FINITE SEMIFIELDS AND RELATED STRUCTURES IN FINITE GEOMETRY

Michel Lavrauw

1. Introduction

These notes are part of the course "Finite Geometry" given at Ghent University by Professor F. De Clerck. They consist of material for approximately four hours, depending on the level of the students.

Remark 1.1. *As indicated in the title we will only be concerned with* **finite** *semifields. So whenever we mention projective planes, semifields, or other structures, it should be clear to the reader that these structures are always finite, even when the word finite is omitted.*

Remark 1.2. *In the earlier literature ($\leq \pm$ 1970) semifields were often called division algebras, division rings or distributive quasifields.*

The first part is aimed at understanding the connection between translation planes and spreads; the so-called André-Bruck-Bose construction. Next we define the notion of semifields (first studied by L. E. Dickson (1906)) and use a vector space representation to obtain semifield spreads, after which we can apply the André-Bruck-Bose construction to obtain semifield planes. We define the notion of isotopy and mention the important result proved by A. A. Albert in 1960, which says that two semifields are isotopic if and only if the corresponding planes are isomorphic. All of this belongs to, let's say, the "classical" part of the notes. It gives us the necessary background to be able to deal with some more recent topics in Finite Geometry which have received quite a lot of attention during the last two decades. We start this "modern" part with a particular configuration of subspaces which turns out to be equivalent to semifields, i.e., each such configuration can be used to construct a semifield and vice versa, each semifield gives rise to such a configuration. We conclude these notes with an example of a well-studied structure in finite geometry related to semifields, namely translation ovoids (sometimes called semifield ovoids) of the parabolic quadric in 4-dimensional projective space. We show their correspondence with semifields by constructing them from a particular case of the configuration of subspaces mentioned above.

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2. Translation planes and spreads: the André-Bruck-Bose construction

2.1. **Recap: perspectivities and translations.** We recall that a *perspectivity* in a projective plane has a *centre* and an *axis* and is called an *elation* if the centre is on the axis and a *homology* otherwise. A pespectivity is determined by the image of single non-fixed element, and the existence of a perspectivity gives rise to the famous *configuration of Desargues* (*minor* in case of a homology). A projective plane is said to be (V, ℓ)-*transitive* if the group of perspectivities with centre V and axis ℓ acts transitive on the points different from V of each line through V different from ℓ. A projective plane is called a *translation plane* if there exists a line ℓ such that the plane is (V, ℓ)-transitive for each point V on ℓ. In this case ℓ is called a *translation line* and the elations with ℓ as axis are called *translations*. Often a translation plane is identified with the associated affine plane with respect to the translation line, and the group of translations is called the *translation group*. The translation group of a translation plane of order n is elementary abelian of order n^2.

2.2. **Spreads.** Let $\mathcal{S}$ be a set of $(t-1)$-dimensional subspaces of $\mathrm{PG}(n-1,q)$. Then $\mathcal{S}$ is called a $(t-1)$-*spread* of $\mathrm{PG}(n-1,q)$ if every point of $\mathrm{PG}(n-1,q)$ is contained in exactly one element of $\mathcal{S}$. If $\mathcal{S}$ is a set of subspaces of $V(n,q)$ of rank t, then $\mathcal{S}$ is called a t-*spread* of $V(n,q)$ if every vector of $V(n,q) - \{0\}$ is contained in exactly one element of $\mathcal{S}$.

