INCIDENCE GEOMETRY AND BUILDINGS

MICHEL LAVRAUW

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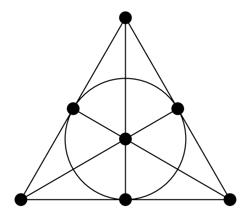

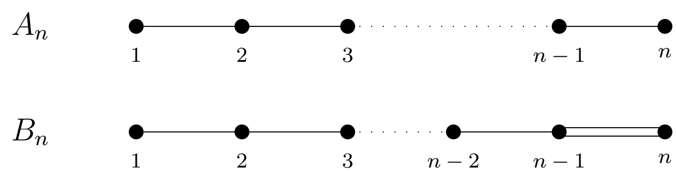

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Introduction

The notion of "Incidence Geometry" covers a wide range of topics ranging from elementary objects like graphs and projective planes to more advanced topics like polar spaces, diagram geometries and buildings. An incidence geometry is a rather general concept and many mathematical objects can be considered as, or are related to, such a geometry: it is a collection of *objects* (think of points, lines, planes, solids, etc.) and a collection of *axioms* (properties, relations between these objects in terms of incidence). To fix ideas, we give a first example.

As objects we take "points" and "lines", and we consider the following axioms:

(1) for each two distinct points there exists at most one line passing through both of them;
(2) given a point p and a line L not through p, there is exactly one line passing through p and intersecting L.

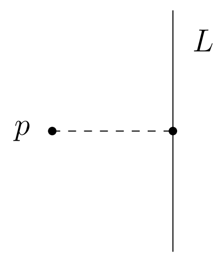
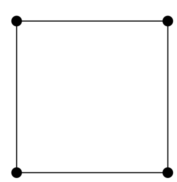


FIGURE 1. Exactly one line through p intersecting L

This example might look trivial, but as we will see later on, it is the starting point of an important class of geometries: the so-called *generalised quadrangles*. The terminology is motivated by the fact that a trivial example of such a geometry is given by the vertices and sides of a quadrangle.



A trivial yet important observation is that the axiom represented in Figure 1 does not allow triangles.

Using a more abstract approach, we can consider the "geometry" consisting of points and lines from the example as a triple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ (set of points $\mathcal{P}$, set of lines $\mathcal{L}$ and incidence relation $\mathcal{I}$, which we consider as a subset of $\mathcal{P} \times \mathcal{L} \cup \mathcal{L} \times \mathcal{P}$) satisfying the axioms

(1) $\forall x, y \in \mathcal{P} \ : \ (x = y) \vee (\exists_{\leq 1} L \ : \ x \ \mathcal{I} \ L \ \mathcal{I} \ y)$,
(2) $\forall x \in \mathcal{P} \ \forall L \in \mathcal{L} \ : \ (x \ \mathcal{I} \ L) \vee (\exists! M \in \mathcal{L} \ \exists y \in \mathcal{P} \ : \ x \ \mathcal{I} \ M \ \mathcal{I} \ y \ \mathcal{I} \ L)$.

Notice that the meaning of the word incidence in the context of *incidence geometry* deviates from the present meaning in the English language. In the context of incidence geometry, two distinct lines can of course have a point in common, but they are never incident!

Incidence geometry grew as a natural generalisation or abstraction of classical geometry (euclidean, affine, projective) and was to a great extend motivated by group theory. Its origin can be traced back to the initial axiomatic treatment of classical geometries ([Pasch 1882], [Hilbert 1899], [Veblen and Young 1910]) which was later picked up by Jacques Tits [Tits 1955] taking the subject to a higher level of abstraction. This would eventually lead to the theory of buildings [Tits 1974].

While the mathematical developments in incidence geometry and the theory of buildings has been developing with great speed since the 1970's, the mathematical community and its institutions, on the other hand, act at a much slower pace, and has not really been able to fully recognise the incidence geometry and buildings as a subject area. De facto, its topics of study are often scattered over the subject areas Combinatorics, Geometry, Field theory (finite fields) and Group theory. Consulting the MSC2010 database (Mathematics subject classification from the American Mathematical Society) we get the following search results. For "buildings" we get: 20-XX Group theory and generalizations; 20Exx Structure and classification of infinite or finite groups; 20E42 Groups with a BN-pair, buildings; and 51-XX Geometry; 51Exx Finite geometry and special incidence structures; 51E24 Buildings and the geometry of diagrams. For "incidence geometry" we get: 51-XX Geometry; 51Axx Linear incidence geometry; 51Bxx Nonlinear incidence geometry; and 51Exx Finite geometry and special incidence structures.

I hope these notes will convince the reader of the fact that incidence geometry is a beautiful part of mathematics: elementary and elegant in its presentation and therefore attractive to the beginning mathematician; and yet powerful and at the disposal of the more experienced mathematician as a useful and flexible tool in the study of mathematical situations of the highest complexity.

Lecture 1. Projective planes

Unquestionably, the main impetus for the study of incidence geometry came from projective geometry. Projective spaces provides the ambient geometry for many different incidence geometries (which are then said to be *embedded* in the ambient projective space) and they also provide the first examples of buildings.

In these notes K will denote a field or skewfield, and K^d the d-dimensional vector space over K.

The *classical* projective plane $\mathrm{PG}(2, K)$ is obtained by considering the subspaces of K^3, of dimension 1 as "points" and of dimension 2 as "lines". The following three simple properties of the obtained point-line geometry are taken as the axioms for the definition of a projective plane.

An incidence geometry is called *thick* when each line contains at least three points.

Definition 1. *A* projective plane *is a thick point-line geometry satisfying:*
(a1) for every two distinct points there exists a unique line containing both;
(a2) every two distinct lines meet in a unique point;
(a3) there exist 4 points, no three of which are collinear.

One of the most famous pictures in geometry must be the Fano plane, Figure 2.

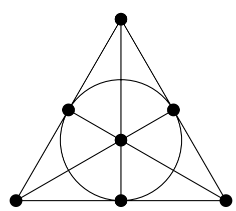


Figure 2. The Fano plane

It contains 7 points and 7 lines, satisfying the axioms of a projective plane. In particular, every line contains 3 points, every point is contained in 3 lines, every two points are contained in a unique common line, and every two lines intersect in a point. It can be constructed in many different ways and has many connections to other mathematical objects. It can be constructed from a 3-dimensional vector space over the field with two elements. Another way is to start with a difference set modulo 7. The Fano plane also gives us a nice error-correcting code. If we consider the incidence matrix of the Fano plane as the generator matrix for a binary linear code, then we obtain the 4-dimensional Hamming code of length 7 and the lines correspond to the smallest weight codewords.

An *isomorphism* between projective spaces is defined in the usual way as a bijection preserving the structure, in this case incidence. In particular an isomorphism α between two projective planes $\Pi = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ and $\Pi' = (\mathcal{P}', \mathcal{L}', \mathcal{I}')$ is a bijection from $\mathcal{P} \cup \mathcal{L}$ to $\mathcal{P}' \cup \mathcal{L}'$ such that incidence is preserved, i.e.

$$p\mathcal{I}L \iff p^\alpha \mathcal{I}' L^\alpha.$$

If there exists an isomorphism between the projective planes Π and Π' then Π and Π' are called *isomorphic* and we write $\Pi \cong \Pi'$. It is very common to use the word *collineation* instead of isomorphism.

The action of the general linear group $\mathrm{GL}(3, K)$ induces a natural action on the projective plane $\mathrm{PG}(2, K)$. The image is called the *projective linear group* $\mathrm{PGL}(3, K)$. Its elements are also called *projectivities*. The kernel of the action is isomorphic to the multiplicative group of K. An important fact is that given any 4 points p_0, p_1, p_2, p_3 in $\mathrm{PG}(2, K)$, no three on a line, there always exists a basis of K^3 with respect to which the points are given by homogeneous coordinates $p_0(1, 0, 0)$, $p_1(0, 1, 0)$, $p_2(0, 0, 1)$, and $p_3(1, 1, 1)$. Such a 4-tuple of points is called a *frame* of $\mathrm{PG}(2, K)$ and the above can be reworded as the fact that the group $\mathrm{PGL}(3, K)$ acts transitively on the set of frames of $\mathrm{PG}(2, K)$. This is a very useful tool in proofs involving coordinates (which will be illustrated in the next two proofs).

The following two theorems are classical results and are of particular historical interest in the study of projective planes, which became very popular in the 20th century, with the discovery of many so-called *non-classical* or *non-Desarguesian* projective planes. These are projective planes which are not constructed from a 3-dimensional vector space and in which Desargues' Theorem does not hold.

