Tensors in Finite Geometry[1]

Michel Lavrauw

Sabancı University
http://people.sabanciuniv.edu/mlavrauw

Throughout the talk V, W $V_1, \ldots, V_m$ denote finite-dimensional vector spaces over some field $\mathbb{F}$.

Outline of this talk:

1. Tensors
2. Groups
3. Geometry
4. Finite Geometry
5. Recent results

Tensors

Tensor spaces

A tensor space will be one of the following:

- $V_1 \otimes V_2 \otimes \ldots \otimes V_m$, (tensors of format $(\dim V_1, \dim V_2, \ldots, \dim V_m)$)
- the space of symmetric tensors $S^d V$,
- the space of alternating tensors $\Lambda^d V$,
- partially symmetric or alternating, i.e. $S^d V \otimes W$, $\Lambda^d V \otimes W$.

Example in $V \otimes V \otimes V$ with $V = \langle e_0, e_1 \rangle$

$$T = e_0 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_1 - e_1 \otimes e_0 \otimes e_1 + e_1 \otimes e_1 \otimes e_0$$

If e_0, e_1 are linearly independent then T can be represented by the hypermatrix

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right)$$

with respect to the basis of $V \otimes V \otimes V$ consisting of the tensors $e_{ijk} = e_i \otimes e_j \otimes e_k$, $i, j, k \in \{0, 1\}$.

Tensor products are ubiquitous in science

- Matrices are tensors in $\mathbb{F}^m \otimes \mathbb{F}^n$
- Multilinear algebra
- Algebraic geometry
- Computational complexity theory
- Quantum mechanics
- Data analysis (chemistry, biology, physics, ...)
- Signal processing, source separation

There is a huge amount of research activity related to tensors. See e.g.
[Kolda and Bader (2009) *Tensor decompositions and applications*] and
[Landsberg (2012) *Tensors: Geometry and Applications*].

Decomposition

One of the central questions concerns the decomposition of a tensor into the sum of pure tensors:

$$T = \sum_i v_{1i} \otimes \ldots \otimes v_{mi} \tag{1}$$

- ▶ Generalizes Singular Value Decomposition (SVD)
- ▶ "PARAFAC", "CANDECOMP", "CP decomposition", ...
- ▶ For general tensors no efficient decomposition algorithms exist.

Main problems

$$T = \sum_{i=1}^{r} v_{1i} \otimes \ldots \otimes v_{mi} \qquad (1)$$

Four important problems:

- ▶ Is there an algorithm to compute a decomposition?
- ▶ Uniqueness: do some tensors have a unique decomposition?
- ▶ Existence: given T and r, does (1) exist? $\rightarrow$ rank(T) ([Hitchcock 1927], [Kruskal 1977])
- ▶ Orbits: how many "different" tensors are there in a given tensor space?

These problems are known to be very hard for $m \geq 3$ (theoretical and computational). ([Håstad 1990] *Tensor rank is NP-complete*)

Two examples of tensor rank

A first example

What is the rank of T?

$$T = e_0 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_1 - e_1 \otimes e_0 \otimes e_1 + e_1 \otimes e_1 \otimes e_0$$

▶ T has rank 3 over $\mathbb{R}$:

$$T = (e_0 - e_1) \otimes e_0 \otimes e_0 + (e_0 + e_1) \otimes e_1 \otimes e_1 + e_1 \otimes (e_0 + e_1) \otimes (e_0 - e_1)$$

▶ T has rank 2 over $\mathbb{C}$:

$$T = \left(\frac{1}{2} e_0 + \frac{1}{2i} e_1 \right) \otimes (e_0 + i e_1) \otimes (e_0 + i e_1)$$

$$+ \left(\frac{1}{2} e_0 - \frac{1}{2i} e_1 \right) \otimes (e_0 - i e_1) \otimes (e_0 - i e_1)$$

▶ T has rank 2 over $\mathbb{F}_q$ iff $q \equiv 1 \bmod 4$

An important example in $\mathbb{F}^4 \otimes \mathbb{F}^4 \otimes \mathbb{F}^4$

The tensor

$$M = e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_3 \otimes e_1 + e_1 \otimes e_2 \otimes e_2 + e_2 \otimes e_4 \otimes e_2$$

$$+ e_3 \otimes e_1 \otimes e_3 + e_4 \otimes e_3 \otimes e_3 + e_3 \otimes e_2 \otimes e_4 + e_4 \otimes e_4 \otimes e_4$$

has rank ≤ 7.