Theorem 2.1. *There exists a $(t-1)$-spread in $\mathrm{PG}(n-1,q)$ if and only if t divides n.*

Proof. Suppose there exists a $(t-1)$-spread $\mathcal{S}$ of $\mathrm{PG}(n-1,q)$. Since every point of $\mathrm{PG}(n-1,q)$ is contained in exactly one element of $\mathcal{S}$, the number of points of a $(t-1)$-space has to divide the number of points of $\mathrm{PG}(n-1,q)$. In other words $\frac{q^t-1}{q-1}$ divides $\frac{q^n-1}{q-1}$. This implies that t divides n.
Conversely, suppose t divides n, and put $n = rt$. We construct a $(t-1)$-spread of $\mathrm{PG}(rt-1,q)$ as follows. The points of $\mathrm{PG}(r-1,q^t)$ are the subspaces of rank 1 of $V(r,q^t)$. If we look at $\mathrm{GF}(q^t)$ as a vector space of rank t over $\mathrm{GF}(q)$ then $V(r,q^t)$ becomes a vector space of rank rt over $\mathrm{GF}(q)$, $V(rt,q)$. A subspace of rank 1 in $V(r,q^t)$ induces a subspace of rank t in $V(rt,q)$. So the points of $\mathrm{PG}(r-1,q^t)$ induce subspaces of rank t in $V(rt,q)$. In this way it is clear that the points of $\mathrm{PG}(r-1,q^t)$, can be seen as the elements of a $(t-1)$-spread $\mathcal{S}$ of $\mathrm{PG}(rt-1,q)$. □

The $(t-1)$-spread constructed in the proof of Theorem 2.1 is called a *Desarguesian spread*.

Exercise 2.2. *Let $\mathcal{D}$ be a Desarguesian $(t-1)$-spread of $\mathrm{PG}(rt-1,q)$. Show that $\mathcal{D}$ induces a Desarguesian spread in every subspaces which is spanned by elements of $\mathcal{D}$.*

Let $\mathcal{S}$ be a $(t-1)$ spread of $\mathrm{PG}(rt-1,q)$. If $t=1$ then $\mathcal{S}$ is just the set of points of $\mathrm{PG}(r-1,q)$. If $t=2$ then $\mathcal{S}$ is called a *line spread* or a *spread of lines*. If $t=3$ then $\mathcal{S}$ is called a *plane spread* or a *spread of planes*.

2.3. **André-Bruck-Bose.** Let $\mathcal{S}$ be a $(t-1)$-spread in $\mathrm{PG}(2t-1,q)$. Consider $\mathrm{PG}(2t-1,q)$ as a hyperplane of $\mathrm{PG}(2t,q)$. We define an incidence structure $(\mathcal{P},\mathcal{L},\mathcal{I})$ as follows. The pointset $\mathcal{P}$ consists of all points of $\mathrm{PG}(2t,q) \setminus \mathrm{PG}(2t-1,q)$ and the lineset $\mathcal{L}$ consists

of all t-spaces of $\mathrm{PG}(2t,q)$ intersecting $\mathrm{PG}(2t-1,q)$ in an element of $\mathcal{S}$. The incidence relation $\mathcal{I}$ is containment.

THEOREM 2.3. *The incidence structure $(\mathcal{P},\mathcal{L},\mathcal{I})$ is a translation plane of order q^t. Moreover every translation plane can be constructed in this way.*

EXERCISE 2.4. *Prove that the incidence structure $(\mathcal{P},\mathcal{L},\mathcal{I})$ is an affine plane of order q^t.*

This is a fundamental theorem in the Finite Geometry; it says that there is one-to-one correspondence between translation planes and spreads (where the rank of the elements of the spread is half the rank of the vector space). The construction is called the *André-Bruck-Bose construction*. The next theorem motivates the choice of the term Desarguesian spread.

THEOREM 2.5. *A $(t-1)$-spread of $\mathrm{PG}(2t-1,q)$ is Desarguesian if and only if the corresponding translation plane is Desarguesian, i.e. isomorphic to $\mathrm{PG}(2,q^t)$.*

Two spreads are said to be *equivalent* if there exists a collineation of the projective space mapping one spread onto the other.

THEOREM 2.6. *Two translation planes are isomorphic if and only if the corresponding spreads are equivalent.*

2.4. **A geometric construction of a non-Desarguesian translation plane.** Consider three mutually skew lines A,B,C in $\mathrm{PG}(3,q)$.