Theorem 2 (Desargues). *Consider three concurrent lines ℓ_1, ℓ_2 and ℓ_3 in $\mathrm{PG}(2, K)$, and two triangles Δabc and $\Delta a'b'c'$ whose vertices a, b, c and a', b', c' lie on the lines ℓ_1, ℓ_2 and ℓ_3, respectively. Then the points $aa' \cap bb'$, $aa' \cap cc'$ and $bb' \cap cc'$ are collinear.*

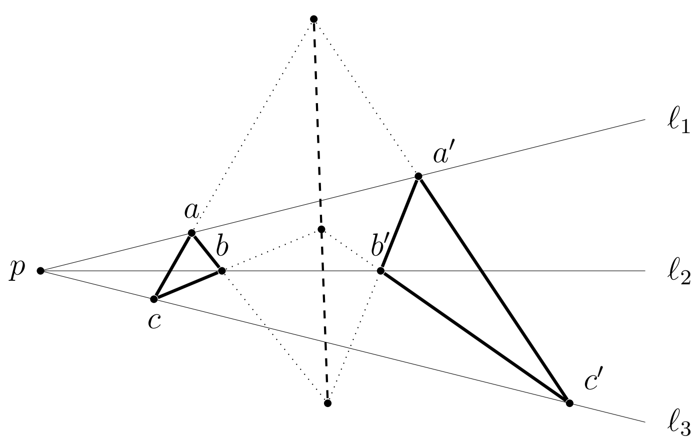


Figure 3. Desargues' configuration

Proof. Let p denote the common intersection of the lines ℓ_1, ℓ_2 and ℓ_3. Then the ordered 4-tuple (a, b, c, p) forms a frame of $\mathrm{PG}(2, K)$ and without loss of generality (by using the transitivity properties of the projectivity group) we may assume that $a(1, 0, 0)$, $b(0, 1, 0)$, $c(0, 0, 1)$ and $p(1, 1, 1)$. Since a' lies on $\ell_1 = pa$, b' on $\ell_2 = pb$, and c' on $\ell_3 = pc$, there exist $\alpha, \beta, \gamma \in K \neq \{0\}$ such that $a'(1 + \alpha, 1, 1)$, $b'(1, 1 + \beta, 1)$ and $c'(1, 1, 1 + \gamma)$. Calculating the intersection $aa' \cap bb'$ gives the point with coordinates $(\alpha, -\beta, 0)$. Similarly we obtain the points with coordinates $(0, \beta, -\gamma)$ and $(\alpha, 0, -\gamma)$. Since

$$\det \begin{bmatrix} \alpha & -\beta & 0 \\ 0 & \beta & -\gamma \\ \alpha & 0 & -\gamma \end{bmatrix} = 0,$$

these three points are collinear. □

Following the axiomatic approach to geometry introduced by Pasch in 1882, already in 1899 Hilbert constructed a projective plane which is not "classical", i.e. not coming from a 3-dimensional vector space over a (skew-)field, in his famous "Grundlagen der Geometrie". A simpler construction of a "non-classical" projective plane was given by Moulton in 1902, by starting from an ordinary Euclidean plane, and replacing the lines with negative slope by "broken" lines, whose slope doubles when passing the X-axis.

EXERCISE 3. *Prove that the Moulton plane is a projective plane, but that the Theorem of Desargues does not hold true.*

Another important theorem is the following.

THEOREM 4 (Pappus). *Consider two lines ℓ_1 and ℓ_2 in* $\mathrm{PG}(2, K)$*, and two triples (a, b, c) and (a', b', c') of points on the lines ℓ_1 and ℓ_2, respectively, but different from the intersection $\ell_1 \cap \ell_2$. Then the points $ac' \cap ca'$, $ab' \cap ba'$ and $bc' \cap cb'$ are collinear.*

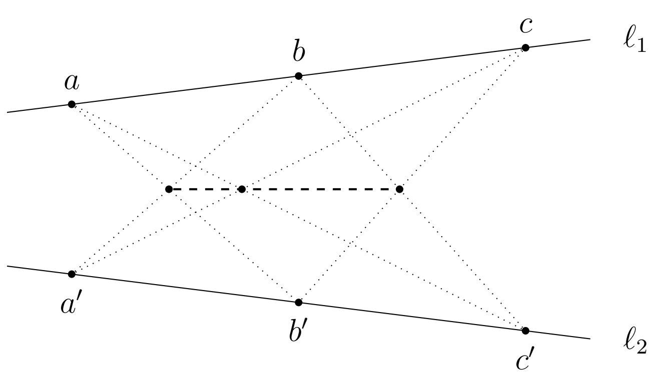


FIGURE 4. Pappus' configuration

Proof. It follows from the transitivity properties of the projectivity group that we may assume w.l.o.g. that the points of the frame (a, b, a', b') have coordinates $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(1, 1, 1)$, respectively. Then the points c and c' must have coordinates $(1, \alpha, 0)$ and $(\beta, \beta, 1)$ for some $\alpha, \beta \in K \setminus \{0\}$. Calculating the points of intersection we obtain $(0, 1, 1)$, $(\beta, \alpha\beta, \alpha)$ and $(\beta, 1 - \alpha + \alpha\beta, 1)$. Since

$$\det \begin{bmatrix} 0 & 1 & 1 \\ \beta & \alpha\beta & \alpha \\ \beta & 1 - \alpha + \alpha\beta & 1 \end{bmatrix} = \det \begin{bmatrix} 0 & 1 & 1 \\ \beta & \alpha\beta & \alpha \\ 0 & 1 - \alpha & 1 - \alpha \end{bmatrix} = 0$$

these points are collinear. □

The configurations of Pappus and Desargues turn out to be extremely important in the theory of projective planes due to the following fundamental results.

THEOREM 5. (Hilbert 1899)
If the Theorem of Desargues holds in a projective plane Π*, then* $\Pi \cong \mathrm{PG}(2, K)$ *for some skewfield* K*.*

This is why the classical projective plane $\mathrm{PG}(2, K)$, with K a (skew)field is often called the *Desarguesian projective plane*. Another fundamental result characterises the planes $\mathrm{PG}(2, K)$, with K a field.

THEOREM 6. *If the theorem of Pappus holds in a projective plane* Π*, then* $\Pi \cong \mathrm{PG}(2, K)$ *for some field* K*.*

The proof of the above theorem is a combination of the result by Hessenberg (1905) which says that Pappus implies Desargues, and a result by Hilbert that if the Theorem of Pappus holds in $\mathrm{PG}(2, K)$ then K is a field.

These theorems are a beautiful illustration of the interplay between geometry and algebra. In fact much more can be said about this interaction and there is a whole theory about projective planes, based on the coordinatisation method. Well studied planes include translation planes, nearfield planes, semifield planes, and Moufang planes, and the corresponding algebraic structures are quasifields, nearfields, semifields, and alternative division rings. It would lead us to far to go into more details here.

We end this lecture with some important properties of projective planes, and the relation between projective and affine planes, and if time permits we will give geometric construction of translation planes.

THEOREM 7. *(i) Every line in a projective plane has the same number of points.*
(ii) Each point in a projective plane lies on the same number of lines.
(iii) The number of points on a line is equal to the number of lines through a point.

Proof. Pick any two lines M and L, and a point p not contained in any of them. For each point x on L you get a point on M by intersecting the line px with M. This defines a bijection between the set of points on L and the set of points on M. This proves (i) and also shows that the number of points on L is equal to the number of lines through p. A similar argument proves (ii) (you can also use duality). Part (iii) now easily follows. □

If the plane has a finite number of points and lines, then straightforward counting leads to the following.

THEOREM 8. *If* $\Pi = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ *is a finite projective plane then there exists a constant* $n \geq 2$ *such that each line contains* $n + 1$ *points, each point is on* $n + 1$ *lines, and* $|\mathcal{P}| = |\mathcal{L}| = n^2 + n + 1$*.*

Proof. Straightforward counting. □

The number n is called the *order* or the finite projective plane.

By the construction of the classical plane and the fact that for each prime power q there exists a finite field of size q, it follows that there exists a projective plane of order q for each prime power q. But what about other non-classical planes? Does there exist a projective plane which is not of prime power order? This is a very interesting open question, which has been solved only for values of $n < 12$. There does not exist a plane of order 6 (6 officers) and neither does there exist a projective plane of order 10 (using coding theory and computer-aided computations). The existence of a projective plane of order $n = 12$ is not known.

Although affine planes are part of projective planes, we still want to mention them, because the terminology is sometimes useful in other purposes in different parts of mathematics.

DEFINITION 9. *An* affine plane *is a thick point-line geometry satisfying the following three axioms:*
(b1) for every two distinct points there exists a unique line containing both;
(b2) given a non-incident point-line pair (x, M), there exists a unique line L passing through x and not intersecting M;
(b3) there exists a triangle.

EXERCISE 10. *Draw a picture of an affine plane with 9 points. How many lines does this affine plane have?*

THEOREM 11. *(i) Every line in an affine plane has the same number of points.*
(ii) Each point in an affine plane lies on the same number of lines.
(iii) The number of points on a line is one less than the number of lines through a point.

THEOREM 12. *If $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a finite affine plane then there exists a constant $n \geq 2$ such that each line contains n points, each point is on $n + 1$ lines, $|\mathcal{P}| = n^2$, and $|\mathcal{L}| = n^2 + n$.*

An affine plane is obtained from a projective plane by deleting a line and its points: $\Pi \to \Pi \setminus L$.

