda_i)q(\lambda_i)$ denote inner product in $\mathbb{R}_3[x]$, $p, q \in \mathbb{R}_3[x]$. Then

$$\pi_0 = (-1)^0(\lambda_0 - \lambda_1)(\lambda_0 - \lambda_2)(\lambda_0 - \lambda_3) = 48,$$

$$H_0 = \frac{1}{\pi_0 g_0}(x - \lambda_1)(x - \lambda_2)(x - \lambda_3) = \frac{1}{12}(x^3 + 3x^2 - x - 3)$$

We want to compute polynomials p_0, p_1, p_2, and p_3 from Proposition 14.19.

First we will compute polynomials q_0, q_1, q_2, and q_3. We know that $q_0 = 1$. Polynomial q_1 is orthogonal projection of H_0 onto $\mathbb{R}_1[x]$ so firs we must find $\mathbb{R}_1^{\perp}[x]$ and then express H_0 as linear combination of polynomials from $\mathbb{R}_1[x]$ and $\mathbb{R}_1^{\perp}[x]$.

Basis for $\mathbb{R}_3[x]$ is $\{1, x, x^2, x^3\}$. Since $\mathbb{R}_1[x] = \text{span}\{1, x\}$, $\dim(\mathbb{R}_1^{\perp}[x]) = 2$, $\mathbb{R}_1^{\perp}[x] = \{r \in \mathbb{R}_3[x] : r \perp \mathbb{R}_1[x]\}$ we want to find scalars α, β, γ and δ such that $\langle \alpha x^3 + \beta x^2 + \gamma x + \delta, 1\rangle = 0$ and $\langle \alpha x^3 + \beta x^2 + \gamma x + \delta, x\rangle = 0$. We have

$$\langle \alpha x^3 + \beta x^2 + \gamma x + \delta, 1\rangle = 5\beta + \delta,$$

$$\langle \alpha x^3 + \beta x^2 + \gamma x + \delta, x\rangle = 41\alpha + 5\gamma.$$

Now it is not hard to compute that $\mathbb{R}_1^{\perp}[x] = \text{span}\{x^3 - \frac{41}{5}x, x^2 - 5\}$, and

$$H_0 = 1 \cdot 1 + \frac{3}{5} \cdot x + \frac{1}{12} \cdot (x^3 - \frac{41}{5}x) + \frac{1}{4} \cdot (x^2 - 5)$$

therefore

$$q_1(x) = 1 + \frac{3}{5}x.$$

Next, polynomial q_2 is orthogonal projection of H_0 onto $\mathbb{R}_2[x]$ so firs we want to find $\mathbb{R}_2^{\perp}[x]$ and then expres H_0 as linear combination of polynomials from $\mathbb{R}_2[x]$ and $\mathbb{R}_2^{\perp}[x]$. Since $\mathbb{R}_2[x] = \text{span}\{1, x, x^2\}$, $\dim(\mathbb{R}_2^{\perp}[x]) = 1$, $\mathbb{R}_2^{\perp}[x] = \{r \in \mathbb{R}_3[x] : r \perp \mathbb{R}_2[x]\}$ we will find scalars α, β, γ and δ such that $\langle \alpha x^3 + \beta x^2 + \gamma x + \delta, 1\rangle = 0$, $\langle \alpha x^3 + \beta x^2 + \gamma x + \delta, x\rangle = 0$ and $\langle \alpha x^3 + \beta x^2 + \gamma x + \delta, x^2\rangle = 0$. We have

$$\langle \alpha x^3 + \beta x^2 + \gamma x + \delta, 1\rangle = 5\beta + \delta,$$

$$\langle \alpha x^3 + \beta x^2 + \gamma x + \delta, x\rangle = 41\alpha + 5\gamma,$$

$$\langle \alpha x^3 + \beta x^2 + \gamma x + \delta, x^2\rangle = 41\beta + 5\delta.$$

Now it is not hard to compute that $\mathbb{R}_2^{\perp}[x] = \text{span}\{x^3 - \frac{41}{5}x, \}$, and

$$H_0 = -\frac{1}{4} \cdot 1 + \frac{3}{5} \cdot x + \frac{1}{4} \cdot x^2 + \frac{1}{12} \cdot (x^3 - \frac{41}{5}x)$$

therefore

$$q_2(x) = -\frac{1}{4} + \frac{3}{5}x + \frac{1}{4}x^2.$$

Since $q_3(x) = H_0$ we have

$$p_0 = 1,$$

$$p_1 = \frac{3}{5}x,$$

$$p_2 = -\frac{5}{4} + \frac{1}{4}x^2,$$

$$p_3 = -\frac{41}{60}x + \frac{1}{12}x^3.$$

On the end notice that we have following properties

(i) $p_0 + p_1 + p_2 + p_3 = H_0$;

(ii) $\|p_0\|^2 = 1 = p_0(3)$, $\|p_1\|^2 = 9/5 = p_1(3)$, $\|p_2\|^2 = 1 = p_2(3)$, $\|p_3\|^2 = 1/5 = p_3(3)$;

(iii) In space $\mathbb{R}[x]/\langle Z\rangle$ where $Z = (x-\lambda_0)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$ we have

$$x\begin{pmatrix}p_0\\p_1\\p_2\\p_3\end{pmatrix} = \underbrace{\begin{pmatrix}0 & 5/3 & 0 & 0\\ 3 & 0 & 12/5 & 0\\ 0 & 4/3 & 0 & 3\\ 0 & 0 & 3/5 & 0\end{pmatrix}}_{=\boldsymbol{R}}\begin{pmatrix}p_0\\p_1\\p_2\\p_3\end{pmatrix};$$

(iv) The entries of the recurrence matrix $\boldsymbol{R}$ associated to $(p_k)_{0\le k\le 3}$, satisfy $a_k + b_k + c_k = \lambda_0$, for any $k = 0, 1, 2, 3$. ♢

(14.21) Definition (canonical orthogonal system)

The sequence of polynomials $(p_k)_{0\le k\le d}$, defined as

$$p_0 := q_0 = 1, \quad p_1 := q_1 - q_0, \quad p_2 := q_2 - q_1, ...,$$

$$p_{d-1} := q_{d-1} - q_{d-2}, \quad p_d := q_d - q_{d-1} = H_0 - q_{d-1}.$$

will be called the canonical orthogonal system associated to $(\mathcal{M}, g)$. ⋄

(14.22) Proposition

Let r_0, r_1, ..., r_{d-1}, r_d be an orthogonal system with respect to the scalar product associated to $(\mathcal{M}, g)$. Then the following assertions are all equivalent:

(a) $(r_k)_{0\le k\le d}$ is the canonical orthogonal system associated to $(\mathcal{M}, g)$;

(b) $r_0 = 1$ and the entries of the recurrence matrix $\boldsymbol{R}$ associated to $(r_k)_{0\le k\le d}$, satisfy $a_k + b_k + c_k = \lambda_0$, for any $k = 0, 1, ..., d$;

(c) $r_0 + r_1 + ... + r_d = H_0$;

(d) $\|r_k\|^2 = r_k(\lambda_0)$ for any $k = 0, 1, ..., d$.

Proof: Let $(p_k)_{0\le k\le d}$ be the canonical orthogonal system associated to $(\mathcal{M}, g)$. Notice that $p_k, r_k \in \mathbb{R}_k[x] \cap \mathbb{R}_{k-1}^{\perp}[x]$. The space $\mathbb{R}_k[x] \cap \mathbb{R}_{k-1}^{\perp}[x]$ has dimension one (see Problem 14.16), and hence the polynomials r_k, p_k are proportional: $r_k = \xi_k p_k$. Let $\boldsymbol{j} := (1, 1, ..., 1)^\top$.