Proof.

$$M = (e_1 + e_4) \otimes (e_1 + e_4) \otimes (e_1 + e_4) + (e_2 + e_4) \otimes e_1 \otimes (e_2 - e_4)$$

$$+ e_1 \otimes (e_3 - e_4) \otimes (e_3 + e_4) + e_4 \otimes (-e_1 + e_2) \otimes (e_1 + e_3)$$

$$+ (e_1 + e_3) \otimes e_4 \otimes (-e_1 + e_3) + (-e_1 + e_2) \otimes (e_1 + e_3) \otimes e_4$$

$$+ (e_3 - e_4) \otimes (e_2 + e_4) \otimes e_1. \quad \square$$

Important? Why?

Observing that

$$M = (e_1 \otimes e_1 + e_2 \otimes e_3) \otimes e_1 + (e_1 \otimes e_2 + e_2 \otimes e_4) \otimes e_2$$

$$+(e_3 \otimes e_1 + e_4 \otimes e_3) \otimes e_3 + (e_3 \otimes e_2 + e_4 \otimes e_4) \otimes e_4$$

can be rewritten as (where $e_1 = e_{11}$, $e_2 = e_{12}$, $e_3 = e_{21}$, $e_4 = e_{22}$)

$$M = (e_{11} \otimes e_{11} + e_{12} \otimes e_{21}) \otimes e_{11} + (e_{11} \otimes e_{12} + e_{12} \otimes e_{22}) \otimes e_{12}$$

$$+(e_{21} \otimes e_{11} + e_{22} \otimes e_{21}) \otimes e_{21} + (e_{21} \otimes e_{12} + e_{22} \otimes e_{22}) \otimes e_{22}$$

hints at the multiplication of 2×2-matrices.

Indeed, the tensor $M \in (\mathbb{F}^{2 \times 2})^{\otimes 3}$ represents the matrix algebra $\mathbb{F}^{2 \times 2}$.

Connection to complexity theory

In general, given a tensor T in $V \otimes V \otimes V^\vee$ we can turn $V^\vee$ into an algebra A. For example, for a pure tensor $u \otimes v \otimes w \in V \otimes V \otimes V^\vee$ define the multiplication

$$V^\vee \times V^\vee \to V^\vee \ : \ (a, b) \mapsto a(u)b(v)w.$$

This connection is a well-studied and important topic in complexity theory: the rank of the tensor T corresponds to the complexity of multiplication in the algebra A.

[Bürgisser - Clausen - Shokrollahi (1997) *Algebraic Complexity Theory*]

[von zur Gathen - Gerhard (2013) *Modern Computer Algebra*]

Rank(M) = 7

The fact that M has rank ≤ 7 implies that two 2×2-matrices can be multiplied, by performing ≤ 7 instead of 8 multiplications: one multiplication for each pure tensor in the decomposition of M.

This was discovered by [Strassen 1969]. Soon after that [Hopcroft and Kerr 1971] and [Winograd 1971] proved that M has rank 7.

This decomposition can also be used to multiply $n \times n$-matrices (by adding zeros and block decomposition of $2^k \times 2^k$-matrices). ($\rightarrow$ exponent ω of matrix multiplication, see [Bürgisser et al., Chapter 15]).

Algebraic varieties related to tensors

The problems of decomposition and rank have natural geometric interpretations, and the following connections with algebraic geometry are well-known.

▶ Pure tensors corresponds to the set of points on a Segre variety in $\mathbb{P}^N$ where $N = \prod \dim V_i - 1$.

▶ Pure symmetric tensors in $S^d V$ corresponds to the points of a Veronese variety in $\mathbb{P}^N$ where $N = \binom{d + \dim V - 1}{d} - 1$.

▶ Pure alternating tensors in $\Lambda^r V$ corresponds to the points of a Grassmann variety $\mathbb{P}^N$ where $N = \binom{\dim V}{r}$ (Plücker embedding)

▶ Tensors in $S^d V \otimes W$ corresponds to linear systems of hypersurfaces of degree d in $\mathbb{P}^{\dim V - 1}$.

Groups

The main problems from before

- decomposition
- rank
- uniqueness
- orbits

are invariant under certain *natural* group actions.