THEOREM 13. *Two affine planes $\Pi \setminus L$ and $\Pi' \setminus L'$ are isomorphic if and only if there exists an isomorphism between Π and Π' mapping L to L'.*

Conversely, a projective plane is obtained from an affine plane by *projective completion*: $\mathcal{A} \to \Pi$. The extra points are the parallel classes of the lines of the affine plane. These form the *line at infinity.*

THEOREM 14. *If $\mathcal{A}$ is an affine plane, then there exists, up to isomorphism, a unique projective plane Π such that $\mathcal{A} = \Pi \setminus L$.*

We end this lecture with the construction of translation planes. Given a partition $\mathcal{S}$ of $\mathrm{PG}(3, K)$ by lines (a *spread*), embed $\mathrm{PG}(3, K)$ as a hyperplane in $\mathrm{PG}(4, K)$ and consider the incidence structure $\mathcal{A}(\mathcal{S}) = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ with points the points of $\mathrm{PG}(4, K) \setminus \mathrm{PG}(3, K)$ and as lines the planes of $\mathrm{PG}(4, K)$ intersecting $\mathrm{PG}(3, K)$ in a line of $\mathcal{S}$. Incidence $\mathcal{I}$ is natural.

Theorem 15. *$\mathcal{A}(\mathcal{S})$ is an affine plane.*

Proof. The axioms of an affine plane are readily verified. □

Let $\Pi(\mathcal{S})$ denote the projective completion of $\mathcal{A}(\mathcal{S})$. The plane $\Pi(\mathcal{S})$ is a *translation plane*. It would take us too far to go into details, but there exist spreads $\mathcal{S}$ for which $\Pi(\mathcal{S})$ is non-Desarguesian, i.e. not isomorphic to any $\mathrm{PG}(2, K)$.

Lecture 2. Generalised polygons

Generalised polygons were introduced by J. Tits in 1959. Here we will give a slightly different (but equivalent definition).

But first we start with an example of a generalised polygon, precisely a generalised 4-gon. You should find lot's of quadrangles and pentagons, but no triangles!

It is another "smallest" example in its class of geometries: the GQ$(2,2)$, the generalised quadrangle with three points on each line and three lines through each point (Figure 8). It has in total 15 points (only the black points count) and 15 lines.

Contrary to the Fano plane, the GQ$(2,2)$ does no longer satisfy the property that each two points lie on a common line and each two lines intersect. A much weaker property holds: every two points lie on at most one common line, and each two lines intersect in at most one common point. There is another property that is maybe less obvious: given a point p and a line L not containing that points, there is exactly one line which contains p and intersects L. This is a fundamental property of generalised quadrangles. This property is also responsible for the name of the GQ$(2,2)$. It implies that the smallest polygon contained in GQ$(2,2)$ is a quadrangle.

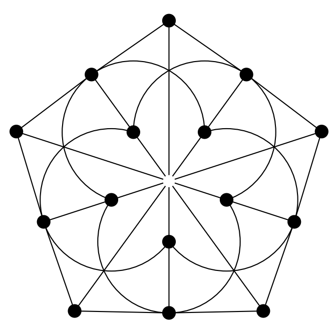


Figure 5. The generalised quadrangle GQ$(2,2)$

It can be constructed in the following way: the points are the 2-subsets of the set $\{1,2,3,4,5,6\}$ and the lines are the 15 3-subsets of 2-subsets. For example the lines $\{\{1,2\},\{3,4\},\{5,6\}\}$ and $\{\{1,2\},\{3,5\},\{4,6\}\}$ meet in the point $\{1,2\}$. If we consider the anti-flag $(p,L) = (\{1,2\},\{\{1,3\},\{2,4\},\{5,6\}\})$, then the unique point on L collinear with p is $\{5,6\}$ and the unique line through p meeting L is $\{\{1,2\},\{5,6\},\{3,4\}\}$.

An *ordinary n-gon* in a point-line geometry $(\mathcal{P},\mathcal{L},\mathcal{I})$ is a sequence

$$(x_1, L_1, x_2, L_2, \ldots, x_n, L_n)$$

of $2n$ distinct elements of $\mathcal{P} \cup \mathcal{L}$ such that each element of the sequence is incident with its neighbours, where it is understood that the neighbours of L_n are x_n and x_1.

Definition 16. *A* generalised m-gon *is a thick point-line geometry satisfying the following properties.*

(GP1) *There is no ordinary n-gon for $n < m$.*
(GP2) *Every two elements belong to an ordinary m-gon.*
(GP3) *There exists an ordinary $m+1$-gon.*

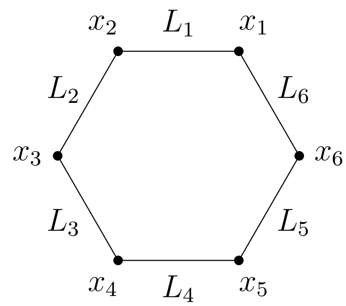


FIGURE 6. An ordinary 6-gon

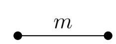
To exclude degenerate examples we assume $m \geq 3$. Generalised polygons are know as generalised *triangles*, *quadrangles*, *hexagons*, and *octagons* for $m = 3, 4, 6, 8$. In the context of buildings, an ordinary m-gon will be called an *apartment*. The *chambers* of the apartment are the incident point-line pairs (called *flags*).

Two elements in a generalised m-gon are called *opposite* if they are at distance is m. For example, in a projective plane, opposite elements are at distance 3. This means that an opposite pair of elements in a projective plane corresponds to an *anti-flag*.

In a generalised quadrangle (a GQ) a pair of opposite elements are at distance 4, so they correspond to two non-intersecting lines, or two non-collinear points.

In general, if m is even then opposite elements are of the same type, while if m is odd, then opposite elements are of different type.

A concise representation of a generalised m-gon is by its so-called *Coxeter diagram*



obtained by the link between geometry and Coxeter groups (see later).

The following theorem shows that the notion of a generalised polygon can be viewed as a generalisation of the notion of a projective plane.

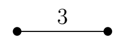
THEOREM 17. *A projective plane is a generalised 3-gon.*

Proof. We first show that the axioms (a1), (a2), (a3) follow from the axioms (GP1), (GP2), and (GP3). Consider two points x, y. Then by (GP2) they are contained in a triangle, and so there exists a line L containing both x and y. It there would be a second line M containing x and y, then M and L would form a 2-gon, which by (GP1) does not exist. This proves (a1). Similarly, one easily verifies (a2) (or you can use duality). Axiom (a3) is exactly (GP3). We can conclude that the the axioms (a1), (a2), (a3) follow from the axioms (GP1), (GP2), and (GP3).

The converse is also easy. The axiom (GP1) follows from (a1). To prove (GP2), we need to show that each two elements are in a triangle. There are 4 cases to consider: two points, two lines, a flag, and an anti-flag. Since two points are contained in a unique line (by (a1)), a flag is contained in a line, and an anti-flag is contained in two lines, it is enough to show that two lines are contained in a triangle. So consider two lines M and L. By (a2) they meet in a point p. Pick another point x on L, and a line $N \neq L$ through

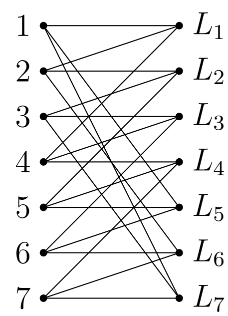
x. Then by (a2) the lines N and M meet in a point y, and that point must be different from p by (a1). Now (p, L, x, N, y, M) forms a triangle containing L and M $\square$

Using Coxeter diagrams, a projective plane will be represented by

 instead of .

The *incidence* graph of a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is the graph with vertices $\mathcal{P} \cup \mathcal{L}$ and edges the flags of Γ. The *distance* $\delta(v, w)$ between to elements of an incidence geometry Γ is the length of the shortest path from v to w in the incidence graph of Γ.

The incidence graph of the Fano plane is a 3-regular graph (a cubic graph) with 14 vertices and 21 edges. The smallest circuit has length 6 (*girth* 6). Any labelling of the points and lines will give the same graph, but if we choose the lines to be $L_i = \{i, i+1, i+3\}$, where the numbers are modulo 7, then we obtain the following bipartite graph.



Every two points are at distance two, and the maximum distance in the graph (called the *diameter*) is three. The shortest circuit (called the *girth*) has length 6. This graph is known as the *Heawood graph*, and it is the *6-cage*: and every cubic graph with less than 14 vertices has shorter circuits. The usual way to draw the Heawood graph is as in Figure 7.

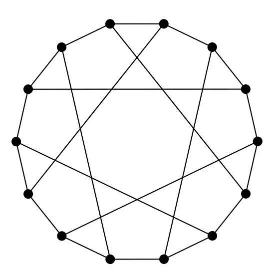
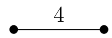


FIGURE 7. The Heawood graph: incidence graph of the Fano plane

Another graph associated to a point-line geometry is the *collinearity graph*, whose vertices are the points and two points are adjacent if they are on a common line.

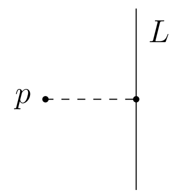
An alternative definition of a generalised polygon is using the incidence graph. A point-line geometry $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a generalised m-gon if its incidence graph is connected, has *diameter* m and *girth* $2m$, and each vertex has *degree* at least three.