(a) ⇒ (b): We have $r_0 = p_0 = 1$. Consider the recurrence matrix $\boldsymbol{R}$ (Proposition 13.03) associated to the canonical orthogonal system $(r_k)_{0\le k\le d} = (p_k)_{0\le k\le d}$

$$x\boldsymbol{p} := x\begin{pmatrix}p_0\\p_1\\p_2\\\vdots\\p_{d-2}\\p_{d-1}\\p_d\end{pmatrix} = \begin{pmatrix}a_0 & c_1 & 0 & & & & \\ b_0 & a_1 & c_2 & 0 & & & \\ 0 & b_1 & a_2 & \dots & \dots & & \\ & 0 & \vdots & \vdots & \dots & 0 & \\ & & \vdots & \dots & a_{d-2} & c_{d-1} & 0\\ & & & 0 & b_{d-2} & a_{d-1} & c_d\\ & & & & 0 & b_{d-1} & a_d\end{pmatrix}\begin{pmatrix}p_0\\p_1\\p_2\\\vdots\\p_{d-2}\\p_{d-1}\\p_d\end{pmatrix} = \boldsymbol{Rp}.$$

Then, computing xq_d in $\mathbb{R}[x]/\langle Z\rangle$ in two different ways we get:

$$xq_d = x\sum_{k=0}^{d} p_k = x\boldsymbol{j}^\top\boldsymbol{p} = \boldsymbol{j}^\top\boldsymbol{Rp} = (a_o + b_0, c_1 + a_1 + b_1, ..., c_d + a_d)^\top\boldsymbol{p} = \sum_{k=0}^{d}(a_k + b_k + c_k)p_k;$$

$$xq_d = xH_0 = \lambda_0 H_0 = \sum_{k=0}^{d} \lambda_0 p_k,$$

(because $(x - \lambda_0)H_0 = \frac{1}{g_0\pi_0}Z$ and in $\mathbb{R}[x]/\langle Z\rangle$ this mean that $(x-\lambda_0)H_0 = 0$) and, from the linear independence of the polynomials p_k, we get $a_k + b_k + c_k = \lambda_0$.

$(b) \Rightarrow (c)$: Working in $\mathbb{R}[x]/\langle Z\rangle$ and from $x\boldsymbol{r} = \boldsymbol{R}\boldsymbol{r}$, we have:

$$0 = \boldsymbol{j}^\top(x\boldsymbol{r} - \boldsymbol{R}\boldsymbol{r}) = x(\boldsymbol{j}^\top\boldsymbol{r}) - \boldsymbol{j}^\top\boldsymbol{R}\boldsymbol{r} = x(\boldsymbol{j}^\top\boldsymbol{r}) - \lambda_0\boldsymbol{j}^\top\boldsymbol{r} = (x-\lambda_0)\boldsymbol{j}^\top\boldsymbol{r} = (x-\lambda_0)\sum_{k=0}^{d} r_k.$$

Therefore (notice that $(x-\lambda_0)\sum_{k=0}^d r_k = 0$, $(x-\lambda_0)H_0 = 0$ and $H_0, \sum_{k=0}^d r_k \in \mathbb{R}_d[x]$) there exists ξ such that $\sum_{k=0}^d r_k = \xi H_0 = \sum_{k=0}^d \xi p_k$. Since, also, $\sum_{k=0}^d r_k = \sum_{k=0}^d \xi_k p_k$, where $\xi_0 = 1$ (since by assumption we have $r_0 = 1$), it turns out that $\xi_0 = \xi_1 = ... = \xi_d = \xi = 1$.
Consequently, $\sum_{k=0}^d r_k = H_0$.

$(c) \Rightarrow (d)$: $\|r_k\|^2 = \langle r_k, r_k\rangle = \langle r_k, r_0 + r_1 + ... + r_d\rangle = \langle r_k, H_0\rangle = r_k(\lambda_0)$.

$(d) \Rightarrow (a)$: From $r_k = \xi_k p_k$, we have $\xi_k^2\|p_k\|^2 = \|r_k\|^2 = r_k(\lambda_0) = \xi_k p_k(\lambda_0) = \xi_k\|p_k\|^2$. Whence $\xi_k = 1$ and $r_k = p_k$. □

15 Characterizations involving the spectrum

Of course, it would be nice to have characterizations of distance-regularity involving only the spectrum. The first question is: Can we see from the spectrum of a graph whether it is distance-regular? In this context, it has been known for a long time that the answer is 'yes' when $D \le 2$ and 'not' if $D \ge 4$. Indeed, a graph with diameter $D = 2$ is strongly regular iff it is regular (a property that can be identified from the spectrum) and has three distinct eigenvalues ($d = 2$). Some time, the only undecided case has been $D = 3$, but Haemers gave also a negative answer constructing many Hoffman-like counterexamples for this diameter. Thus, in general the spectrum is not sufficient to ensure distance-regularity and, if we want to go further, we must require the graph to satisfy some additional conditions.

To make characterization of DRG which involve the spectrum we first introduce a local version of the predistance polynomials and enunciate a key result involving them: Namely, an upper bound for their value at λ_0 and the characterization of the case when the bound is attained. To construct such polynomials we use diagonal entries of idempotents $\boldsymbol{E}_i$ defined earlier, that is the crossed uv-local multiplicities when $u = v$.

(15.01) Proposition

Let $\mathcal{X}$ and $\mathcal{Y}$ be complementary subspaces of a vector space $\mathcal{V}$. Projector P onto $\mathcal{X}$ along $\mathcal{Y}$, is orthogonal if and only if

$$\langle Pu, v\rangle = \langle u, Pv\rangle \text{ for all } u, v \in \mathcal{V}.$$

Proof: First recall some basic definitions from Linear algebra. Subspaces $\mathcal{X}$, $\mathcal{Y}$ of a space $\mathcal{V}$ are said to be <u>complementary</u> whenever

$$\mathcal{V} = \mathcal{X} + \mathcal{Y} \quad \text{and} \quad \mathcal{X} \cap Y = \{\boldsymbol{0}\},$$

in which case $\mathcal{V}$ is said to be the <u>*direct sum*</u> of $\mathcal{X}$ and $\mathcal{Y}$, and this is denoted by writing $\mathcal{V} = \mathcal{X} \oplus \mathcal{Y}$. This is equivalent to saying that for each $v \in \mathcal{V}$ there are unique vectors $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ such that $v = x + y$. The vector x is called the <u>*projection*</u> of v onto $\mathcal{X}$ along $\mathcal{Y}$. The vector y is called the projection of v onto $\mathcal{Y}$ along $\mathcal{X}$. Operator P defined by $Pv = x$ is unique linear operator and is called the <u>*projector*</u> onto $\mathcal{X}$ along $\mathcal{Y}$. Vector m is called the orthogonal projection of v onto $\mathcal{M}$ if and only if $v = m + n$ where $m \in \mathcal{M}$, $n \in \mathcal{M}^\perp$ and $\mathcal{M} \subseteq \mathcal{V}$. The projector $P_\mathcal{M}$ onto $\mathcal{M}$ along $\mathcal{M}^\perp$ is called the orthogonal projector onto $\mathcal{M}$.

($\Rightarrow$) Suppose first that projector P onto $\mathcal{X}$ along $\mathcal{Y}$, is orthogonal, that is $\mathcal{X} \perp \mathcal{Y}$. In another words

$$\langle x, y \rangle = 0 \text{ for every choice of } x \in \mathcal{X} \text{ and } y \in \mathcal{Y}.$$

Then, since $Pu \in \mathcal{X}$ and $(I - P)u \in \mathcal{Y}$ for every vector $u \in \mathcal{V}$,

$$\langle Pu, (I-P)v \rangle = 0 \quad \text{and} \quad \langle (I-P)u, Pv \rangle = 0 \quad \text{for every choice of } u, v \in \mathcal{V}.$$

Finally

$$\langle Pu, v \rangle = \langle Pu, Pv + (I-P)v \rangle = \langle Pu, Pv \rangle + \langle Pu, (I-P)v \rangle = \langle Pu, Pv \rangle$$

and

$$\langle u, Pv \rangle = \langle Pu + (I-P)u, Pv \rangle = \langle Pu, Pv \rangle + \langle (I-P)u, Pv \rangle = \langle Pu, Pv \rangle$$

for every choice of $u, v \in \mathcal{V}$. Therefore

$$\langle Pu, v \rangle = \langle u, Pv \rangle \quad \forall u, v \in \mathcal{V}.$$