Group actions I

▶ An element $(g_1, g_2, \ldots g_m)$ of $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \ldots \times \mathrm{GL}(V_m)$ acts on $V_1 \otimes V_2 \otimes \ldots \otimes V_m$ as follows:

$$v_1 \otimes v_2 \otimes \ldots \otimes v_m \mapsto v_1^{g_1} \otimes v_2^{g_2} \otimes \ldots \otimes v_m^{g_m}.$$

▶ If $V_i = V$ for k of the i's ($1 \leq k \leq m$), then we also have a natural action of $Sym(k)$ on $V_1 \otimes V_2 \otimes \ldots \otimes V_m$. For example, for $k = m$ we have the action of $\pi \in Sym(m)$ defined by

$$\pi : v_1 \otimes v_2 \otimes \ldots \otimes v_m \mapsto v_{\pi(1)} \otimes v_{\pi(2)} \otimes \ldots \otimes v_{\pi(m)}.$$

▶ A combination of these gives the action of a wreath product ($\mathrm{GL}(V) \wr Sym(m)$ in the above example) on $V_1 \otimes V_2 \otimes \ldots \otimes V_m$.

Group actions II

Notation: $V^{\otimes d} = V \otimes \ldots \otimes V$ (d factors)

- ▶ The space $S^d V$ of symmetric tensors in $V^{\otimes d}$ consists of all fixed points of the action of $Sym(d)$ on $V^{\otimes d}$.
- ▶ The space $\Lambda^d V$ of alternating tensors in $V^{\otimes d}$ consists of all $T \in V^{\otimes d}$ for which $X^\pi = sgn(\pi)X$, $\forall \pi \in Sym(d)$.
- ▶ An element g of $\mathrm{GL}(V)$ acts on $V^{\otimes d}$ as follows:

$$g : v_1 \otimes \ldots \otimes v_d \mapsto v_1^g \otimes \ldots \otimes v_d^g$$

- ▶ Action of $\mathrm{GL}(V)$ on both subspaces $S^d V$ and $\Lambda^d V$.
- ▶ Action of $\mathrm{GL}(V) \times \mathrm{GL}(W)$ on $S^d V \otimes W$ and $\Lambda^d V \otimes W$.

For the purpose of this talk the acting group will usually be denoted by G.

Tensor spaces with a finite number of G-orbits over $\mathbb{C}$

These spaces have been classified by [Victor Kac 1980].

(1) $V_1 \otimes \ldots \otimes V_m$ and $G = \mathrm{GL}(V_1) \times \ldots \times \mathrm{GL}(V_m)$ $(m \geq 2)$

 (a) $m = 2$
 (b) $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^a$
 (c) $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^a$

(2) $S^d V$ with $G = \mathrm{GL}(V)$

 (a) $S^2\mathbb{C}^a$
 (b) $S^3\mathbb{C}^2$

(3) $S^d V \otimes W$ with $G = \mathrm{GL}(V) \times \mathrm{GL}(W)$

 (a) $S^2\mathbb{C}^2 \otimes \mathbb{C}^a$
 (b) $S^2\mathbb{C}^3 \otimes \mathbb{C}^2$

Further results

1. There is a list of tensor spaces over $\mathbb{C}$ whose G-orbits can be parametrized. These cases correspond to group actions (G, V) where the GIT quotient $G//V$ (which is the image of V under the map $\varphi : v \mapsto (f_1(v), \ldots, f_m(v))$ where $f_1, \ldots, f_m \in \mathbb{C}[V]$ are generators for the ring of invariants $\mathbb{C}[V]^G$) for which each fiber $\varphi^{-1}(w)$, $w \in V//G$, consists of finitely many G-orbits.

2. There are many results known on G-orbits on $\Lambda^r(\mathbb{F}^n)$, for various $\mathbb{F}$, e.g. algebraically closed fields $\mathbb{F} = \overline{\mathbb{F}}$, finite fields $\mathbb{F} = \mathbb{F}_q$, and fields of cohomological dimension ≤ 1.

([Schouten 1933] [Dieudonné 1955] [Gurevich 1964] [Cohen and Helminck 1988] [DeBruyn and Kwiatkowski 2013] [Cardinali, Giuzzi, and Pasini 2017])

Classification

Since we are primarily interested in tensors over finite fields, with a classification of the G-orbits we mean a list of representatives, one for each G-orbit.[2] And of course we want a complete proof!