While a projective plane is full of triangles, a generalised quadrangle does not contain any triangle. The Coxeter diagram would be 

, but the diagram with a double line

is often used instead.

Theorem 18. *Given an anti-flag (p, L) in* GQ*, there exists exactly one line through p meeting L.*



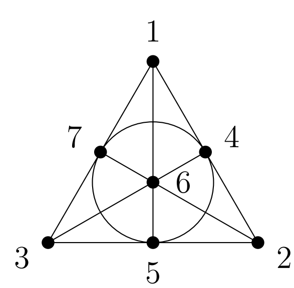
Proof. By axiom (GP2) there exists an ordinary quadrangle containing the elements L and p. One of the two lines containing p must meet L. If there would be two line through p meeting L, then we would have a triangle, which is not allowed by (GP1). □

To finish this lecture we give an example of the construction of a linear code from a geometry. The code obtained in this way from the Fano plane will be given in detail.

A $\mathbb{F}_q$-linear code C is a subspace of $\mathbb{F}_q^n$ (here $\mathbb{F}_q$ denotes the finite field with q elements). If the dimension of C is k then C is called a linear $[n, k]$-code, and n is called the *length* of the code. Elements of C are called *codewords*. The distance between two codewords $u, v \in C$ is the number of nonzero coordinates of $u - v$. The *minimum distance* of C is $\min\{d(u, v) \ : \ u, v \in C, u \neq v\}$. If C has minimum distance d, then C is called a linear $[n, k, d]$-code. The *weight* of a codeword is the number of nonzero coordinates.

To construct a code from a point-line geometry $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ we make use of the *incidence matrix*: the rows are indexed by the lines and the columns are indexed by the points, and the (L, p)-entry is 1 or 0 depending on whether (L, p) is a flag or an anti-flag. This will be an $(m \times n)$-matrix where $m = |\mathcal{L}|$ and $n = |\mathcal{P}|$. The code C obtained from the point-line geometry is then defined as the subspace of $\mathbb{F}_2^n$ generated by the rows of the incidence matrix.

So for the Fano plane, if we label the points as follows then the lines are $L_i = \{i, i+1, i+3\}$,



where the numbers are modulo 7, and we obtain the incidence matrix

$$G = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This is a (7×7)-matrix (here $m = n$) and has rank 4 (verify this!). Therefore the code C obtained from the Fano plane has length 7 and dimension 4. The code C consists of $2^4 = 16$ codewords. To find the minimum distance of C, observe that the sum of two rows of G gives a codeword of C corresponding to the complement of a line in the Fano plane. There are 7 lines, so there are also 7 complements of lines. Also notice that the sum of the rows corresponding to the three lines through a point gives the all-one vector $(1, 1, 1, 1, 1, 1, 1)$. Since C is a subspace it also contains the zero vector $(0, 0, 0, 0, 0, 0, 0)$. This amounts to $7 + 7 + 1 + 1 = 16$. So we have all the codewords of C. It is now easy to see that the minimum distance of C is 3 (it is equal to the minimum weight of the nonzero codewords). We conclude that C is a linear $[7, 4, 3]$-code. It is known as the 4-dimensional Hamming code of length 7.

Lecture 3. Some combinatorics of generalised polygons

A finite generalised polygon with $s+1$ points on a line and $t+1$ lines through each point is said to have *order* (s,t), and we usually write GP(s,t).

Theorem 19. *The dual of a* GP(s,t) *is a* GP(t,s).

Proof. The definition of a GP given in terms of its incidence graph does not distinguish between points and lines, and is therefore self-dual. □

To warm up we count the number of points and lines in a generalised quadrangle.

Theorem 20. *A* GQ(s,t) $\mathcal{G}$ *has* $s^2t+st+s+1$ *points and* $st^2+st+t+1$ *lines.*

Proof. Pick any line L. It follows from the axioms that every point of $\mathcal{G}$ not incident with L lies on a unique line intersecting L. This gives

$$(s+1)+(s+1)ts.$$

By duality $\mathcal{G}$ has $st^2+st+t+1$ lines. □

We have seen an example of a GQ$(2,2)$ with 15 points and 15 lines.

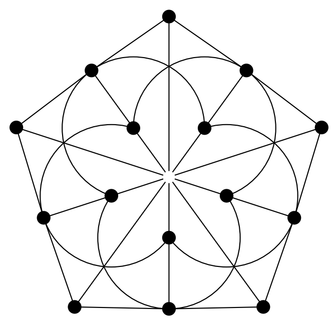


Figure 8. The generalised quadrangle GQ$(2,2)$

Next we will show that every generalised polygon has an order. We already proved this for projective planes: every line has the same number of points and every point lies on the same number of lines.

We will use the following two lemmas.

Lemma 21. *Every path of length at most* $m+1$ *consisting of distinct elements in a generalised* m*-gon* $\mathcal{G}$ *is contained in an ordinary* $(m+1)$*-gon.*

Proof. Consider any path of lenght at most $m+1$ and if necessary extend it to a path of length $m+1$:

$$\gamma=(x,x_1,x_2,\dots,x_{m+1}).$$

Then by (GP1) $\delta(x,x_m)=m$, so x and x_m are opposite. It follows that every element opposite to x has the same type as x_m. Therefore x_{m+1} is not opposite to x. Hence $\delta(x,x_{m+1}=m-1$, giving us an ordinary n-gon, call it α, containing γ. Since $\mathcal{G}$ is thick there exists an element $z\mathcal{I}x_{m+1}$ and $y\mathcal{I}x$ which are not contained in α. By (GP1) the distance between x and z must be m and the distance between z and y must therefore be $m-1$. Joining the path γ with a path of length $m-1$ from y to z together with the edges (x,y) and (z,x_{m+1}) give us an ordinary $(m+1)$-gon containing γ. □

LEMMA 22. *In a generalised m-gon $\mathcal{G}$, elements of the same type have the same degree and if m is odd then all elements of $\mathcal{G}$ have the same degree.*

Proof. (i) First consider two opposite elements x and y. By (GP1) there is a bijection between the neighbours of x and the neighbours of y. Therefore x and y have the same degree.

(ii) If x and y are two elements of the same type, which are not opposite, then consider a $(m+1)$-gon γ containing x and y. Since the incidence graph is connected, it is enough to prove that elements at distance 2 have the same degree. So we may as well assume that $\delta(x,y)=2$. Let u be the element in γ which is opposite to both x and y. Apply (i) to (x,u) and (y,u).

(iii) If m is odd, then opposite vertices are of different type, and therefore, by (i) and (ii), all elements have the same degree. □

Now that we have an *order* (s,t) for a generalised polygon $\mathcal{G}$ we can also count the number of points and lines in $\mathcal{G}$. This is done step by step. Starting with a point x first we count its neighbours (the lines through x), then we count the number of points on these lines (elements at distance two), then elements at distance three, etc. We use the notation $\Gamma_i(x)$ to denote the set of elements of $\mathcal{G}$ at distance i from x (with the convection that $\Gamma(x)=\Gamma_1(x)$).

THEOREM 23. *If x is a point in a generalised m-gon $\mathcal{G}$ of order (s,t) then for $i<m$ we have*

$$|\Gamma_i(x)| = (t+1)s^{\lfloor \frac{i}{2} \rfloor}t^{\lfloor \frac{i-1}{2} \rfloor}.$$

and the number of elements opposite to x is

$$|\Gamma_m(x)| = \begin{cases} s^{\lfloor \frac{m}{2} \rfloor}t^{\lfloor \frac{m-1}{2} \rfloor} & \text{for } m \text{ even,} \\ s^{m-1} & \text{for } m \text{ odd.} \end{cases}$$

Proof. There are $t+1$ lines through x and s points different from x on each of them. This amounts to

$$|\Gamma(x)| = (t+1) \text{ and } |\Gamma_2(x)| = (t+1)s.$$

If $m=3$ (projective plane) then each line not through x meets every line through x in a point different from x. So in this case there are st further lines (i.e. $|\Gamma_m(x)|=st$). This amounts to a total of $st+t+1$ lines and $(t+1)s+1$ points, as it should be.

If $m>3$, then through each of the $(t+1)s$ points at distance two from x there are t further lines which do not contain x, and all these lines must be distinct, since otherwise we would have a triangle, contradicting $m>3$. So in this case we have

$$|\Gamma_3(x)| = (t+1)st \ (m \geq 4).$$

If $m=4$, then fix a line L through x. The points opposite to x (at distance 4) are each on a unique line meeting L in one of its s points distinct from x. So this gives us sts points opposite to x, i.e. $|\Gamma_m|=s^2t$. In total we obtain $1+(t+1)s+sts$ points, as required.

Now we move up the game.

If $i < m$ then we claim that

$$|\Gamma_{i-1}(x)|\alpha = |\Gamma_i(x)|$$

where α is s or t depending on whether elements at distance i from x are points or lines.

To prove this claim, we will count flags (y, z) with $y \in \Gamma_{i-1}(x)$ and $z \in \Gamma_i(x)$.