($\Leftarrow$) Conversely, if

$$\langle Pu, v \rangle = \langle u, Pv \rangle \text{ for every choice of } u, v \in \mathcal{V}$$

is in force and $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, then

$$\langle x, y \rangle = \langle Px, y \rangle = \langle x, Py \rangle = \langle x, \mathbf{0} \rangle = 0.$$

□

(15.02) Proposition

Let z_{ui} represents the orthogonal projection of the u-canonical vector $e_u = (0, 0, ..., 0, 1, 0, ..., 0)^\top$ on $\mathcal{E}_i = \ker(A - \lambda_i i)$, that is $z_{ui} := \boldsymbol{E}_i e_u$. Then (u, v) entry of the principal idempotent $\boldsymbol{E}_i$ correspond to the scalar products $\langle z_{ui}, z_{vi} \rangle$ that is

$$(\boldsymbol{E}_i)_{uv} = \langle z_{ui}, z_{vi} \rangle \ \ (u, v \in V).$$

Proof: First we will notice that, from Proposition 5.02(i)

$$\boldsymbol{E}_i^2 = \boldsymbol{E}_i, \tag{30}$$

from Theorem 11.06 that

$$\boldsymbol{E}_i\text{'s are orthogonal projectors onto } \mathcal{E}_i \tag{31}$$

and from Proposition 15.01

$$\langle \boldsymbol{E}_i u, v \rangle = \langle u, \boldsymbol{E}_i v \rangle \text{ for all } u, v \in \mathbb{F}^n. \tag{32}$$

Entries of $\boldsymbol{E}_i$'s, just this time, we will denote by e^i_{uv} that is $\boldsymbol{E}_i = \begin{bmatrix} e^i_{11} & e^i_{12} & \dots & e^i_{1n} \\ e^i_{21} & e^i_{22} & \dots & e^i_{2n} \\ \vdots & \vdots & & \vdots \\ e^i_{n1} & e^i_{n2} & \dots & e^i_{nn} \end{bmatrix}$. For arbitrary $u, v \in V$ we have

$$(\boldsymbol{E}_i)_{uv} = e_u^\top \begin{bmatrix} e^i_{1v} \\ e^i_{2v} \\ \vdots \\ e^i_{uv} \\ \vdots \\ e^i_{nv} \end{bmatrix} = e_u^\top \boldsymbol{E}_i e_v = \langle e_u, \boldsymbol{E}_i e_v \rangle \overset{(30)}{=} \langle e_u, \boldsymbol{E}_i^2 e_v \rangle =$$

$$= \langle e_u, \boldsymbol{E}_i(\boldsymbol{E}_i e_v)\rangle \overset{(32)}{=} \langle \boldsymbol{E}_i e_u, \boldsymbol{E}_i e_v\rangle = \langle z_{ui}, z_{vi}\rangle.$$

□

(15.03) Example

Let $\Gamma = (V, E)$ denote regular graph with λ_0 as his largest eigenvalue. Then multiplicity of λ_0 is 1 and $\boldsymbol{j} = (1, 1, ..., 1)^\top$ is appropriate eigenvalue for λ_0 (see Proposition 4.18). So $U_0 = \frac{1}{\sqrt{n}}\boldsymbol{j}$, and

$$\boldsymbol{E}_0 e_u = U_0 U_0^\top e_u = U_0 \begin{bmatrix} \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \frac{1}{\sqrt{n}} U_0 = \frac{1}{n}\boldsymbol{j}$$

From this it follow $(\boldsymbol{E}_0)_{uv} = \langle \frac{1}{n}\boldsymbol{j}, \frac{1}{n}\boldsymbol{j}\rangle = \frac{1}{n^2}n = 1/n$ for any $u, v \in V$, and hence

$$\boldsymbol{E}_0 = \frac{1}{n}\boldsymbol{J}.$$

◊

(15.04) Proposition (spectral decomposition)

Let $\Gamma = (V, E)$ ($|V| = n$) be a graph with eigenvalues $\lambda_0(= \lambda) > \lambda_1 > ... > \lambda_d$, $\boldsymbol{A}$ be the adjacency matrix of Γ, $\{\boldsymbol{e}_1, \boldsymbol{e}_2, ...\boldsymbol{e}_n\}$ be the canonical base of $\mathbb{R}^n$ and let $(\lambda, \boldsymbol{v})$ be the eigenpair from Perron-Frobenius theorem such that $\boldsymbol{v} = (v_1, v_2, ..., v_n)$ is normalize in such a way that $\min_{i\in V} v_i = 1$. Then for a given vertex $i \in V$ we have the spectral decomposition

$$\boldsymbol{e}_i = \sum_{\ell=0}^{d} z_{i\ell} = \frac{v_i}{\|\boldsymbol{v}\|^2}\boldsymbol{v} + z_i$$

where $z_{i\ell} \in \ker(\boldsymbol{A} - \lambda_\ell I)$ and $z_i \in \boldsymbol{v}^\perp$.

Proof: Let $\mathcal{E}_i$ denote the eigenspace $\mathcal{E}_i = \ker(\boldsymbol{A} - \lambda_i I)$, and let $\dim(\mathcal{E}_i) = m_i$, for $0 \le i \le d$. Since $\boldsymbol{A}$ is real symmetric matrix, it is diagonalizable (Lemma 2.09), and for diagonalizable matrices we have

$$m_0 + m_1 + ... + m_d = n \tag{33}$$

(Lemma 4.02).

Matrix $\boldsymbol{A}$ is symmetric $n \times n$ matrix, so $\boldsymbol{A}$ have n distinct eigenvectors $\mathcal{B} = \{u_1, u_2, ..., u_n\}$ which form orthonormal basis for $\mathbb{R}^n$ (Lemma 2.06). Notice that for every vector $u_i \in \mathcal{B}$ there exist $\mathcal{E}_j$ such that $u_i \in \mathcal{E}_j$. Since $\mathcal{E}_i \cap \mathcal{E}_j = \emptyset$ for $i \ne j$, it is not possible that eigenvector u_i $(1 \le i \le n)$ belongs to different eigenspace. So, by Equation (33), we can divide set $\mathcal{B}$ to sets $\mathcal{B}_0$, $\mathcal{B}_1$, ...,$\mathcal{B}_d$ such that

$$\mathcal{B}_i \text{ is a basis for } \mathcal{E}_i, \quad \mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup ... \cup \mathcal{B}_d \quad \text{and} \quad \mathcal{B}_i \cap \mathcal{B}_j = \emptyset.$$

Let U_i $(1 \le i \le d)$ be a matrices which columns are orthonormal basis for $\ker(\boldsymbol{A} - \lambda_i I)$ i.e. which columns are vectors from $\mathcal{B}_i$, and consider matrix $P = [U_0|U_1|...|U_d]$. We have

$$P^\top P = PP^\top = I,$$

$$I = PP^\top = [U_0|U_1|...|U_d] \begin{bmatrix} U_0^\top \\ \hline U_1^\top \\ \hline \vdots \\ \hline U_d^\top \end{bmatrix} = U_0U_0^\top + U_1U_1^\top + ... + U_dU_d^\top = \boldsymbol{E}_0 + \boldsymbol{E}_1 + ... + \boldsymbol{E}_d,$$

that is

$$\boldsymbol{E}_0 + \boldsymbol{E}_1 + ... + \boldsymbol{E}_d = I.$$

From Definition 4.03 we have that matrices $\boldsymbol{E}_i$ $(1 \le i \le d)$ are known by name principal idempotents. Every of $\boldsymbol{E}_i$ $(1 \le i \le d)$ is $n \times n$ matrix, and if we columns of $\boldsymbol{E}_k$ denote by z_{1k}, z_{2k}, ..., z_{nk} we have

$$\begin{bmatrix} | & | & & | \\ z_{10} & z_{20} & ... & z_{n0} \\ | & | & & | \end{bmatrix} + \begin{bmatrix} | & | & & | \\ z_{11} & z_{21} & ... & z_{n1} \\ | & | & & | \end{bmatrix} + ... + \begin{bmatrix} | & | & & | \\ z_{1d} & z_{2d} & ... & z_{nd} \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \boldsymbol{e}_1 & \boldsymbol{e}_2 & ... & \boldsymbol{e}_n \\ | & | & & | \end{bmatrix}$$

From this it follow that

$$\boldsymbol{e}_i = \sum_{\ell=0}^{d} z_{il}.$$

Let u_{k1}, u_{k2}, ..., u_{kj} be columns of $U_k^\top$. Since

$$\begin{bmatrix} | & | & & | \\ z_{1k} & z_{2k} & ... & z_{nk} \\ | & | & & | \end{bmatrix} = \boldsymbol{E}_k = U_k U_k^\top = U_k \begin{bmatrix} | & | & & | \\ u_{k1} & u_{k2} & ... & v_{kj} \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ U_k u_{k1} & U_k u_{k2} & ... & U_k u_{kj} \\ | & | & & | \end{bmatrix}$$

we have $z_{ik} = U_k u_{ki} \in \ker(A - \lambda_k I)$ and first equalities follow.