EXERCISE 2.7. *Prove that there are exactly $q+1$ lines in $\mathrm{PG}(3,q)$ meeting each of the lines A,B,C.*

This set of $q+1$ lines, say $\mathcal{R}$, is called a *regulus* of $\mathrm{PG}(3,q)$. Similarly, any three lines of $\mathcal{R}$ determine a regulus.

EXERCISE 2.8. *Prove that each set of three lines of $\mathcal{R}$ determines the same regulus $\mathcal{R}'$.*

The reguli $\mathcal{R}$ and $\mathcal{R}'$ are called *opposite*. An element of the opposite regulus of a regulus $\mathcal{R}$ is called a *transversal* of $\mathcal{R}$. If a linespread of $\mathrm{PG}(3,q)$ is such that every regulus containing three lines of the spread is contained in the spread, then this spread is called *regular*.

EXERCISE 2.9. *Show that the Desarguesian linespread of $\mathrm{PG}(3,q)$ is regular.*

We can construct a non-Desarguesian translation plane as follows. Take a regulus $\mathcal{R}$ in a Desarguesian linespread of $\mathrm{PG}(3,q)$, and replace $\mathcal{R}$ by its opposite regulus. This method to construct a new translation plane from an old one by replacing a regulus by its opposite regulus is called *derivation*.

3. Finite semifields

3.1. **Recap: coordinatisation and quasifields.** We recall that given a projective plane $\pi(\mathcal{P}, \mathcal{L}, I)$, there are several methods of labeling the points and lines of the plane (*coordinatisation*), using a set R of n symbols (where n is the *order* of the plane) and one extra symbol (usually ∞). This coordinatision depends on the choice of a *frame* (four points, no three collinear), and the labeling of one of the *sides* of the frame. Assume we have coordinatised the plane π, and we obtained a the set of points

$$\mathcal{P} = \{(a, b) \ : \ a, b \in R\} \cup \{(m) \ : \ m \in R \cup \{\infty\}\}$$

and the set of lines

$$\mathcal{L} = \{[m, k] \ : \ m, k \in R\} \cup \{[k] \ : \ k \in R \cup \{\infty\}\},$$

where the symbols $0, 1 \in R$, the frame consists of the points $\{(\infty), (0), (0, 0), (1, 1)\}$, the line $[\infty]$ contains the points (0) and (∞), the line $[0]$ contains (∞) and $(0, 0)$, and the line $[0, 0]$ contains (0) and $(0, 0)$, and so on ...

Once the plane has been coordinatised one can introduce a *ternary operation* T on R as follows; if $a, b, c \in R$, define $T(a, b, c) = k$ if and only if (b, c) is on $[a, k]$. A set R with a ternary operation is called a *ternary ring* and denoted by (R, T). It follows from the definition that T satisfies certain properties, and in fact one can list the necessary and sufficient properties that a ternary ring has to satisfy in order to be a ternary ring obtained by coordinatising a projective plane (by "inverse coordinatisation", i.e. constructing the plane starting from the ternary ring). In this case (R, T) is called a *planar ternary ring*, usually abbreviated to PTR.

Let (R, T) be a PTR. In the coordinatisation we mentioned the choice of a frame and the special symbols 0 and 1 and now we use these symbols to define two operations (denoted by $+$ and $\circ$) as follows; for $a, b \in R$ put $a + b := T(1, a, b)$, and $a \circ b := T(a, b, 0)$. It can be shown that this turns $(R, +)$ and $(R, \circ)$ into loops, with respective identities 0 and 1. With this setup one is able to connect the algebraic properties of the PTR with the geometric properties of the plane, or more correctly, with the properties of the automorphism group of the plane π. A PTR is called *linear* if $T(a, b, c) = a \circ b + c$, $\forall a, b, c \in R$. A *cartesian group* is a linear PTR with associative addition. A *(left) quasifield* is a cartesian group in which the left distributive law holds. A linear PTR is a cartesian group if and only if π is $((\infty), [\infty])$-transitive. A cartesian group is a quasifield if and only if π is $((0), [\infty])$-transitive, and in this case π is a translation plane and $(R, +)$ is abelian.