Observe that if we first choose y there are $|\Gamma_{i-1}(x)|$ such choices for y. If y is a line, then for each such y there are s points z incident with y which are not on the unique path of length $i-1$ from y to x. If y is a point, then for each such y there are t lines z incident with y which are not on the unique path of length $i-1$ from y to x. This explains the left hand side of the equation. On the other hand if we first choose $z \in \Gamma_i(x)$, then there is a unique element y incident with z and at distance $i-1$ from x, namely the neighbour of z on the unique path of length i going from z to x.

This proves that for $i < m$ we have

$$|\Gamma_i(x)| = (t+1)s^{\lfloor \frac{i}{2} \rfloor} t^{\lfloor \frac{i-1}{2} \rfloor}.$$

If $i = m$, then we claim that

$$|\Gamma_{m-1}(x)|s = |\Gamma_m(x)|(t+1)$$

where $\{\alpha, \beta\} = \{s, t\}$. Again we count pairs (y, z). Note that, since opposite elements have the same degree, either z a point, or $s = t$. This time $y \in \Gamma_{m-1}(x)$ and $z \in \Gamma_m(x)$. The left hand side (starting with y) is the same as before, except that now we know that $\alpha = s$, since y is either a line or $s = t$. For the right hand side, once we have chosen z, the choice of y is no longer unique. This time there are $t+1$ paths from z to x of length m and on each of these the neighbour of z is a valid choice for y. This proves the claim.

Using the formula for $i = m-1$ we get.

$$|\Gamma_m(x)|(t+1) = (t+1)s^{\lfloor \frac{m+1}{2} \rfloor} t^{\lfloor \frac{m-2}{2} \rfloor},$$

and therefore

$$|\Gamma_m(x)| = s^{\lfloor \frac{m+1}{2} \rfloor} t^{\lfloor \frac{m-2}{2} \rfloor}.$$

This gives

$$|\Gamma_m(x)| = \begin{cases} s^{\frac{m}{2}} t^{\frac{m}{2}-1} & \text{for } m \text{ even} \\ s^{m-1} & \text{for } m \text{ odd} \end{cases}$$

as required. □

We finish the lecture by mentioning the following remarkable result proved by Feit and Higman in 1964.

Theorem 24. *Finite generalised m-gons exist only for $m \in \{3, 4, 6, 8\}$.*

We are therefore left with projective planes, GQ's, generalised hexagons and generalised octagons. All of these exist. We already saw examples of the first two, and Figure 9 is an example of a generalised hexagon (thanks to Stephen Glasby!). Generalised octagons also exist, but a construction is beyond the scope of these notes.

There are many different constructions for the split Cayley hexagon, it is the unique generalised hexagon of order $(2, 2)$. A construction using only the Fano plane goes as

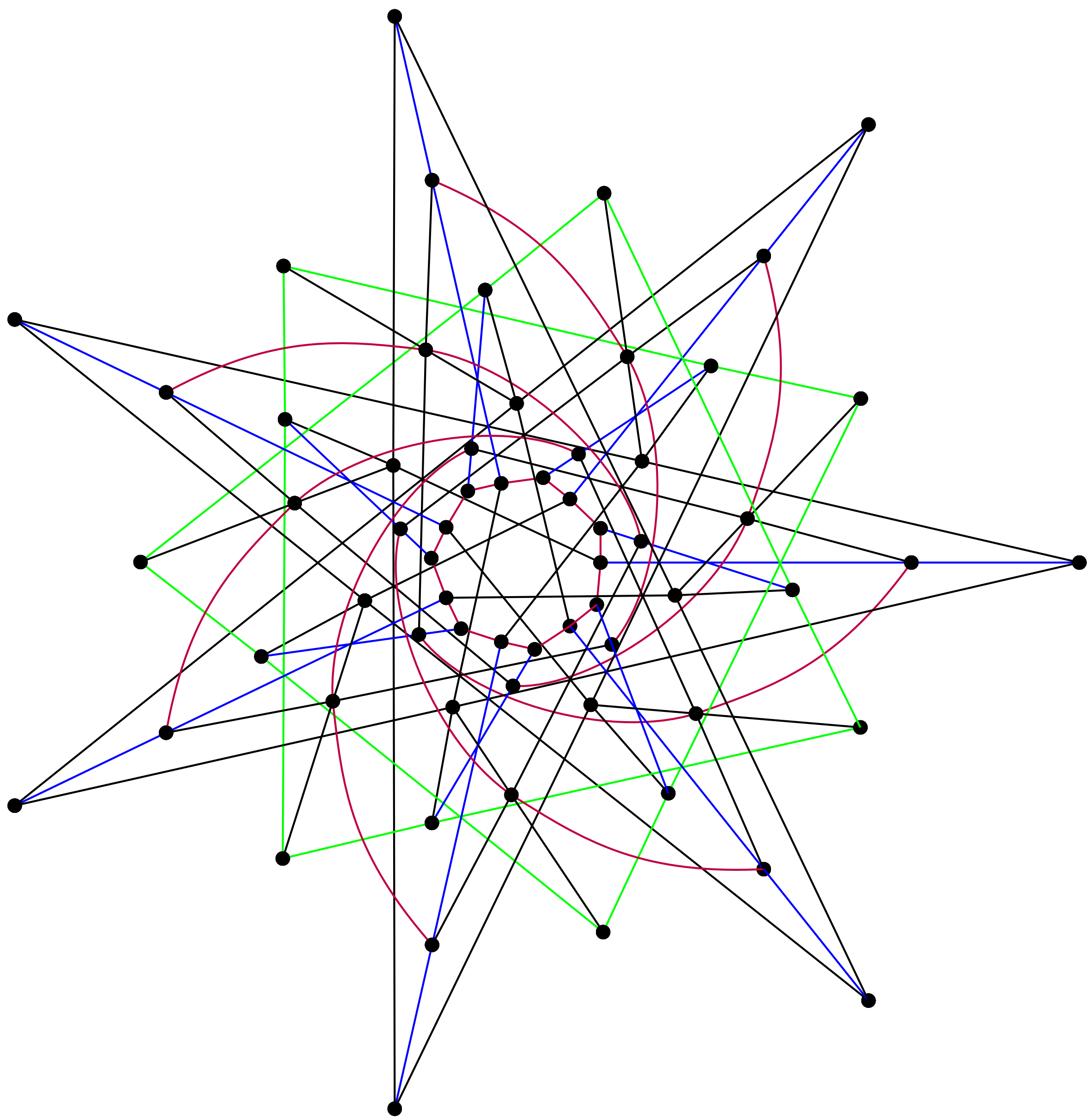


Figure 9. The split Cayley hexagon

follows. Define the set of "points" $\mathcal{P}$ as the set of points, lines, flags and anti-flags of the Fano plane. For each flag (p, L) define the "lines" $\{p, L, (p, L)\}$, $\{(p, L), (y, M), (x, N)\}$, and $\{(p, L), (x, M), (y, N)\}$ where L, M, N are the lines through p and p, x, y are the points on L. Incidence $\mathcal{I}$ is defined as symmetric containment. Then $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a generalised hexagon, precisely the split Cayley hexagon of order $(2, 2)$.

Lecture 4. Definition of a building.

Buildings are incidence geometries which may have more than just two types of objects. Generalised polygons are point-line geometries, they have two types of objects, and we say that such a geometry has *rank two*. The rank two geometries are the construction material we need to make buildings: the "building blocks". Recall that our geometries are always thick and connected.

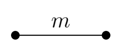
We are not ready to understand the following definition of a building yet, but it will serve as a guidance to develop some idea about what we are looking for. The examples and terminology which we will introduced below should clarify the following definition.

Definition 25. *A* building Δ *is an incidence geometry with a system of* apartments *such that the following properties hold:*

- (B1) *every apartment is isomorphic to a given Coxeter complex;*
- (B2) *any two elements are contained in a common apartment;*
- (B3) *for any two apartments* Σ *and* Σ' *and elements* $C, D \in \Sigma \cap \Sigma'$*, there exists an isomorphism* $\Sigma \to \Sigma'$ *which is the identity on* C, D*.*

The *type* of the buildings is defined as the type of the Coxeter complex (see below).

Since the rank 2 geometries are building blocks for our geometries of higher rank, it will be usual to have a concise representation for them. We use the following convention. A generalised m-gon with $m > 4$ is represented as



a generalised quadrangle (4-gon) is represented by

a projective plane (3-gon) by

and a digon (2-gon) by

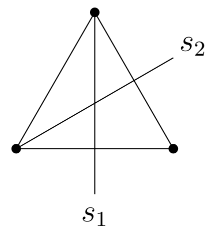
Our first aim is to understand the buildings of rank two.

Theorem 26. *A building of rank two of type*   *is a generalised* m*-gon.*

Recall that the apartments of a generalised m-gon are ordinary m-gons. It follows that axioms (B2) and (GP2) coincide. Also axiom (B3) is readily verified. We will now construct a so-called Coxeter complex associated to a generalised m-gon.