For second equation firs notice that λ have geometric multiplicity equal to 1, so $\mathcal{E}_0 = \text{span}\{\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\}$. From this notice that

$$\boldsymbol{E}_0 = U_0 U_0^\top = \frac{1}{\|\boldsymbol{v}\|^2}\boldsymbol{v}\boldsymbol{v}^\top = \frac{1}{\|\boldsymbol{v}\|^2}\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\begin{bmatrix} v_1 & v_2 & ... & v_n \end{bmatrix} = \frac{1}{\|\boldsymbol{v}\|^2}\begin{bmatrix} v_1^2 & v_1 v_2 & ... & v_1 v_n \\ v_1 v_2 & v_2^2 & ... & v_2 v_n \\ \vdots & \vdots & & \vdots \\ v_1 v_n & v_2 v_n & ... & v_n^2 \end{bmatrix}.$$

Since

$$\boldsymbol{e}_i = z_{i0} + z_{i1} + ... + z_{id} = \frac{1}{\|\boldsymbol{v}\|^2} v_i \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \underbrace{z_{i1} + ... + z_{id}}_{=z_i}$$

we have

$$\boldsymbol{e}_i = \frac{v_i}{\|\boldsymbol{v}\|^2}\boldsymbol{v} + z_i.$$

□

(15.05) Definition (uv-local multiplicities, u-local multiplicity of λ_i)

The entries of the idempotents $m_{uv}(\lambda_i) = (\boldsymbol{E}_i)_{uv}$ are called <u>*crossed uv-local multiplicities*</u>. In particular, when $u = v$, $m_u(\lambda_i) = m_{uv}(\lambda_i)$ are the <u>*local multiplicities*</u> of vertex u. ◇

(15.06) Lemma

Let $m_u(\lambda_i)$ be u-local multiplicity of λ_i and $m_{uv}(\lambda_i)$ be uv-local multiplicities. Then

(i) $(\boldsymbol{A}^k)_{uv} = \sum_{i=0}^{d} \lambda_i^k m_{uv}(\lambda_i)$ (the number of closed walks of length k going through vertex u, can be computed in a similar way as the whole number of such rooted walks in Γ is computed by using the "global" multiplicities);

(ii) $\sum_{i=0}^{d} m_u(\lambda_i) = 1$ *(for each vertex u, the u-local multiplicities of all the eigenvalues add up to 1);*

(iii) $\sum_{u \in V} m_u(\lambda_i) = m_i$ *for $i = 0, 1, ..., d$ (the multiplicity of each eigenvalue λ_i is the sum, extended to all vertices, of its local multiplicities).*

Proof: From Proposition 5.02(*iii*) we have

$$\boldsymbol{A}^k = \sum_{i=0}^{d} \lambda_i^k \boldsymbol{E}_i$$

and result for (*i*) follow.

From (*iv*) of the same proposition we have

$$\sum_{i=0}^{d} \boldsymbol{E}_i = I$$

and result for (*ii*) follow.

From Proposition 4.06 we have

$$\text{trace}(\boldsymbol{E}_i) = m(\lambda_i) \quad (i = 0, 1, ..., d)$$

and the result for (*iii*) follow. □

Using the local multiplicities as the values of the weight function, we can now define the (u-)local scalar product and (u-local) predistance polynomials.

(15.07) Definition ((u-)local scalar product, (u-local) predistance polynomials)

Let $\Gamma = (V, E)$ be a simple connected graph with $|V| = n$ and with spectrum $\text{spec}(\Gamma) = \text{spec}(\boldsymbol{A}) = \{\lambda_0^{m_0}, \lambda_1^{m_1}, ..., \lambda_d^{m_d}\}$. We define the *(u-)local scalar product*

$$\langle p, q \rangle_u = (p(\boldsymbol{A})q(\boldsymbol{A}))_{uu} = \sum_{i=0}^{d} m_u(\lambda_i)p(\lambda_i)q(\lambda_i).$$

with normalized weight function $\rho_i := m_u(\lambda_i)$, $0 \le i \le d$, since $\sum_{i=0}^{d} \rho_i = 1$. Associated to this product, we define a new orthogonal sequence of polynomials $\{p_k^u\}_{0 \le k \le d_u}$ (where d_u is the number of eigenvalues $\lambda_i \ne \lambda_0$ such that $m_u(\lambda_i) \ne 0$) with $\text{dgr}\, p_k^u = k$, called the *(u-local) predistance polynomials* normalized in such a way that $\|p_k^u\|_u^2 = p_k^u(\lambda_0)$. ◇

(15.08) Lemma

The scalar product defined in Definition 11.07

$$\langle p, q \rangle = \frac{1}{n}\text{trace}(p(\boldsymbol{A})q(\boldsymbol{A})) = \frac{1}{n}\sum_{k=0}^{d} m_k\, p(\lambda_k)q(\lambda_k)$$

is simply the average, over all vertices, of the local scalar products

$$\langle p, q \rangle = \frac{1}{n}\sum_{u \in V} \langle p, q \rangle_u$$

Proof: We have

$$\frac{1}{n}\sum_{u\in V}\langle p,q\rangle_u = \frac{1}{n}(\langle p,q\rangle_u + \langle p,q\rangle_v + ... + \langle p,q\rangle_z) =$$

$$= \frac{1}{n}\left(\sum_{i=0}^{d} m_u(\lambda_i)p(\lambda_i)q(\lambda_i) + \sum_{i=0}^{d} m_v(\lambda_i)p(\lambda_i)q(\lambda_i) + ... + \sum_{i=0}^{d} m_z(\lambda_i)p(\lambda_i)q(\lambda_i)\right) =$$

$$= \frac{1}{n}\sum_{i=0}^{d}(m_u(\lambda_i) + m_v(\lambda_i) + ... + m_z(\lambda_i))p(\lambda_i)q(\lambda_i) =$$

$$= \frac{1}{n}\sum_{i=0}^{d}\underbrace{\left(\sum_{u\in V} m_u(\lambda_i)\right)}_{=\text{trace}(\boldsymbol{E}_i)=m(\lambda_i)} p(\lambda_i)q(\lambda_i) = \frac{1}{n}\sum_{i=0}^{d} m_i\, p(\lambda_i)q(\lambda_i) = \langle p,q\rangle.$$

□

(15.09) Observation

Because of Proposition 13.06 (u-local) predistance polynomials, satisfying the same properties as the predistance polynomials. For instance,

$$\langle p_k^u, p_\ell^u\rangle_u = \delta_{kl}p_k^u(\lambda_0).$$ ◊

Before presenting the main property of these polynomials, we need to introduce a little more notation. Let $N_k(u)$ be the set of vertices that are at distance not greater than k from u, the so-called <u>k-neighborhood</u> of u (that is $N_k(u) = \Gamma_0(u) \cup \Gamma_1(u) \cup ... \cup \Gamma_k(u) =$ $= \{v : \partial(u,v) \leq k\}$). For any vertex subset U, let $\boldsymbol{\rho}U$ be the <u>*characteristic vector*</u> of U; that is $\boldsymbol{\rho}U := \sum_{u\in U}\boldsymbol{e}_u$ (mapping $\boldsymbol{\rho}$ we had define in Definition 8.03).