3.2. **Semifields, semifield spreads and semifield planes.** A finite *semifield* $(\mathbb{S}, +, \circ)$ is an algebraic system satisfying the following axioms:

(A1) $(\mathbb{S}, +)$ is a group, with identity element 0
(A2) $(\mathbb{S}, \circ)$ has no zero divisors
(A3) $(\mathbb{S}, +, \circ)$ satisfies the distributive law
(A4) $(\mathbb{S}, \circ)$ has an identity element

Exercise 3.1. *Suppose $(\mathbb{S}, +, \circ)$ is a finite semifield. Prove that $(\mathbb{S}, +)$ is an abelian group. (Hint: consider $(a + b) \circ (c + d)$.)*

In fact $(\mathbb{S}, +)$ is elementary abelian and the additive order of the non-zero elements is a prime and is called the *characteristic* of $\mathbb{S}$.

If an algebraic system $(\mathbb{S}, +, \circ)$ satisfies all of (A1), (A2), and (A3), then $(\mathbb{S}, +, \circ)$ is called a *pre-semifield*.

EXERCISE 3.2. *Show that if we define a new multiplication $*$ by $(a \circ u) * (u \circ b) = a \circ b$, then we obtain a semifield with unit $u \circ u$.*

Of course every finite field is also a finite semifields, but not a *proper* one. A proper example of a semifield odd order q^{2k} due to L. E. Dickson (1906) is the following.

$$(\mathbb{F}_{q^k}^2, +, \circ) \begin{cases} (x,y) + (u,v) & = (x+u, y+v) \\ (x,y) \circ (u,v) & = (xu + \alpha y^q v^q, xv + yu) \end{cases}$$

where α is a non-square in $\mathbb{F}_{q^k}$.

EXERCISE 3.3. *Prove that the above multiplication defines a semifield.*

Let $\mathbb{S}$ be a pre-semifield, and let $\mathbb{F}_p$ be the finite field of order p, where p is the characteristic of $\mathbb{S}$. Then we can consider $\mathbb{S}$ as a vectorspace over $\mathbb{F}_p$; say of dimension n.

EXERCISE 3.4. *For each $x \in \mathbb{S}$ consider the set of vectors $S_x := \{(y, y \circ x) : y \in \mathbb{S}\}$, and prove that $\mathcal{S} := \{S_x : x \in \mathbb{S}\} \cup \{(0, y) : y \in \mathbb{S}\}$ is a spread of $V(2n, p) \cong \mathbb{F}_{p^n}^2$.*

Such a spread is called a *semifield spread*. The corresponding translation plane (constructed by applying the André-Bruck-Bose construction) is called a *semifield plane*. Semifield planes are translation planes for which the dual is also a translation plane.

EXERCISE 3.5. *Use this to prove that the automorphism group G of a semifield spread contains a "special element" such that G (as a subgroup of $P\Gamma L(2n, p)$) fixes that special element pointwise and acts transitively on the other elements of the spread.*

The converse of this exercise is also true; if a spread has such a special element and automorphism group then it is a semifield spread.

3.3. **Isotopic semifields and isomorphic planes.** If we are interested in constructing new translation planes or new semifields, we need to know what exactly is meant by "new". In the case of translation planes we know that we are interested in planes which are not isomorphic to any of the known planes. In the case of semifields, we can define *isomorphic semifields* in the usual way, but it turns out that non-isomorphic semifields can correspond to isomorphic planes, suggesting that we should weaken the equivalence relation on the set of semifields, in order to obtain equivalence classes of semifields which are compatible with the isomorphism classes of projective planes. This is done by defining *isotopy*.