To fix ideas, we first consider the case that $m = 3$, in which case an apartment is a triangle. If we draw the triangle as an equiangular one, then it has the dihedral group $D_6 \cong S_3$ as automorphism group and this also gives a well-defined action of the group S_3 on the set of 6 flags contained in Σ. In fact we can reconstruct the apartment Σ as a chamber complex from the group.

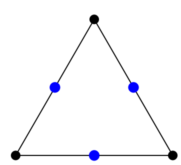
We do this by fixing two transpositions s_1 and s_2 in S_3 (two reflexions in D_3). Put $S = \{s_1, s_2\}$. In the figure below, the reflexions s_1 and s_2 are represented by their respective lines of fixpoints.



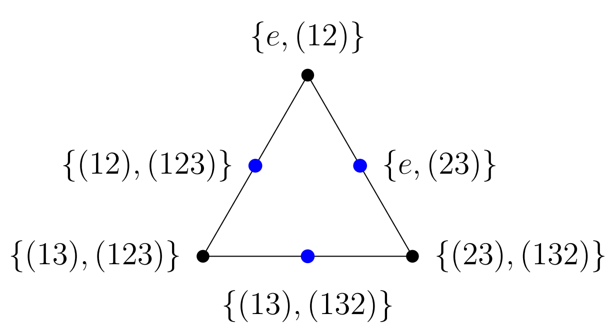
We define the subgroups generated by elements of S as *special subgroups*, and their left cosets as *special cosets*. We also include the singletons $\{w\}$, with $w \in S_3$, and identify $\{w\}$ and w. Define a partial order on the set of special coset by saying that $A \leq B$ if and only if $B \subseteq A$. So the maximal elements are the singletons x with $x \in S_3$, and we call them the *chambers*. Two chambers x, y are adjacent if $x \in \{ys_1, ys_2\}$. The special cosets are called the *simplices*.

We make everything explicit. Put $s_1 = (12)$ and $s_2 = (23)$. We have the six special cosets $\{e, (12)\}$, $\{(13), (123)\}$, $\{(23), (132)\}$, $\{e, (23)\}$, $\{(12), (123)\}$, and $\{(13), (132)\}$ of the special subgroups $\langle s_1 \rangle$ and $\langle s_2 \rangle$ (call these 6 cosets *elements*), together with the six chambers $\{\{w\} : w \in S_3\}$. Each *element* is contained in two chambers, and each chamber contains two *elements*.

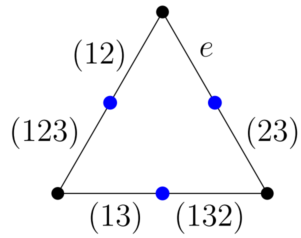
To facilitate the labeling of the flags on our picture we replace each "line" by a vertex and connect it to the points it contains. So we get the incidence graph. The flags correspond to the 6 line segments.



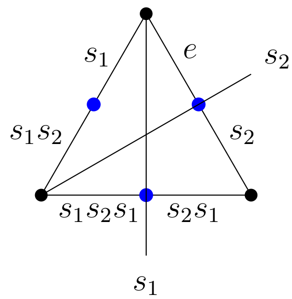
We can also divide the "elements" we just defined above into two sets of three elements. Let us call the cosets of $\langle s_1 \rangle$ elements *of type 1* and cosets of $\langle s_2 \rangle$ elements *of type 2*. It should now be straightforward to define a bijection, mapping the elements of type 1 to the vertices of Σ, elements of type 2 to the sides of Σ, and chambers to the flags contained in Σ.



The corresponding labeling of the flags is then as follows



Which is exactly what it should be, namely consistent with the images of the line segments under the reflexion group $\langle s_1, s_2 \rangle$.



Note that in the action of D_3 on the chambers of Σ, no element of D_3 fixes a chamber. Also, if two chambers in the apartment Σ are mapped onto each other by the reflexion r_i, then the two chambers share a point or a line. In other words the chambers are adjacent.

The group $D_6 \cong S_3$ is an example of a finite *Coxeter group*, and the construction of the chamber complex from S_3 explained above works in general. First we give the definition of a Coxeter group and then we will construct its associated *Coxeter complex* Σ.

Put $I = [n]$. A group W with presentation

$$\langle r_1, \ldots, r_n \mid (r_i r_j)^{m_{ij}} = 1 \text{ for all } i, j \in I \rangle$$

with $m_{ij} \in \mathbb{Z} \cup \{\infty\}$, $m_{ii} = 1$ for all i, $m_{ji} = m_{ij} \geq 2$ for all $i \neq j$ is called a *Coxeter group*. When a Coxeter group W is given together with a set of generating involutions $S = \{r_1, \ldots, r_n\}$, the pair (W, S) is called a *Coxeter system*.

As in the example we define the *chambers* to be the elements of W and the *simplices* to the special cosets of (W, S), where *special cosets* are cosets of subgroups generated by elements of S. Also recall two chambers x, y are called i-adjacent if and only if $x = yr_i$.

Throughout these notes, by $[n]$ we denote the set $\{1, \ldots, n\}$. A *chamber system* $\mathcal{C}$ *over* $[n]$ is a set equipped with n equivalence relations R_i, $i = 1, \ldots, n$. The elements of $\mathcal{C}$ called *chambers* and if $(C, D) \in R_i$ then C and D are said to be i*-adjacent*, in which case we will write $C \sim_i D$. The *rank* of $\mathcal{C}$ is n.

Given a Coxeter system (W, S), for each $i \in I$ define the relation $\sim_i$ on W by

$$x \sim_i y \iff x = yr_i.$$

LEMMA 27. *A Coxeter system* (W, S) *together with the relations* $\sim_i$ *as above forms a chamber system.*

Proof. Show that each $\sim_i$ is an equivalence relation on W. □

A *gallery* $\mathcal{G}$ is a finite sequence of adjacent chambers $\mathcal{G} = (C_0, \ldots, C_k)$. If C_{j-1} is i_j-adjacent to C_j then the $\mathcal{G}$ is said to have type $i_1 i_2 \ldots i_k$ (considered as a "word" over I). If $J \subseteq I$ and each $i_j \in J$ then $\mathcal{G}$ is also called a J-gallery.

A chamber system $\mathcal{C}$ is called *connected* (J-connected) if each two chambers are contained in a common gallery (J-gallery). The J-connected components are called *J-residues.* The subset J is called the *type* of the J-residue, and its *co-type* is defined as $I \setminus J$. The residues of co-type i are considered as the *elements of type i*.

The *Coxeter complex* $\mathcal{C}(W, S)$ is defined as the poset of special cosets of the Coxeter system (W, S).

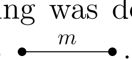
Exercise 28. *Construct the Coxeter complex of type*

and label an apartment of a generalised quadrangle accordingly.

Lecture 5. Buildings of type A_n and B_n

The building of type A_n. The standard example of a building is a projective space $\mathrm{PG}(n, K)$. Recall that the points of $\mathrm{PG}(n, K)$ (elements of type 1) are the 1-dimensional subspaces of K^{n+1}, the lines of $\mathrm{PG}(n, K)$ (elements of type 2) are the 2-dimensional subspaces of $\mathrm{PG}(n, K)$, etc. The $(n-1)$-dimensional subspaces of K^{n+1} are also called *hyperplanes* of $\mathrm{PG}(n, K)$ (they are elements of *type* n).

Consider a geometry $\mathcal{G}$ of rank 3, i..e a geometry consisting of three different types of objects. We call them points, lines and planes. A flag in $\mathcal{G}$ is a set of elements $\mathcal{G}$ of distinct types, which are pairwise incident. For example, an incident point-line pair is a flag, or a point and a plane containing it, or a triple (p, L, π) where $p\mathcal{I}L\mathcal{I}\pi$. The last type of flag is called a *maximal flag*, it contains one element of each type. Maximal flags will be called *chambers* of the building. As an example, consider the chamber $C = (p, L, \pi)$, consisting of elements of type 1 (the point p), type 2 (the line L), and type 3 (the plane π) in a projective space $\mathrm{PG}(3, K)$.

Before we discuss the buildings associated to $\mathrm{PG}(n, K)$ we explain another way to interpret the type of a building. Recall that the type of a building was defined as the Coxeter diagram. For example, the type of a generalised m-gon is 

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The *shadow* of an element x (notation $\mathrm{shad}(x)$) is the set of elements incident with x together with the induced incidence relation. So the shadow is a geometry itself. For example the shadow of the point p in $\mathrm{PG}(3, K)$ is the geometry of lines and planes through p, which is a projective plane.

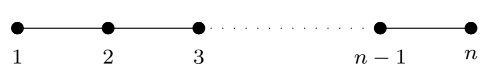
Exercise 29. *Let p be a point of $\mathrm{PG}(3, K)$. Show that $\mathrm{shad}(p)$ is a projective plane.*

The shadow of L consists of all points on L and all planes through L, which is a digon: every two elements of distinct types are incident. The shadow of π is obviously a projective plane, since it consists of all points and lines in π. Using the representation of a digon and of projective plane introduced in the previous lecture, it follows that the projective space $\mathrm{PG}(3, K)$ can be represented by the diagram

where the relation between two nodes is determined by the shadow of an element of the third node.