If possible we would also like to have

- ▶ the stabiliser group of each representative,
- ▶ the size of each orbit,
- ▶ an algorithm to determine the orbit of a given tensor,
- ▶ Schreier elements for constructive recognition,
- ▶ combinatorial invariants for the orbits,
- ▶ or even better, a complete invariant for the orbits.

---

[2]There are many examples in mathematics of important "classifications" which do not satisfy these conditions.

Geometry

Geometry

Summarising the above we obtain the following geometric setting.

The pure tensors correspond to points of classical algebraic varieties in projective space $\mathbb{P}^N$

- $V_1 \otimes V_2 \otimes \ldots \otimes V_m \rightarrow$ Segre variety with $N = (\prod \dim V_i) - 1$
- $S^d V \rightarrow$ Veronese variety with $N = \binom{\dim V + d - 1}{d} - 1$
- $\Lambda^d V \rightarrow$ Grassmann variety with $N = \binom{\dim V}{d} - 1$

In each case the group G induces a subgroup K of $\mathrm{PGL}(\mathbb{P}^N)$ leaving the relevant variety invariant.

The tensor rank of T corresponds to the minimal number of points on the variety which are needed to span a subspace containing the corresponding point $\langle T \rangle$ in $\mathbb{P}^N$.

Points belonging to the same K-orbit have the same rank. So one can speak of the rank of a K-orbit.

(More generally, one speaks of the $\mathcal{X}$-rank of a point in the ambient space of an algebraic variety $\mathcal{X}$.)

Contractions

A contraction of a tensor

$$T \in V_1 \otimes \ldots \otimes V_m$$

is a tensor in

$$V_1 \otimes \ldots \otimes \hat{V}_i \otimes \ldots \otimes V_m$$

obtained as the image of T under $u_i^\vee \in V_i^\vee$, defined by

$$u_i^\vee(v_1 \otimes \ldots \otimes v_m) = u_i^\vee(v_i)v_1 \otimes \ldots v_{i-1} \otimes v_{i+1} \otimes \ldots v_m.$$

The i-th contraction space of T is the subspace

$$C_i(T) := \{u_i^\vee(T) \; : \in u_i^\vee \in V_i^\vee\} \le V_1 \otimes \ldots \otimes \hat{V}_i \otimes \ldots \otimes V_m.$$

Contraction spaces and subspace rank

A tensor $T \in V_1 \otimes \ldots \otimes V_m$ is called *i-concise* if $\dim C_i(T) = \dim V_i$ and concise if T is *i-concise* for all $i \in \{1, \ldots, m\}$.

Each contraction space admits a natural action of the group G_i.

The rank of a subspace U of $V_1 \otimes \ldots \otimes V_m$ is defined as the minimal number of pure tensors needed to span a subspace containing U.

Two useful lemma's

Lemma
Two tensors $V_1 \otimes \ldots \otimes V_m$ are G-equivalent if and only if the their i-th contraction spaces $C_i(T)$ and $C_i(S)$ are G_i-equivalent.

In particular, two G-equivalent tensors have contraction spaces of the same dimension, giving us a combinatorial invariant $[\dim C_1(T), \ldots, \dim C_m(T)]$ for the G-orbits.

Lemma
The rank of a tensor T is equal to the rank of the i-th contraction space of T, i.e.

$$rank(T) = rank(C_i(T))$$
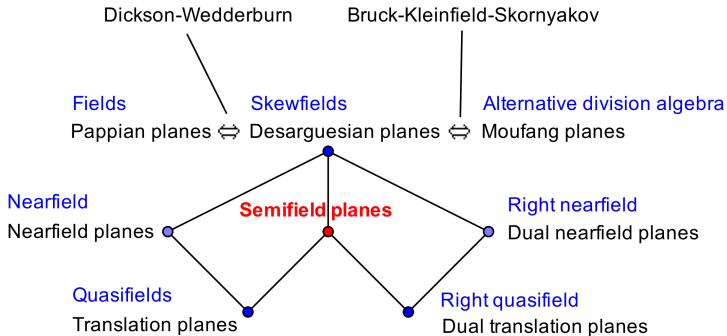
for $i \in \{1, \ldots, m\}$.