(15.10) Lemma

Let $N_k(u)$ be k-neighborhood of vertex u and let $\boldsymbol{\rho}U$ be the characteristic vector of U. Then
(i) $\boldsymbol{\rho}N_k(u)$ is just the u column (or row) of the sum matrix $I + \boldsymbol{A} + ... + \boldsymbol{A}_k$;
(ii) $\|\boldsymbol{\rho}N_k(u)\|^2 = s_k(u) := |N_k(u)|$.

Proof: (i) It is not hard to see that $\boldsymbol{\rho}\Gamma_0(u) = \sum_{v\in\Gamma_0(u)}\boldsymbol{e}_v = (I)_{*u}$, $\boldsymbol{\rho}\Gamma_1(u) = \sum_{v\sim u}\boldsymbol{e}_v = (\boldsymbol{A})_{*u}$, $\boldsymbol{\rho}\Gamma_2(u) = \sum_{v\in\Gamma_2(u)}\boldsymbol{e}_v = (\boldsymbol{A}_2)_{*u}$, ..., $\boldsymbol{\rho}\Gamma_k(u) = \sum_{v\in\Gamma_k(u)}\boldsymbol{e}_v = (\boldsymbol{A}_k)_{*u}$, and the result follow.

(ii) $\|\boldsymbol{\rho}N_k(u)\|^2 = \langle\boldsymbol{\rho}N_k(u), \boldsymbol{\rho}N_k(u)\rangle = (I + \boldsymbol{A} + ... + \boldsymbol{A}_k)_{*u}^\top(I + \boldsymbol{A} + ... + \boldsymbol{A}_k)_{*u} = |N_k(u)|$. □

(15.11) Lemma

Let u be an arbitrary vertex of a simple graph Γ, and let $p \in \mathbb{R}_k[x]$. Then
(i) there exists scalars α_v, $v \in N_k(u)$, such that $p(\boldsymbol{A})\boldsymbol{e}_u = \sum_{v\in N_k(u)}\alpha_v\boldsymbol{e}_v$;
(ii) $\|p\|_u = \|p(\boldsymbol{A})\boldsymbol{e}_u\|$.

Proof: Just this time elements of $p(\boldsymbol{A})$ we will denote by $s_{11}, s_{12}, ..., s_{nn}$ that is

$$p(\boldsymbol{A}) = \begin{bmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \vdots & \vdots & \dots & \vdots \\ s_{n1} & s_{n2} & \dots & s_{nn} \end{bmatrix}. \text{ From this it follow that } p(\boldsymbol{A})\boldsymbol{e}_u = \begin{bmatrix} s_{1u} \\ s_{2u} \\ \vdots \\ s_{nu} \end{bmatrix}.$$

(i) By Lemma 3.01, (u,v)-entry of the matrix $\boldsymbol{A}^\ell$ is the number of walks of length ℓ joining u to v, and because of this, if v is vertex such that $\partial(u,v) > k$ we have $(p(\boldsymbol{A})\boldsymbol{e}_u)_v = 0$. The result follows.

(ii) We have

$$\|p(\boldsymbol{A})\boldsymbol{e}_u\|^2 = \langle p(\boldsymbol{A})\boldsymbol{e}_u, p(\boldsymbol{A})\boldsymbol{e}_u\rangle = \langle \begin{bmatrix} s_{1u} \\ s_{2u} \\ \vdots \\ s_{nu} \end{bmatrix}, \begin{bmatrix} s_{1u} \\ s_{2u} \\ \vdots \\ s_{nu} \end{bmatrix} \rangle = (s_{1u}, s_{2u}, ..., s_{nu})^{\top} \begin{bmatrix} s_{1u} \\ s_{2u} \\ \vdots \\ s_{nu} \end{bmatrix} =$$

$$= (p(\boldsymbol{A})p(\boldsymbol{A}))_{uu} = \|p\|_u.$$

□

(15.12) Lemma

Let u be a fixed vertex of a regular graph Γ. Then, for any polynomial $q \in \mathbb{R}_k[x]$,

$$\frac{q(\lambda_0)}{\|q\|_u} \le \|\boldsymbol{\rho} N_k(u)\|$$

and equality holds if and only if

$$\frac{1}{\|q\|_u} q(\boldsymbol{A})\boldsymbol{e}_u = \frac{1}{\|\boldsymbol{\rho} N_k(u)\|}\boldsymbol{\rho} N_k(u),$$

where $N_k(u)$ is k-neighborhood of vertex u ($N_k(u) = \{v : \partial(u, v) \le k\}$).

Proof: Let $q \in \mathbb{R}_k[x]$ and if we set $p = \frac{q}{\|q\|_u}$ we have $\|p\|_u = 1$. By Lemma 15.11(i) there exists scalars α_j, $j \in N_k(u)$, such that $p(\boldsymbol{A})\boldsymbol{e}_u = \sum_{j\in N_k(u)} \alpha_j \boldsymbol{e}_j$. Then

$$1 = \|p\|_u^2 = \|p(\boldsymbol{A})\boldsymbol{e}_u\|_u^2 = \langle p(\boldsymbol{A})\boldsymbol{e}_u, p(\boldsymbol{A})\boldsymbol{e}_u\rangle = \langle \sum_{j\in N_k(u)} \alpha_j \boldsymbol{e}_j, \sum_{v\in N_k(u)} \alpha_v \boldsymbol{e}_v\rangle = \sum_{j\in N_k(u)} \alpha_j^2$$

that is $\sum_{j\in N_k(u)} \alpha_j^2 = 1$. Next, we want to make projection of $p(\boldsymbol{A})\boldsymbol{e}_u = \sum_{j\in N_k(u)} \alpha_j \boldsymbol{e}_j$ onto $\ker(\boldsymbol{A} - \lambda_0 I)$. By Proposition 2.15, $(\lambda_0, \boldsymbol{j})$ is an eigenpair, so if we use Proposition 15.04 we get

$$p(\boldsymbol{A})\boldsymbol{e}_u = p(\boldsymbol{A})(\frac{1}{n}\boldsymbol{j} + z_u) = \frac{1}{n}p(\boldsymbol{A})\boldsymbol{j} + p(\boldsymbol{A})z_u = \frac{1}{n}p(\lambda_0)\boldsymbol{j} + p(\boldsymbol{A})z_u$$

and

$$\sum_{j\in N_k(u)} \alpha_j \boldsymbol{e}_j = \sum_{j\in N_k(u)} \alpha_j(\frac{1}{n}\boldsymbol{j} + z_j) = \frac{1}{n}\sum_{j\in N_k(u)} \alpha_j \boldsymbol{j} + \sum_{j\in N_k(u)} \alpha_j z_j.$$

Thus, projecting onto $\ker(\boldsymbol{A} - \lambda_0 I)$ we get

$$\frac{1}{n}p(\lambda_0) = \frac{1}{n}\sum_{j\in N_k(u)} \alpha_j \quad \text{whence} \quad p(\lambda_0) = \sum_{j\in N_k(u)} \alpha_j.$$

With $N_k = \{j_1, j_2, ..., j_s\}$ notice that problem of maximize value $p(\lambda_0)$, is equivalent to the following constrained optimization problem:

- maximize $f(j_1, j_2, ..., j_s) = \sum_{j\in N_k(u)} \alpha_j$
- subject to $\sum_{j\in N_k(u)} \alpha_j^2 = 1$.

The absolute maximum turns out to be $\sqrt{\sum_{j\in N_k(u)} 1} = \sqrt{|N_k(u)|} = \|\boldsymbol{\rho} N_k(u)\|$, and it is attained at $\alpha_j = \dfrac{1}{\sqrt{\sum_{j\in N_k(u)} 1}} = \dfrac{1}{\sqrt{|N_k(u)|}} = \dfrac{1}{\sqrt{s_k(u)}}$.