The *shadow of a flag* is the set of elements which are incident with all elements of the flag, together with the induced geometry.

It is not difficult to see that the projective space $\mathrm{PG}(n, K)$ can be represented by the diagram where we have added numbers below the nodes to indicate the *type* of the elements.

A_n 

There are n types of elements in $\mathrm{PG}(n, K)$, which is therefore a geometry of *rank* n. The diagram itself is of type A_n.

This is our first example of a building. The n-dimensional projective space is a building of type A_n.

We will first investigate our example of a building $\Delta(A_n)$ of type A_n. The *chambers* of Δ are the maximal flags of $\mathrm{PG}(n, K)$ and the *apartments* are the sets of $(n+1)$ points spanning $\mathrm{PG}(n, K)$, together with all flags that can be formed with these $(n+1)$ points. For example, in a projective plane, an apartment is a triangle together with all flags contained in it. If the triangle has vertices x, y, z and sides L, M, N (where x is on L and M, and y is on M), then the flags are x, y, z, L, M, N, (x, L), (x, M), (y, M), (y, N), (z, L), (z, N).

The set of flags in an apartment satisfies the properties of what is called a *chamber complex*. A pair (S, X) consisting of a set S with a set of subsets X of S is called a *simplicial complex* if for each x the set $\{x\}$ belongs to X and every subset of an element of X belongs to X. If $A \in X$, then any subset of A is called a *face* of A. We call (S, X) a *chamber complex* if all its maximal simplices (the *chambers*) have the same cardinality, and any two can be connected by a *gallery*, a sequence of chambers $(C_1, \ldots, C_k)$, where each C_{i+1} can be obtained from C_i by changing one element. The chambers C_i and C_{i+1} are said to be *adjacent*.

EXERCISE 30. *Convince yourself that an apartment of* $\mathrm{PG}(n, K)$ *is a chamber complex.*

To show that $\mathrm{PG}(n, K)$ (together with this system of apartments, which we have just introduced) is a building, we should verify that each such apartment is isomorphic to a Coxeter complex. Moreover according to our statement that it is a building of type A_n, it should be the Coxeter complex of the Coxeter group with diagram A_n, i.e. the symmetric group on $n+1$ letters S_{n+1} or $Sym(n+1)$.

EXERCISE 31. *Show that the Coxeter group with Coxeter diagram* A_n *is isomorphic to* S_{n+1}.

Fix an apartment Σ in $\Delta(A_n)$. Consider the simplicial complex determined by Σ, this is the set of flags consisting of elements in Σ. Number the points in Σ as $p_1, \ldots, p_{n+1}$. Define an action of $\sigma \in S_{n+1}$ on the points of Σ by

$$\sigma \ : \ p_i \mapsto p_{\sigma(i)}.$$

This extends to an action of the permutation of σ to the set of all flags contained in the apartment: a flag F

$$p_{i_1} \subseteq \langle p_{i_1}, p_{i_2} \rangle \subseteq \ldots \subseteq \langle p_{i_1}, p_{i_2}, \ldots, p_{i_k} \rangle$$

has image F^σ

$$p_{\sigma(i_1)} \subseteq \langle p_{\sigma(i_1)}, p_{\sigma(i_2)} \rangle \subseteq \ldots \subseteq \langle p_{\sigma(i_1)}, p_{\sigma(i_2)}, \ldots, p_{\sigma(i_k)} \rangle.$$

EXERCISE 32. *Show that the above defined action of* S_{n+1} *on* Σ *is transitive, and that its restriction to the set of chambers of* Σ *is sharply transitive (i.e. regular).*

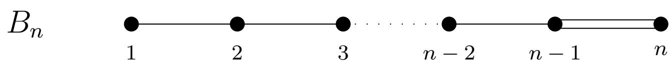
Now let us fix a chamber C in Σ (call it the *fundamental* chamber) and label it with the identity e of S_{n+1}. Then for every $\sigma \in S_{n+1}$ we obtain a chamber C^σ which we label with σ. This labels all the chambers (uniquely!), since the group S_{n+1} acts (sharply!) transitive on the set of chambers of Σ.

Put $S = \{s_1, \ldots, s_n\}$ equal to the set of generators of the Coxeter group $W = S_{n+1}$ associated to the diagram of type A_n, i.e. $s_i = (i, i+1)$.

Now consider a panel P of C, say it is the flag of co-type i contained in C. Then P is also contained in the chamber C^{s_i}. So, we can identify P with the special coset $\{e, s_i\}$ of Coxeter system (W, S).

In this way each of the panels, and in fact each of the flags, of Σ can be associated to a special coset of the Coxeter system (W, S), resulting in the necessary isomorphism between the apartment Σ and the Coxeter complex from (W, S).

The building of type B_n. The Coxeter diagram of type B_n is So the shadows of the flags



of co-type $\{i, j\}$ in the building of type B_n are projective planes for $j = i + 1$, $i < n - 1$, digons for $|i - j| > 1$. This is exactly the same as for the building of type A_n. The difference between the building of type A_n and of type B_n lies in the fact that a flag of co-type $\{n - 1, n\}$ in B_n is a generalised quadrangle.

We will give a construction for $n = 3$. Consider the quadratic form

$$f(X) = X_0X_1 + X_2X_3 + X_4X_5,$$

where $X = (X_0, X_1, \ldots, X_5)$, and the associated quadric $\mathcal{Z}(f)$ in $\mathrm{PG}(5, K)$. Define Δ as the incidence geometry consisting of points, lines and planes contained in $\mathcal{Z}(f)$, where incidence is symmetric containment. Then Δ is of type B_3, as in incidence geometry, i.e. the shadow of a point is a generalised quadrangle, the shadow of a line is a digon and the shadow of a plane is projective plane.

The apartments in Δ are as follows. Consider the basis B consisting of $u_i = e_{2(i-1)+1}$, $v_i = e_{2(i-1)+2}$, for $i = 1, 2, 3$, where $e_1, \ldots, e_6$ is the standard basis of K^6. Let β denote the symmetric bilinear form on K^6 defined by the quadratic form f. So

$$\beta(u, v) = g(u, v)$$

where

$$g(X, Y) = f(X + Y) - f(X) - f(Y).$$

Then $\beta(u_i, v_j) = \delta_{ij}$, $\beta(u_i, u_j) = \beta(v_i, v_j)$ for all $i, j \in \{1, 2, 3\}$.

Exercise 33. *Show that the line of $\mathrm{PG}(5, K)$ defined by the vectors u and v of K^6 is contained in $\mathcal{Z}(f)$ if and only if $f(u) = f(v) = 0$ and $b(u, v) = 0$.*

Define the apartment Σ as the set of subspaces of $\mathrm{PG}(5, K)$ which are spanned by subsets of B and are contained in $\mathcal{Z}(f)$.

Exercise 34. *Draw an octahedron and label the vertices by the points of Σ such that two vertices are adjacent if and only the line they span (as points of $\mathrm{PG}(5, K)$) is contained in $\mathcal{Z}(f)$.*

Exercise 35. *The symmetry group of the octahedron is isomorphic to the Coxeter group of type B_3.*

For each set of 6 points spanning $\mathrm{PG}(5,K)$ and satisfying the conditions $\beta(u_i,v_j)=\delta_{ij}$, $\beta(u_i,u_j)=\beta(v_i,v_j)$ for all $i,j\in\{1,2,3\}$, we define an apartment of the incidence geometry Δ as above. Then Δ together with this system of apartments is a building of type B_3.

Lecture 6. Buildings and Groups

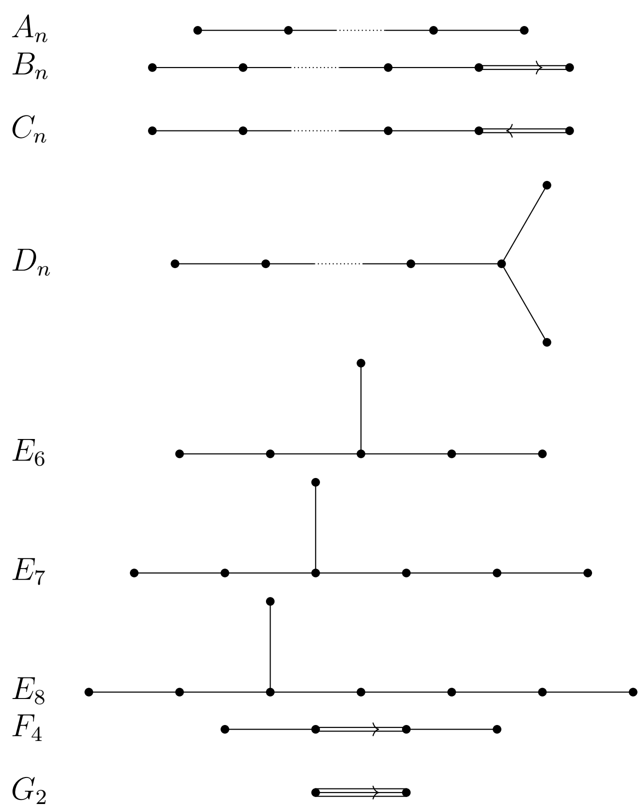
One of the major achievements in mathematics in the 20th century is the classification of finite simple groups.