An *isotopy* between two semifields $\mathbb{S}$ and $\hat{\mathbb{S}}$ of order p^n is a triple (F, G, H) of non-singular linear transformations of the vector space $V(n, p)$ such that $x^F \hat{\circ} y^G = (x \circ y)^H$. In the case such an isotopy exists we call the two semifields *isotopic*.

With this definition we can state an important result in the theory of semifield planes.

THEOREM 3.6 (A. A. Albert 1960). *Two semifield planes are isomorphic if and only if the corresponding semifields are isotopic.*

The proof of this theorem is not extremely difficult but quite technical and a bit "out of our way" for the purpose of these notes.

EXERCISE 3.7. *Define isotopy between two pre-semifields in the same way and construct and isotopy between a pre-semifield and a semifield.*

3.4. **A geometric construction of finite semifields.** In what follows we explain part of recent research concerning finite semifields. We will see that every finite semifield can be constructed from a particular (and fairly simple) configuration of two subspaces with respect to a Desarguesian spread. In what follows, the set of elements of the Desarguesian spread intersecting a set of points T is denoted by $B(T)$.

A. The configuration $C(U, W)$

Choose any $r \geq 2$. Put $\Sigma := \mathrm{PG}(rn-1, q)$

- Let $\mathcal{D}$ be a Desarguesian $(n-1)$-spread of Σ.
- Let
$$\begin{cases} U \subset \Sigma \text{ , } \dim(U) = n-1 \\ W \subset \Sigma \text{ , } \dim(W) = rn-n-1 \end{cases}$$

If no elements of $\mathcal{D}$ intersect both U and W (i.e. $B(U) \cap B(W) = \emptyset$), then (U, W) is called a *semifield pair* of subspaces (with respect to $\mathcal{D}$). The configuration is denoted by $C(U, W)$.

B. The construction

Suppose (U, W) is a semifield pair of subspaces with respect to $\mathcal{D}$ in Σ.

- Embed Σ in $\Gamma := \mathrm{PG}(rn+n-1, q)$ and extend $\mathcal{D}$ to Γ $(\rightarrow \overline{\mathcal{D}})$.
- Let A be an n-space with $A \cap \Sigma = U$.
- Let $\mathcal{S}(U, W)$ be the set of subspaces defined by $B(A)$ in the quotient geometry Γ/W, i.e.
$$\mathcal{S}(U, W) = \{\langle R, W\rangle / W \ : \ R \in B(A)\}.$$

EXERCISE 3.8. *Prove that $\mathcal{S}(U, W)$ is an $(n-1)$-spread in $\Gamma/W \cong \mathrm{PG}(2n-1, q)$.*

It can be shown that $\mathcal{S}(U, W)$ is a semifield spread (guess what the special element is). Applying the André-Bruck-Bose construction we obtain a semifield plane $\pi(U, W)$. We let $\mathbb{S}(U, W)$ denote the semifield corresponding to the semifield plane $\pi(U, W)$. Recently it has been shown that each finite semifield can be obtained from a semifield pair of subspaces. This gives us a correspondence between the set of finite semifields and the set of semifield pairs of subspaces.

3.5. **Generalised twisted field.** Let q be an odd prime power. Let $\mathbb{F}$ be the finite field of q^n elements, let $\mathbb{F}_0$ be the subfield of q elements and assume $n \geq 2$. The *generalised twisted field* $(\mathbb{F}, +, \circ)$ has multiplication defined by

$$y \circ x = yx - \eta y^\sigma x^\alpha,$$

where σ and α are automorphisms of $\mathbb{F}$ with fixed field $\mathbb{F}_0$ and $\eta \in \mathbb{F} \setminus \{a^{q-1} \mid a \in \mathbb{F}\}$.