If we set $\alpha_j = \frac{1}{\sqrt{s_k(u)}}$ in equation $p(\boldsymbol{A})\boldsymbol{e}_u = \sum_{j\in N_k(u)} \alpha_j \boldsymbol{e}_j$ we get

$$\frac{q(\boldsymbol{A})}{\|q\|_u}\boldsymbol{e}_u = \sum_{j\in N_k(u)} \frac{1}{\sqrt{s_k(u)}}\boldsymbol{e}_j \overset{Lemma\,15.10}{=\!=\!=} \frac{1}{\|\boldsymbol{\rho} N_k(u)\|}\boldsymbol{\rho} N_k(u)$$

and the result follows. □

(15.13) Lemma

Let $\{p_k^u\}_{0\le k\le d_u}$ be sequence of (u-local) predistance polynomials, let $q_k^u := \sum_{h=0}^k p_h^u$ and let $s_k(u) := |N_k(u)|$. Then

(i) $q_k^u(\lambda_0) = \|q_k^u\|_u^2$;

(ii) $q_k^u(\lambda_0) = s_k(u)$ if and only if $q_k^u(\boldsymbol{A})\boldsymbol{e}_u = \boldsymbol{\rho} N_k(u)$.

Proof: (i) We have

$$q_k^u(\lambda_0) = \sum_{h=0}^k p_h^u(\lambda_0) \overset{def\,\{p_h^u\}}{=\!=\!=} \sum_{h=0}^k \|p_h^u\|_u^2 \overset{\{p_h^u\}\,orthog.}{=\!=\!=} \left\|\sum_{h=0}^k p_h^u\right\|_u^2 = \|q_k^u\|_u^2.$$

(ii) By Lemma 15.12, for arbitrary $q \in \mathbb{R}_k[x]$ we have $\dfrac{q(\lambda_0)}{\|q\|_u} = \|\boldsymbol{\rho} N_k(u)\|$ if and only if $\dfrac{1}{\|q\|_u}q(\boldsymbol{A})\boldsymbol{e}_u = \dfrac{1}{\|\boldsymbol{\rho} N_k(u)\|}\boldsymbol{\rho} N_k(u)$. If we q replace by q_k^u we get

$$\frac{q_k^u(\lambda_0)}{\|q_k^u\|_u} = \|\boldsymbol{\rho} N_k(u)\| \quad \text{iff} \quad \frac{1}{\|q_k^u\|_u}q_k^u(\boldsymbol{A})\boldsymbol{e}_u = \frac{1}{\|\boldsymbol{\rho} N_k(u)\|}\boldsymbol{\rho} N_k(u).$$

Since $q_k^u(\lambda_0) = \|q_k^u\|_u^2$ we have $\|q_k^u\|_u = \|\boldsymbol{\rho} N_k(u)\|$ and from this it follow

$$q_k^u(\lambda_0) = \|q_k^u\|_u^2 \quad \text{iff} \quad q_k^u(\boldsymbol{A})\boldsymbol{e}_u = \boldsymbol{\rho} N_k(u)$$

thus

$$q_k^u(\lambda_0) = s_k(u) \quad \text{iff} \quad q_k^u(\boldsymbol{A})\boldsymbol{e}_u = \boldsymbol{\rho} N_k(u)$$

□

(15.14) Proposition

Let Γ denote a simple connected graph with predistance polynomials $\{p_k\}_{0\le k\le d}$. If Γ is distance-regular then $p_k(\boldsymbol{A}) = \boldsymbol{A}_k$ for any $0 \le k \le d$ (predistance polynomials are, in this case, distance polynomials).

Proof: Since Γ is distance-regular we have $d = D$ (Corollary 8.10) and there exists polynomials r_k of degree k, $0 \le k \le D$, such that $\boldsymbol{A}_k = r_k(\boldsymbol{A})$ (Proposition 8.05). Polynomials r_k are distance polynomials of regular graph, and they are orthogonal (Proposition 10.07). By Problem 14.16, $\mathbb{R}_k[x] \cap \mathbb{R}_{k-1}[x]^\perp$ has dimension 1 and since $p_k, r_k \in \mathbb{R}_k[x]$ and $p_k, r_k \in \mathbb{R}_{k-1}[x]^\perp$ it follow $p_k, r_k \in \mathbb{R}_k[x] \cap \mathbb{R}_{k-1}[x]^\perp$, that is $\exists \xi_k$ such that $r_k = \xi_k p_k$ for every $k = 0, 1, .., d$. We know that $\|r_k\|^2 = r_k(\lambda_0)$ and $\|p_k\|^2 = p_k(\lambda_0)$ so

$$\xi_k^2\|p_k\|^2 = \|r_k\|^2 = r_k(\lambda_0) = \xi_k p_k(\lambda_0) = \xi_k\|p_k\|^2$$

and we may conclude $\xi_k = 1$. Therefore $\{p_k\}_{0\le k\le d}$ are distance polynomials. □

(15.15) Theorem (characterization J)

A regular graph Γ with n vertices and predistance polynomials $\{p_k\}_{0\leq k\leq d}$ is distance-regular if and only if

$$q_k(\lambda_0) = \frac{n}{\sum_{u\in V}\frac{1}{s_k(u)}} \qquad (0 \leq k \leq d),$$

where $q_k = p_0 + ... + p_k$, $s_k(u) = |N_k(u)| = |\Gamma_0(u)| + |\Gamma_1(u)| + ... + |\Gamma_k(u)|$.

Proof: ($\Rightarrow$) Assume that Γ is distance-regular. Then predistance polynomials $\{p_k\}_{0\leq k\leq d}$ are in fact distance polynomials (Proposition 15.14) and by Proposition 10.07 we have $\|p_h\|^2 = |\Gamma_h(u)|$. Now, the number of vertices at distance not greater than k from any given vertex u is a constant since

$$s_k(u) = \sum_{h=0}^{k}|\Gamma_h(u)| = \sum_{h=0}^{k}p_h(\lambda_0) = q_k(\lambda_0).$$

We have

$$\frac{1}{s_k(u)} = \frac{1}{q_k(\lambda_0)}$$

that is

$$\sum_{u\in V}\frac{1}{s_k(u)} = \frac{n}{q_k(\lambda_0)}$$

and the result follows.

($\Leftarrow$) In order to show that the converse also holds, let Γ be a regular graph with predistance polynomials $\{p_k\}_{0\leq k\leq d}$, and consider, for some fixed k, the sum polynomial $q_k := \sum_{h=0}^{k} p_h$ which also satisfies $q_k(\lambda_0) = \|q_k\|^2$. Then, by Lemma 15.12, we have $q_k(\lambda_0)/\|q_k\|_u \leq \|\boldsymbol{\rho} N_k(u)\|$, or

$$\frac{\|q_k\|_u^2}{q_k(\lambda_0)^2} \geq \frac{1}{\|\boldsymbol{\rho} N_k(u)\|^2} = \frac{1}{s_k(u)} \quad (u \in V).$$

Then, by adding over all vertices we get

$$\sum_{u\in V}\frac{1}{s_k(u)} \leq \frac{1}{q_k(\lambda_0)^2}\sum_{u\in V}\|q_k\|_u^2 = \frac{n}{q_k(\lambda_0)^2}\|q_k\|^2 = \frac{n}{g_k(\lambda_0)},$$

where we have used relationship $\langle p, q\rangle = \frac{1}{n}\sum_{u\in V}\langle p, q\rangle_u$ from Lemma 15.08 between the scalar products involved. Thus, we conclude that $q_k(\lambda_0)$ never exceeds the harmonic mean of the numbers $s_k(u)$:

$$q_k(\lambda_0) \leq \frac{n}{\sum_{u\in V}1/s_k(u)}.$$

What is more, equality can only hold if and only if inequality $\frac{\|q_k\|_u^2}{q_k(\lambda_0)^2} \geq \frac{1}{s_k(u)}$ above, is also equality, that is (Lemma 15.12)

$$\frac{q_k(\lambda_0)}{\|q_k\|_u} = \|\boldsymbol{\rho} N_k(u)\| \iff \frac{1}{\|q_k\|_u}q_k(\boldsymbol{A})\boldsymbol{e}_u = \frac{1}{\|\boldsymbol{\rho} N_k(u)\|}\boldsymbol{\rho} N_k(u) \iff$$

$$\iff q_k(\boldsymbol{A})\boldsymbol{e}_u = \frac{\|q_k\|_u}{\|\boldsymbol{\rho} N_k(u)\|}\boldsymbol{\rho} N_k(u).$$