Finite Geometry

For me the tensor story started around 2008 during a visit to Tim Penttila at Colorado State University in Fort Collins, where I gave a talk in the *Rocky Mountain Algebraic Combinatorics Seminar* (7 March 2008) with the title *Semifields and secant varieties to Segre varieties*. This is where I met Robert Liebler (he was one of the organisers of the seminar). Although I actually used tensor products in my PhD thesis to give a construction of a maximum scattered linear set (this must have been in 1998), I had absolutely no idea about all, or at least most, of the above. I think I even forgot the construction in my thesis.

The conversations which I had with Bob led me to study the connection between tensors and semifields and eventually to the paper *Finite semifields and non-singular tensors* (2013) which I presented at the Third Irsee Conference on Finite Geometries in September 2011. Soon I realised that I had stumbled upon a "can of worms". Instead of tensors allowing me a way out, they pulled me (back) in.

We start by giving a very brief intro to semifields via projective planes.

Types of finite translation planes

Dickson-Wedderburn   Bruck-Kleinfield-Skornyakov

Fields   Skewfields   Alternative division algebra

Pappian planes ⇔ Desarguesian planes ⇔ Moufang planes

Nearfield   Right nearfield

Nearfield planes   **Semifield planes**   Dual nearfield planes

Quasifields   Right quasifield

Translation planes   Dual translation planes

[Hughes - Piper, Projective Planes, Springer, 1973]

Some details on semifields and tensors

A semifield is a (finite) possibly non-associative division algebra.

- An n-dimensional semifield $\mathbb{S}$ gives $T_{\mathbb{S}} \in V_1 \otimes V_2 \otimes V_3$, $V_i \cong \mathbb{F}_q^n$
- The tensor $T_{\mathbb{S}} \in V_1 \otimes V_2 \otimes V_3$ is nonsingular (each non-trivial double contraction gives a nonzero vector).
- To every nonsingular tensor $T \in V_1 \otimes V_2 \otimes V_3$ there corresponds a (pre-)semifield $\mathbb{S}$ for which $T = T_{\mathbb{S}}$.
- The map $\mathbb{S} \mapsto T_{\mathbb{S}}$ is injective.
- Semifields $\leftrightarrow$ projective planes (with a lot of symmetry)

Inspired by previous work by [Knuth 1965] and [Liebler1981].

Orbits and isotopism (isomorphism)

- The isomorphism classes (planes) ↔ isotopism classes (semifields) ↔ orbits on tensors.
- The Knuth orbit of a semifield $\mathbb{S}$ is represented in the projective space $\mathbb{P}^{n^3-1}$ as the orbit of $\langle T_{\mathbb{S}} \rangle$ under the group $\mathrm{GL} \wr S_3$.
- The tensor rank of a semifield is an invariant for the Knuth orbit of a semifield

Moral of the story: the tensor $T_{\mathbb{S}}$ is what we, geometers, should be looking at.

For further details, see [ML 2013] *Finite semifields and non-singular tensors*.

Finite geometry and coding theory/cryptography

Considering the impact that the research on semifields has had on the study of MRD codes (as have other topics in finite geometry on the theory of linear codes, or cryptography), it would not surprise me if the research on the geometry of tensors over finite fields has a similar impact on coding theory (or cryptography).

Recent results

1. Tensors
2. Groups
3. Geometry
4. Finite Geometry
5. Recent results
   (A) *G*-orbits
   (B) Tensor rank

(A) G-orbits - theoretical results over $\mathbb{F}_q$

Besides the previously mentioned studies of G-orbits on $\Lambda^r(\mathbb{F}_q^n)$, the G-orbits have been theoretically studied in the following cases:

(i) $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^r$ $r \leq 6$, [ML-Sheekey 2015], [ML-Sheekey 2017], [Alnajjarine-ML 2020]

(ii) $S^2\mathbb{F}_q^3 \otimes \mathbb{F}_q^2$ [ML-Popiel 2019]

(iii) $S^2\mathbb{F}_q^3 \otimes \mathbb{F}_q^3$ [ML-Popiel-Sheekey 2020, 202*]

(iv) $S^2\mathbb{F}_q^3 \otimes \mathbb{F}_q^4$ [Alnajjarine-ML-Popiel 202*]

(v) $S^3\mathbb{F}_q^2 \otimes \mathbb{F}_q^2$ [Günay-ML 202*]

More details on (ii), (iii) and (iv) can be found on the slides of a recent talk in the eSeminar series *Galois Geometries and their applications*.

The following slides concern (i) and (v).