EXERCISE 3.9. *Show that this multiplication defines a semifield.*

We now illustrate that we can find a semifield pair of subspaces (U, W) for this semifield. We will identify subspaces of projective space with subspaces of the corresponding vectorspace. Let $\mathcal{D}$ be the usual Desarguesian $(n-1)$-spread of $\mathrm{PG}(3n-1, q)$, i.e. the element of $\mathcal{D}$ corresponding to the point $P(x, y, z)$ of $\mathrm{PG}(2, q^n)$ is the subspace of $V(3n, q)$ (or $\mathrm{PG}(3n-1, q)$) defined by the set of vectors $\{(ax, ay, az) : a \in \mathbb{F}\}$ in $V(3, q^n)$. Define subspaces $U = \{(0, x, -\eta^{1/\sigma} x^{\alpha/\sigma}) \mid x \in \mathbb{F}\}$ and $W = \{(0, -z^\sigma, z) \mid z \in \mathbb{F}\}$ of $V(2n, q)$. Clearly

$$B(U) = \{\{(0, ax, -a\eta^{1/\sigma} x^{\alpha/\sigma}) \mid a \in \mathbb{F}\} \mid x \in \mathbb{F}^*\}$$

and

$$B(W) = \{\{(0, -by^\sigma, by) \mid b \in \mathbb{F}\} \mid y \in \mathbb{F}^*\}$$

and they are disjoint since η is not a $(q-1)$-th power. Let $v = (1, 0, 0)$. An element of $B(\langle U, v\rangle) \setminus B(U)$ is of the form

$$S_x = \{(y, yx, -y\eta^{1/\sigma} x^{\alpha/\sigma}) \mid y \in \mathbb{F}\},$$

for some $x \in \mathbb{F}$. We can obtain S_x/W by intersecting $\langle S_x, W\rangle$ with a subspace of rank $2n$ which has no non-trivial intersection with W, for example $X_3 = 0$. Now

$$\langle S_x, W\rangle = \{(y, yx - z^\sigma, -y\eta^{1/\sigma} x^{\alpha/\sigma} + z) \mid y, z \in \mathbb{F}\}$$

and so

$$\mathcal{S}(U, W) = \{\{(y, yx - \eta y^\sigma x^\alpha)\} \mid y \in \mathbb{F}\} \mid x \in \mathbb{F}\} \cup \{(0, y) \mid y \in \mathbb{F}\}.$$

This is indeed the semifield spread corresponding to the *generalised twisted field*.

3.6. **Connection with ovoids of $Q(4, q)$.** We now turn our attention to a class of semifields which is of particular interest in Finite Geometry because of their numerous links with other objects in this research area. These are the so-called Rank Two Commutative Semifields (RTCS). Apart from a pure algebraic definition, there are various ways to introduce these semifields. They are connected to translation generalised quadrangles, semifield flocks, eggs in projective spaces, ...
Here we will introduce them geometrically using the subspaces U and W and the Desarguesian spread $\mathcal{D}$ as before with $r = 2$, and explain their connection to ovoids of the polar space $Q(4, q)$.

3.6.1. *Intermezzo: An alternative model for the polar space $Q(4, q)$.* Let $\mathcal{C}$ be a conic in the plane π contained in $\mathrm{PG}(3, q)$. Define the following incidence structure $T_2(\mathcal{C}) = (\mathcal{P}, \mathcal{L}, I)$. There are three different types of points: (i) a point (∞); (ii) the planes of $\mathrm{PG}(3, q)$ which intersect π in a tangent line of $\mathcal{C}$ (called *points of tangent type*); (iii) the points of $\mathrm{PG}(3, q) \setminus \pi$ (called *affine points*). There are two different types of lines: (a) the points of $\mathcal{C}$; (b) the lines of $\mathrm{PG}(3, q)$ intersecting π in a point of $\mathcal{C}$. The point (∞) is incident with the lines corresponding to points of $\mathcal{C}$ and the remaining incidence relations are natural.