But for (u-local) predistance polynomials we have (Lemma 15.13(ii))

$$q_k^u(\boldsymbol{A})\boldsymbol{e}_u = \boldsymbol{\rho} N_k(u) \quad \left(\iff \quad q_k^u(\lambda_0) = s_k(u) \quad \iff \quad q_k^u(\lambda_0) = \frac{n}{\sum_{u\in V} 1/s_k(u)}\right),$$

and, hence, $q_k = \alpha_u q_k^u$ for every vertex $u \in V$ and some constants α_u. Let us see that all these constants are equal to 1. Let u, v be two adjacent vertices and assume $k \geq 1$. Using the second equality in Lemma 15.13(ii) we have that
$(q_k^u(\boldsymbol{A}))_{uv} = (q_k^u(\boldsymbol{A})e_u)_v = (\boldsymbol{\rho} N_k(u))_v = (I + \boldsymbol{A} + ... + \boldsymbol{A}_k)_{uv} = 1$ that is
$(q_k^u(\boldsymbol{A}))_{uv} = (q_k^v(\boldsymbol{A}))_{vu} = 1$, and, therefore, (since $q_k = \alpha_u q_k^u$)

$$\frac{1}{\alpha_u}(q_k(\boldsymbol{A}))_{uv} = \frac{1}{\alpha_v}(q_k(\boldsymbol{A}))_{vu} = 1$$

Hence $\alpha_u = \alpha_v$ and, since Γ is supposed to be connected, $q_k = \alpha q_k^u$ for some constant α and any vertex u. Moreover, using these equalities and Lemma 15.08,

$$\frac{n}{\alpha} q_k(\lambda_0) = \frac{1}{\alpha} q_k(\lambda_0) \sum_{u\in V} 1 = \sum_{u\in V} q_k^u(\lambda_0) = \sum_{u\in V} \|q_k^u\|_u^2 = \frac{1}{\alpha^2} \sum_{u\in V} \|q_k\|_u^2 =$$
$$= \frac{1}{\alpha^2} \sum_{u\in V} \langle q_k, q_k \rangle_u = \frac{n}{\alpha^2} \|q_k\|^2 = \frac{n}{\alpha^2} q_k(\lambda_0),$$

whence $\alpha = 1$ and $q_k = q_k^u$ for any $u \in V$. Consequently, by Lemma 15.13(ii),
$q_k(\boldsymbol{A})\boldsymbol{e}_u = \boldsymbol{\rho} N_k(u)$ for every vertex $u \in V$. Since $\boldsymbol{\rho} N_k(u)$ is the uth column of the sum matrix $I + \boldsymbol{A} + ... + \boldsymbol{A}_k$, we have

$$q_k(\boldsymbol{A}) = I + \boldsymbol{A} + ... + \boldsymbol{A}_k.$$

Then, if we assume that Γ has $d+1$ eigenvalues and the above holds for any $1 \leq k \leq d$ (the case $k = 0$ being trivial since $q_0 = p_0 = 1$), we have that $p_k(\boldsymbol{A}) = q_k(\boldsymbol{A}) - q_{k-1}(\boldsymbol{A}) = \boldsymbol{A}_k$ for any $1 \leq k \leq d$ and, by Theorem 10.08 (characterization D), Γ is a distance-regular graph. □

Alternatively, considering the ”base vertices” one by one, we may give a characterization which does not use the sum polynomials q_k or the harmonic means of the $s_k(u)$'s:

(15.16) Theorem (characterization K)
A graph $\Gamma = (V, E)$ with predistance polynomials $\{p_k\}_{0\leq k\leq d}$ is distance-regular if and only if the number of vertices at distance k from every vertex $u \in V$ is

$$p_k(\lambda_0) = |\Gamma_k(u)| \quad (0 \leq k \leq d).$$

Proof: ($\Leftarrow$) If $p_k(\lambda_0) = |\Gamma_k(u)|$ holds for every $0 \leq k \leq d$, we have
$q_k(\lambda_0) = p_0(\lambda_0) + p_1(\lambda_0) + ... + p_k(\lambda_0) = |\Gamma_0(u)| + |\Gamma_1(u)| + ... + |\Gamma_k(u)| = s_k(u)$ for every vertex u, so

$$\frac{1}{s_k(u)} = \frac{1}{g_k(\lambda_0)} \quad \Rightarrow \quad \sum_{u\in V} \frac{1}{s_k(u)} = \frac{n}{g_k(\lambda_0)}$$

and Theorem 15.15 (characterization J) trivially applies.

Notice also that, in this case, we do not need to assume the regularity of the graph, since it is guaranteed by considering $k = 1$ in $p_k(\lambda_0) = |\Gamma_k(u)|$: $\delta_u = |\Gamma_1(u)| = p_1(\lambda_0)$ for any $u \in V$ (whence $p_1(\lambda_0) = \lambda_0$).

($\Rightarrow$) If Γ is distance-regular then predistance polynomial $\{p_k\}_{0\leq k\leq d}$ are in fact distance polynomials (Proposition 15.14), and from this it follow that $p_k(\lambda_0) = |\Gamma_k(u)|$ for $0 \leq k \leq d$ (Proposition 10.07). □

But, once more, not all the conditions in Characterization J or Characterization K are necessary to ensure distance-regularity. In fact, if the graph is regular (which guarantees the case $k = 1$ since then $p_1 = x$), only the case $k = d - 1$ matters. First we need Lemma 15.17 and Lemma 15.18.

(15.17) Lemma

Let $\Gamma = (V, E)$ *be simple connected graph with predistance polynomials* $\{p_k\}_{0\le k\le d}$ *and spectrum* $\operatorname{spec}(\Gamma) = \{\lambda_0^{m_0}, \lambda_1^{m_1}, ..., \lambda_d^{m_d}\}$. *Then*

(i) $m(\lambda_i) = \dfrac{\phi_0 p_d(\lambda_0)}{\phi_i p_d(\lambda_i)} \quad (0 \le i \le d)$,

(ii) $p_d(\lambda_0) = n\left(\displaystyle\sum_{j=0}^{d} \frac{\pi_0^2}{m(\lambda_j)\pi_j^2}\right)^{-1}$,

where $\phi_i = \prod_{j=0(j\ne i)}^{d}(\lambda_i - \lambda_j)$ *and* π_i*'s are moment-like parameters* ($\pi_i = |\phi_i|$).

Proof: Let us consider polynomials $Z_i^* = \prod_{j=1\, j\ne i}(x - \lambda_j)$, $q \le i \le d$ so that

$$Z_i^*(\lambda_0) = \prod_{j=1\, j\ne i}^{d} (\lambda_0 - \lambda_j) = (\lambda_0 - \lambda_1)(\lambda_0 - \lambda_2)...(\widehat{\lambda_0 - \lambda_i})...(\lambda_0 - \lambda_d) =$$

$$= \frac{(\lambda_0 - \lambda_1)(\lambda_0 - \lambda_2)...(\lambda_0 - \lambda_i)...(\lambda_0 - \lambda_d)}{\lambda_0 - \lambda_i} = \frac{\phi_0}{\lambda_0 - \lambda_i},$$

where $(\widehat{x - \lambda_i})$ denotes that this factor is not present in the product, and

$$Z_i^*(\lambda_i) = \prod_{j=1\, j\ne i}^{d} (\lambda_0 - \lambda_j) = (\lambda_i - \lambda_1)(\lambda_i - \lambda_2)...(\widehat{\lambda_i - \lambda_i})...(\lambda_i - \lambda_d) =$$

$$= \frac{(\lambda_i - \lambda_0)(\lambda_i - \lambda_1)(\lambda_0 - \lambda_2)...(\widehat{\lambda_i - \lambda_i})...(\lambda_i - \lambda_d)}{\lambda_i - \lambda_0} = \frac{\phi_i}{\lambda_i - \lambda_0}$$

Hence, since $\operatorname{dgr} Z_i^* = d - 1$

$$0 = \langle p_d, Z_i^*\rangle = \sum_{j=0}^{d} m(\lambda_j)p_d(\lambda_j)Z_i^*(\lambda_j) = m(\lambda_0)p_d(\lambda_0)Z_i^*(\lambda_0) + m(\lambda_i)p_d(\lambda_i)Z_i^*(\lambda_i) =$$

$$= \frac{p_d(\lambda_0)\phi_0}{\lambda_0 - \lambda_i} + \frac{p_d(\lambda_i)\phi_i}{\lambda_i - \lambda_0} m(\lambda_i)$$

and first result follows.