The orbits in $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$ [ML-Sheekey 2015]

Here is an example of a classification of the G-orbits. The following table gives a representative for each of the 18 G-orbits of tensors in $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$. The last column gives the rank distribution of the first contraction space $C_1(A)$ of the representative A.

Orbit	Canonical form	Condition	$r_1(A)$
o_0	$\mathbf{0}$		$[0, 0, 0]$
o_1	$e_1 \otimes e_1 \otimes e_1$		$[1, 0, 0]$
o_2	$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2)$		$[0, 1, 0]$
o_3	$e_1 \otimes e$		$[0, 0, 1]$
o_4	$e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_1 \otimes e_2$		$[q + 1, 0, 0]$
o_5	$e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2$		$[2, q - 1, 0]$
o_6	$e_1 \otimes e_1 \otimes e_1 + e_2 \otimes (e_1 \otimes e_2 + e_2 \otimes e_1)$		$[1, q, 0]$
o_7	$e_1 \otimes e_1 \otimes e_3 + e_2 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2)$		$[1, q, 0]$
o_8	$e_1 \otimes e_1 \otimes e_1 + e_2 \otimes (e_2 \otimes e_2 + e_3 \otimes e_3)$		$[1, 1, q - 1]$
o_9	$e_1 \otimes e_3 \otimes e_1 + e_2 \otimes e$		$[1, 0, q]$
o_{10}	$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2 + u e_1 \otimes e_2) + e_2 \otimes (e_1 \otimes e_2 + v e_2 \otimes e_1)$	$(*)$	$[0, q + 1, 0]$
o_{11}	$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2) + e_2 \otimes (e_1 \otimes e_2 + e_2 \otimes e_3)$		$[0, q + 1, 0]$
o_{12}	$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2) + e_2 \otimes (e_1 \otimes e_3 + e_3 \otimes e_2)$		$[0, q + 1, 0]$
o_{13}	$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2) + e_2 \otimes (e_1 \otimes e_2 + e_3 \otimes e_3)$		$[0, 2, q - 1]$
o_{14}	$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2) + e_2 \otimes (e_2 \otimes e_2 + e_3 \otimes e_3)$		$[0, 3, q - 2]$
o_{15}	$e_1 \otimes (e + u e_1 \otimes e_2) + e_2 \otimes (e_1 \otimes e_2 + v e_2 \otimes e_1)$	$(*)$	$[0, 1, q]$
o_{16}	$e_1 \otimes e + e_2 \otimes (e_1 \otimes e_2 + e_2 \otimes e_3)$		$[0, 1, q]$
o_{17}	$e_1 \otimes e + e_2 \otimes (e_1 \otimes e_2 + e_2 \otimes e_3 + e_3 \otimes (\alpha e_1 + \beta e_2 + \gamma e_3))$	$(**)$	$[0, 0, q + 1]$

Condition $(*)$ is: $v\lambda^2 + uv\lambda - 1 \neq 0$ for all $\lambda \in \mathbb{F}_q$.

Condition $(**)$ is: $\lambda^3 + \gamma\lambda^2 - \beta\lambda + \alpha \neq 0$ for all $\lambda \in \mathbb{F}_q$.

Orbit identification

This classification of the G-orbits includes various other useful data of the G-orbits, such as tensor rank, geometric description, and rank distribution of the contraction spaces, see
`http://people.sabanciuniv.edu/~mlavrauw/T233/table1.html`.

In some cases one needs to understand more of the geometry to tell orbits apart. For example, for o_{15} and o_{16}.

Lemma
Let x_2 be the unique rank 2 point on $C_1(A)$ and x_1 be a point among the q points of rank 3 on $C_1(A)$. Then, there exists a unique solid V containing x_2 which intersects $S_{3,3}(\mathbb{F}_q)$ in a subvariety $Q(x_2)$ equivalent to a Segre variety $S_{2,2}(\mathbb{F}_q)$. There is no rank one point in $U \setminus Q(x_2)$ for o_{16} where $U := \langle V, x_1 \rangle$, and there is one for o_{15}.

The lemma is implicitly contained in [ML-Sheekey 2015] as explained in [Alnajjarine-ML2020].

Implementation in GAP [Alnajjarine - ML 2020]

(joint work with Nour Alnajjarine)

The geometric and combinatorial data from the classification of tensors in $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$ has been used to implement algorithms in GAP (relying on the FinInG package) which allow orbit identification and rank computation in this tensor space.