Without going into too much detail, the classification basically says that a finite simple group must be either

(i) cyclic of prime order,
(ii) alternating,
(iii) of Lie type, or
(iv) one of the 26 sporadic groups (27 if the Tits group is counted as such).

The *ATLAS* of Finite Group Representations `<http://brauer.maths.qmul.ac.uk/Atlas/v3/>` provides a searchable database of representations and other data for many finite simple groups.

The biggest chunk of the finite simple groups are the *groups of Lie type*. These can be defined in various ways, e.g. as groups of automorphisms of certain Lie algebras. Every semi-simple Lie algebra over an algebraically closed field of characteristic 0 is the direct sum of simple Lie algebras. Finite-dimensional simple Lie algebras are classified by the connected Dynkin diagrams. They fall into the following classes:



The groups of Lie type contain the finite simple groups associated to the classical groups, the Chevalley groups, and the twisted Chevalley groups.

Finite groups of Lie type are characterised by the fact that they all posses a so-called *Tits system*.

A *Tits system* in a group G is a quadruple (B, N, W, S) for which the following conditions hold.

(BN1) B and N are subgroups of G and $G = \langle B, N \rangle$.
(BN2) $T = B \cap N \trianglelefteq G$ and $W = G/T$ is a Coxeter group with distinguished set of generators $S = \{s_1, \ldots, s_n\}$.
(BN3) $BsBwB \subset BwB \cup BswB$ for each $w \in W$ and $s \in S$.
(BN4) $sBs \neq B$ for each $s \in S$.

The integer n is called the *Lie rank* of G. Note that wB, $w \in W$, is well defined. The pair of subgroups (B, N) is also known as a *BN-pair*. It can be shown that the existence of a Tits system in a group G leads to a decomposition of G into the disjoint union $G = \cup BwB$, also called the *Bruhat decomposition*.

Every Tits system defines a building. Let (B, N, W, S) be Tits system for the group G. First we define a chamber system. The chambers are the left cosets of B and two chambers gB and hB are i-adjacent if and only if $g^{-1}h = s_i$.

Next we define the the poset $\Delta(G, B)$. The *special subgroups of* G are defined as subgroups of G containing B, and the *special cosets*, as cosets of special subgroups. The poset $\Delta(G, B)$ is then defined as the set of special cosets together with the partial order defined by reversed containment (c.f. the Coxeter complex).

Finally we define the apartments. The *fundamental apartment* Σ is the set of special cosets $\{wP \;:\; w \in W, P \text{ a special subgroup}\}$. The system of apartments is then defined as the set $\{g\Sigma \;:\; g \in G\}$.

Every building defines a Tits system. The converse is true under certain extra conditions on the building. Let Δ be a building of type W. So W is a Coxeter group and every apartment in Δ is isomorphic to the Coxeter complex associated to W. Moreover assume that Δ is a thick building admitting a group G of automorphisms, whose action on Δ is *strongly transitive*. This means that G acts transtively on the set of pairs (C, Σ) where C is a chamber and Σ is an apartment containing C.

Fix a chamber C and an apartment Σ containing C. Define the group B as the stabiliser of C in G and the group N as the stabiliser of Σ in G. Then $T = B \cap N$ is the kernel of the action of N on Σ, and therefore T is normal in N. The group N/T is isomorphic to the image of the action of N on Σ and is therefore isomorphic to W.

The set S of reflexions corresponds to the co-dimension 1 faces of C. For each face A of C of co-type i (we called this a *panel of* C), there $s_i \in S$ mapping C to s_iC and fixing A.

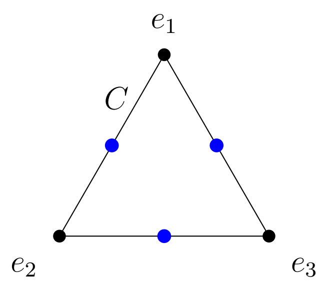
The parabolic subgroups can be obtained as follows. Consider a chamber hC which is i-adjacent to C, and let A be the panel $C \cap hC$. Then $hA = A$ since h is type-preserving, h maps C to hC, and A is the unique panel of type i in both C and hC. Conversely, if $hA = A$, then $C = hC$ or C and hC are i-adjacent. It follows that C and hC are i-adjacent if and only if h belongs to the stabiliser of A. The stabiliser of A is called a *special subgroup* P_{s_i}. Similarly, for each subset $S' \subseteq S$ we define the *special subgroup* $P_{S'}$

as the stabiliser of the face of C of co-type S'. This defines a poset isomorphism between the Coxeter complex and the poset $\Delta(G, S)$ defined above, mapping special subgroups of W to special subgroups of G.

Note that in a given group G, there can be more than one Tits system giving the same building: B is the stabiliser of a chamber so it cannot change, but N can often be replaced by one of its subgroups.

Explicit description of the Tits system associated to a building of type A_2

We return to our favourite example. Fix an apartment Σ corresponding to the basis e_1, e_2, e_3 and a chamber C corresponding to the flag (p, L), where $p = \langle e_1 \rangle$ and $L = \langle e_1, e_2 \rangle$, as in the following illustration.



Define the action of the group $\mathrm{GL}(3, K)$ on the vectors of K^3 as

$$A \,:\, x \mapsto xA^T.$$

This defines an action of $\mathrm{GL}(3, K)$ on the points (and consequently also on lines) of the projective plane $\mathrm{PG}(2, K)$. The kernel of this action is the set of matrices in $\mathrm{GL}(3, K)$ fixing each points of $\mathrm{PG}(2, K)$, i.e. matrices of the form

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix},$$

for some nonzero $a \in K$. The image of this action is the *projectivity group* $\mathrm{PGL}(3, K)$ (or *projective linear group*).

An element g in the stabiliser B of the chamber C must fix the point p and so the first column of an element of B must have a zero in the last two positions. Note that it's first element does not need to be equal to 1 since the point p is a projective point, and so its coordinates *homogeneous*, i.e. determined up to a nonzero scalar factor. The matrix defined g must therefore have the form

$$\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$

The element g of B must also preserve the line L, so the image of a point with last coordinate equal to zero must again have last coordinate equal to zero. To ensure this property it is enough that the entries in positions $(3,1)$ and $(3,2)$ are zero. This gives the form

$$\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}.$$

We conclude that the group B is the subgroup of $\mathrm{PGL}(3,K)$ induced by the upper triangular matrices in $\mathrm{GL}(3,K)$.

The group N is the stabiliser of the apartment Σ. If an element of N fixes each point of Σ then it must be induced by a matrix of the form

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix},$$

for some nonzero $a,b,c \in K$. But an element of N does not need to fix each of the points defined by the basis e_1, e_2, e_3, as it could also permute them. So every element of N is the composition of the element induced by the above matrix together with a permutation matrix. These matrices are called *monomial matrices*; they have exactly one nonzero element in each row and each column. Here are the six possible shapes:

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{bmatrix}, \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & a & 0 \\ b & 0 & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 0 & 0 & a \\ b & 0 & 0 \\ 0 & c & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{bmatrix}$$

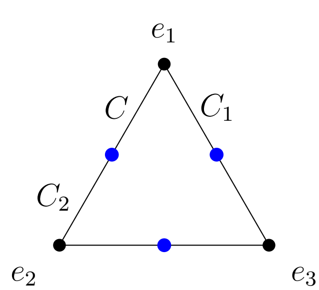
for nonzero $a,b,c \in K$. It is now straightforward to see that the group $T = B \cap N$ consists of the elements induced by the diagonal matrices

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix},$$

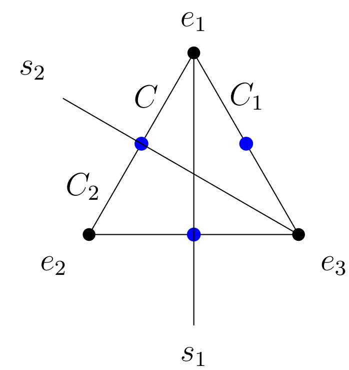
and that N/T is isomorphic to the group induced by the set of monomial matrices where each nonzero entry is equal to 1. In other words N/T is the group S_3 consisting, where its elements are represented by permutation matrices:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

In order to find out the transpositions corresponding to the elements of S, it suffices to find the chambers which are adjacent to the fundamental chamber. These are C_1 and C_2.



The reflexions s_1 and s_2 are therefore corresponding to the permutation matrices



$$s_1 \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } s_2 \approx \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The BN-pair associated to a building of type A_n Similarly, for $G = \mathrm{PGL}(n+1, K)$, B is the subgroup induced by the set of upper triangular matrices and N is the subgroup induced by the set of matrices with one non-zero entry in each row and column (*monomial matrices*). Then $T = B \cap N$ is the subgroup corresponding to the diagonal matrices. In this case $W \cong S_{n+1}$ since N/T consists of the $(n+1) \times (n+1)$ permutation matrices. The resulting building has as chambers the maximal flags of $\mathrm{PG}(n, K)$.