EXERCISE 3.10. *Show that this incidence structure is a generalised quadrangle of order q.*

This incidence structure is isomorphic to $Q(4,q)$. An *ovoid* of $Q(4,q)$ is a set of q^2+1 points no two collinear.

EXERCISE 3.11. *Let $\mathcal{O}$ be a set of q^2+1 points containing the point (∞). Prove that $\mathcal{O}$ is an ovoid of $Q(4,q)$ if and only if the set of points of the conic $\mathcal{C}$ is disjoint from the set of directions determined by $\mathcal{O}\setminus\{(\infty)\}$.*

An ovoid $\mathcal{O}$ of $Q(4,q)$ is called a *translation ovoid* or *semifield ovoid* if there is group G of order q^2 of collineations of $Q(4,q)$, fixing $\mathcal{O}$, a point P in $\mathcal{O}$, and every line through P.

Example: The q^2 affine points of a plane intersecting π in a line external to $\mathcal{C}$ together with (∞) is an ovoid of $Q(4,q)$.

EXERCISE 3.12. *Prove that it is a translation ovoid.*

These ovoids are quite rare in the sense that few examples are known compared to the amount of effort that mathematicians have put in this area of research. They are in one-to-one correspondence with the so called *semifield flocks*, and people have tried to classify them by hand and using a computer. At the moment it is known that if the characteristic is even there are no "proper" examples (1981), and if the characteristic is odd then there is are strong restrictions on the possible values of q (2001). Apart from a finite field and an early example due to Dickson (1906) there was one other example constructed in 1981. For almost twenty years people believed that no other examples could exist until the most recent construction in 1999 with the help of a computer.

3.6.2. *Continuing from before the intermezzo.* Our aim is to put extra hypotheses on the semifield pair (U,W) in order to construct a translation ovoid of $Q(4,q)$.
Suppose n is even, and recall that the elements of the Desarguesian $(n-1)$-spread $\mathcal{D}$ of $\mathrm{PG}(2n-1,q)$ correspond to the points of $\mathrm{PG}(1,q^n)$. Let W_0 be an $(n-1)$-dimensional subspace of $\mathrm{PG}(2n-1,q)$ corresponding to a Baer subline $\mathrm{PG}(1,q^{n/2})$ of $\mathrm{PG}(1,q^n)$, and let U be an $(n-1)$-space of $\mathrm{PG}(2n-1,q)$ such that no elements of $\mathcal{D}$ intersect both U and W_0. Then by definition, the pair (U,W_0) is a semifield pair of subspaces.

REMARK 3.13. *If $n=4$ then all these configurations $C(U,W_0)$ have recently been classified (2005).*

Now put $n=2m$ and consider this configuration in $\mathrm{PG}(3,q^m)$. Then, by definition W_0 is a transversal to a regulus of $\mathrm{PG}(3,q^m)$, and defines a hyperbolic quadric $Q^+(3,q^m)$, and U is skew to the set of elements of $\mathcal{D}$ corresponding to $Q^+(3,q^m)$ in $\mathrm{PG}(3m-1,q)$. Let γ be a plane of $\mathrm{PG}(3,q^m)$ which intersects $Q^+(3,q^m)$ in a non-degenerate conic $\mathcal{C}$, and suppose that U is contained in the $(3m-1)$-dimensional subspace of $\mathrm{PG}(2n-1,q)$ corresponding to γ.

EXERCISE 3.14. *Construct an ovoid of $Q(4,q^m)$ from U (hint: embed $\mathrm{PG}(3m-1,q)$ as a hyperplane in $\mathrm{PG}(3m,q)$, and consider the q^{2m} "affine points" of a $2m$-dimensional subspace of $\mathrm{PG}(3m,q)$ which intersects $\mathrm{PG}(3m-1,q)$ in U).*

In fact it can be shown that this ovoid is a translation ovoid of $Q(4,q)$, and we have reached what we aimed for.