In order to prove (ii), we use the property $p_d(\lambda_0) = \|p_d\|^2$ and the fact that, from (i), $p_d(\lambda_i) = \dfrac{\phi_0 p_d(\lambda_0)}{\phi_i m(\lambda_i)}$, $0 \le i \le d$. We have

$$\|p_d\|^2 = \langle p_d, p_d \rangle = \frac{1}{n}\sum_{j=0}^{d} m(\lambda_j)p_d(\lambda_j)^2,$$

$$p_d(\lambda_0) = \frac{1}{n}\sum_{j=0}^{d} m(\lambda_j)\left(\frac{\phi_0 p_d(\lambda_0)}{\phi_j m(\lambda_j)}\right)^2 = \frac{1}{n}p_d(\lambda_0)^2\sum_{j=0}^{d}\frac{\phi_0^2}{\phi_j^2 m(\lambda_j)},$$

$$n = p_d(\lambda_0)\sum_{j=0}^{d}\frac{\pi_0^2}{m(\lambda_j)\pi_j^2},$$

which yields (ii). □

(15.18) Lemma

For a regular graph Γ *with* n *vertices and spectrum* $\operatorname{spec}(\Gamma) = \{\lambda_0^{m_0}, \lambda_1^{m_1}, ..., \lambda_d^{m_d}\}$ *we have*

$$\frac{\sum_{u\in V} n/(n-k_d(u))}{\sum_{u\in V} k_d(u)/(n-k_d(u))} = \sum_{i=0}^{d}\frac{\pi_0^2}{m(\lambda_i)\pi_i^2} \quad\Longleftrightarrow\quad q_{d-1}(\lambda_0) = \frac{n}{\sum_{u\in V}\frac{1}{s_{d-1}(u)}}$$

where $k_d(u) = |\Gamma_d(u)|$, $q_k = p_0 + ... + p_k$, $s_k(u) = |N_k(u)|$.

Proof: Since $q_d = \sum_{i=1}^{d} p_i = H_0 = \frac{n}{\pi_0}\prod_{i=1}^{d}(x-\lambda_i)$ (Proposition 13.06, Γ is regular, so $g_0 = \frac{1}{n}$), we have $q_{d-1}(\lambda_0) = q_d(\lambda_0) - p_d(\lambda_0) = n - p_d(\lambda_0)$. By Lemma 15.17$(ii)$ the value of $p_d(\lambda_0)$ is $n\left(\sum_{j=0}^{d}\frac{\pi_0^2}{m(\lambda_j)\pi_j^2}\right)^{-1}$. Notice equivalence that follow

$$q_{d-1}(\lambda_0) = \frac{n}{\sum_{u\in V}\frac{1}{s_{d-1}(u)}} \quad\Leftrightarrow\quad n - p_d(\lambda_0) = \frac{n}{\sum_{u\in V}\frac{1}{s_{d-1}(u)}} \quad\Leftrightarrow\quad n - \frac{n}{\sum_{u\in V}\frac{1}{s_{d-1}(u)}} = p_d(\lambda_0)$$

$$\Leftrightarrow\quad \frac{\sum_{u\in V}\frac{n}{s_{d-1}(u)} - n}{\sum_{u\in V}\frac{1}{s_{d-1}(u)}} = p_d(\lambda_0) \quad\Leftrightarrow\quad \frac{\sum_{u\in V}\frac{n}{s_{d-1}(u)} - \sum_{u\in V} 1}{\sum_{u\in V}\frac{1}{s_{d-1}(u)}} = p_d(\lambda_0)$$

$$\Leftrightarrow\quad \frac{\sum_{u\in V}\frac{n-s_{d-1}(u)}{s_{d-1}(u)}}{\sum_{u\in V}\frac{1}{s_{d-1}(u)}} = p_d(\lambda_0) = n\left(\sum_{j=0}^{d}\frac{\pi_0^2}{m(\lambda_j)\pi_j^2}\right)^{-1} \quad\Leftrightarrow\quad \frac{\sum_{u\in V}\frac{n}{n-|\Gamma_d(u)|}}{\sum_{u\in V}\frac{|\Gamma_d(u)|}{n-|\Gamma_d(u)|}} = \sum_{j=0}^{d}\frac{\pi_0^2}{m(\lambda_j)\pi_j^2}$$

and the result follow. □

(15.19) Theorem (characterization J')

A regular graph Γ *with* n *vertices and spectrum* $\operatorname{spec}(\Gamma) = \{\lambda_0^{m(\lambda_0)}, \lambda_1^{m(\lambda_1)}, ..., \lambda_d^{m(\lambda_d)}\}$ *is distance-regular if and only if*

$$\frac{\sum_{u\in V} n/(n-k_d(u))}{\sum_{u\in V} k_d(u)/(n-k_d(u))} = \sum_{i=0}^{d}\frac{\pi_0^2}{m(\lambda_i)\pi_i^2}.$$

where $\pi_h = \prod_{\substack{i=0\\ i\neq h}}^{d}(\lambda_h - \lambda_i)$ *and* $k_d(u) = |\Gamma_d(u)|$.

Proof: If $q_k(\lambda_0) = \dfrac{n}{\sum_{u\in V}\frac{1}{s_k(u)}}$ is satisfied for $k = d-1$, we infer that $q_{d-1}(\boldsymbol{A}) = \sum_{h=0}^{d}\boldsymbol{A}_h$ (from prove of Theorem 15.15 (characterization J)) and so
$p_d(\boldsymbol{A}) = H(\boldsymbol{A}) - q_{d-1}(\boldsymbol{A}) = \boldsymbol{J} - \sum_{i=0}^{d-1}\boldsymbol{A}_i = \boldsymbol{A}_d$, where H is the Hoffman polynomial. Thus, from Theorem 11.15 (characterization D'), the result follow. □

(15.20) Theorem (characterization K')

A regular graph $\Gamma = (V, E)$ *with* n *vertices and spectrum* $\mathrm{spec}(\Gamma) = \{\lambda_0^{m(\lambda_0)}, ..., \lambda_d^{m(\lambda_d)}\}$ *is distance-regular if and only if the number of vertices at (spectrally maximum) distance* d *from each vertex* $u \in V$ *is*

$$k_d(u) = n \left(\sum_{i=0}^{d} \frac{\pi_0^2}{m(\lambda_i)\pi_i^2} \right)^{-1}$$

where $\pi_h = \prod_{\substack{i=0 \\ i \neq h}}^{d} (\lambda_h - \lambda_i)$ *and* $k_d(u) = |\Gamma_d(u)|$.

Proof: We will left this proof like intersting exercise (challenge). Proof can be found in [20] Theorem 4.4. □

Theorem 15.20 was proved by Fiol and Garriga [20], generalizing some previous results. Finally, notice that, since $A_k = p_k(A)$ implies $k_h(u) = p_h(\lambda_0)$ for every $u \in V$ - see Proposition 10.07 - both characterizations (D') and (K') are closely related.

Conclusion

Algebraic graph theory is a branch of mathematics in which algebraic methods are applied to problems about graphs. This is in contrast to geometric, combinatoric, or algorithmic approaches. There are several branches of algebraic graph theory, involving the use of linear algebra, the use of group theory, and the study of graph invariants.

In this thesis we had try to show this connection with linear algebra. We had shown how from given graph obtain adjacency matrices, principal idempotent matrices, distance matrices and predistance polynomials. There are many questions that raise from this, as example: what are the connections between the spectra of these matrices and the properties of the graphs, what we can say about graph from its distance matrices, are there some connection between orthogonal polynomials and properties of the graphs. This question we had tried to answer in Chapter II and III, in case when we have distance-regular graphs.

Further study that would be interesting to explore is use some of this results and make connection with group theory, or to be more precisely with Coding theory (Coding theory is the study of the properties of codes and their fitness for a specific application. Codes are used for data compression, cryptography, error-correction and more recently also for network coding). If we have some distance-regular graph is it possible to make some code that would be, for example, efficient and reliable for data transmission methods.

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