1 Sets and Mathematical Induction

1. Let A be a set of cardinality $n \in \mathbb{Z}^+$, and let $\mathcal{P}(A)$ denote the power set of A. Using mathematical induction, prove the following equality:

$$|\mathcal{P}(A)| = 2^n.$$

2. Prove the following statement: If n books are distributed among m boxes, then at least one box will contain at least $\left\lfloor \frac{n-1}{m} \right\rfloor + 1$ books. Provide a detailed explanation of each step in your proof.

2 Basic Combinatorial Principles: Sum Principle, Product Principle, Equality Principle

3. Suppose we have 5 (identical) oranges and 8 (identical) apples. How many different non-empty collections of fruit can we make if a collection can include at most 3 oranges? Before calculating the numerical value, provide at least three examples of possible collections.

4. How many different five-digit numbers can be formed using the digits 1, 2, 3, and 4 if:

(a) there are no additional restrictions?

(b) the sum of the digits must be even?

(c) there must be more even digits than odd digits?

Provide the result as a numerical value.

5. How many different 4-digit numbers can be formed using the digits 2, 3, 5, 6, 7, and 8 under the following conditions:

(a) the digits in the number are all distinct?

(b) all digits in the number are the same?

(c) the number is greater than 5688?

(d) the number is divisible by 5?

(e) the number is a palindrome[1]?

(f) the number is even and is a palindrome?

Provide the result as a precise numerical value. For all parts of the task, provide at least three examples of possible 4-digit numbers before computing numerical results.

6. In how many ways can we form a sequence of 5 letters using the letters x, y, a, b, u, and v, considering the following conditions:

(i) letters in the sequence can be repeated?

(ii) letters in the sequence cannot be repeated, and the sequence must include the letter x?

(iii) letters in the sequence can be repeated, and the sequence must include the letter x?

7. Let $A = \{-4, -3, -2, -1, 1, 2, 3, 4\}$. In how many ways can we choose four numbers from the set A if the selected numbers must have different absolute values:

(a) and the order of selection matters?

(b) and the order of selection does not matter?

[1] A palindrome is a sequence that reads the same forwards and backwards, e.g., 12321, 1221.

3 The Pigeonhole Principle

8. In the Diocletian family, there are 12 children. Prove that at least two family members were born in the same month.

9. How many students must take the written exam in Discrete Mathematics II to ensure that at least four students receive the same score if the exam is graded on a scale from 0 to 100?

10. Given five points inside an equilateral triangle with side length 2 cm, prove that there exist two points that are at most 1 cm apart.

11. At a party, there are 8 students aged between 18 and 30 years. Justify that it is possible to select two different groups of individuals such that the sum of the ages in one group equals the sum of the ages in the other group.

12. From the set $\{1, 2, \ldots, 2n\}$, $n+1$ distinct numbers are chosen. Prove that among the selected integers, there are always two whose greatest common divisor is 1.

13. Every week, 200 cargo ships arrive at the Port of Koper. Justify that there are at least two ships that arrive at the port less than an hour apart.

14. At a family gathering, nine people aged between 18 and 58 years were present. Justify that it is possible to select two different groups of individuals such that the sum of the ages in one group equals the sum of the ages in the other group.

15. Consider a game involving 10 tennis players (one-on-one matches). Each player plays at least one match, and no two players play against each other more than once (the exact number of matches played by each player is unknown, and two different players may participate in different numbers of matches). Prove that there are at least two players who have played the same number of matches.

16. Consider a standard die with 6 faces numbered from 1 to 6. Select four of its faces. Prove that among these four faces, there are two that are on opposite sides of the die.

All above math problems are taken from the following website:
https://osebje.famnit.upr.si/~penjic/teaching.html.
THE READER CAN FIND ALL SOLUTIONS TO THE GIVEN PROBLEMS ON THE SAME PAGE.