[Alnajjarine - ML 2020] *Determining the rank of tensors in $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$ (2020).*

The GAP code is available from
`http://people.sabanciuniv.edu/mlavrauw/T233/T233_paper.html`.

This research is part of Nour's PhD.

The code is illustrated on the next slide with an example in the 17-dimensional projective space over the field of size 397.

```
gap> q:=397; sv:=SegreVariety([PG(1,q),PG(2,q),PG(2,q)]);
397
Segre Variety in ProjectiveSpace(17, 397)
gap> Size(Points(sv));
9936552395502
gap> pg:=AmbientSpace(sv);;
gap> Size(Points(pg));
1515423214380981479565590114693875696752607
gap> A:=VectorSpaceToElement(pg,[Z(397)^0,Z(397)^336,Z(397)^339,
Z(397)^37,Z(397)^233,Z(397)^56,Z(397)^268,Z(397)^363,Z(397)^342,
Z(397)^297,Z(397)^146,Z(397)^71,Z(397)^57,Z(397)^84,Z(397)^33,
Z(397)^203,Z(397)^229,Z(397)^191]);
gap> OrbitOfTensor(A)[1]; time;
14
94
gap> RankOfTensor(A); time;
3
141
gap> NrCombinations([1..Size(Points(sv))], 3);
1635143718652028814749545561407873423500
```

Orbits in $S^3\mathbb{F}_q^2 \otimes \mathbb{F}_q^2$ [Günay-ML 202*]

(joint work with Gülizar Günay)

The classification of the G-orbits of partially symmetric tensors $S^3\mathbb{F}_q^2 \otimes \mathbb{F}_q^2$ includes the classification of lines in $\mathbb{P}^3$ under the action of a copy of $\mathrm{PGL}(2, q)$ in $\mathrm{PGL}(4, q)$. The underlying geometry is the twisted cubic.

We have thus far determined 10 line-orbits, including their point-orbit distributions and hyperplane-orbit distributions. (There are 4 point orbits and 4 hyperplane orbits.) These 10 line-orbits comprise the set of lines ℓ of $\mathbb{P}^3$ for which ℓ or ℓ^ρ belong to an osculating plane of the twisted cubic. Here ρ is the symplectic polarity of $\mathbb{P}^3$ sending a point on the twisted cubic to its osculating plane.

This research is part of Gülizar's PhD.

G-orbits - computational results for small values of q

There are computational classification results for the G-orbits in the following cases:

1. $\mathbb{F}_q^2 \otimes \mathbb{F}_q^2 \otimes \mathbb{F}_q^2 \otimes \mathbb{F}_q^2$ for $q \in \{2, 3\}$ [Bremner-Stavrou 2013]
2. $S^4 \mathbb{F}_q^2$ for $q \in \{2, 3, 5, 7\}$ [Stavrou 2014]
3. $S^d \mathbb{F}_q^3$ for $d \in \{3, 4\}$ and $q \in \{2, 3\}$, [Stavrou 2015]
4. $\mathbb{F}_2^2 \otimes \mathbb{F}_2^2 \otimes \mathbb{F}_2^2 \otimes \mathbb{F}_2^2 \otimes \mathbb{F}_2^2$ [Stavrou-Low-Hernandez 2016] [Stavrou-Low-Hernandez 2018] [Betten-ML 202*]

Other results?

(B) Tensor ranks

(i) The maximum rank in $\mathbb{F}^3 \otimes \mathbb{F}^3 \otimes \mathbb{F}^3$ is 6 [ML-Pavan-Zanella 2012]
Since it is known that $\mathbb{F}_{q^n}$ has $\mathbb{F}_q$-tensor rank $2n-1$ if $q \geq 2n-2$,
and $> 2n-1$ if $q < 2n-2$ [Winograd 1979], [de Groote 1983], this
shows that $\mathbb{F}_{27}$ has $\mathbb{F}_3$-tensor rank 6.

(ii) All semifields of order 27 have $\mathbb{F}_3$-tensor rank equal to 6
[ML-Pavan-Zanella 2012].

(iii) All semifields of order 16 have $\mathbb{F}_2$-tensor rank equal to 9.

(iv) The field and GTF of order 81 have $\mathbb{F}_3$-tensor rank 9, all other
semifields of order 81 have $\mathbb{F}_3$-tensor rank 8 [ML-Sheekey 202*].
This establishes the tensor rank of a semifield as a non-trivial
invariant of the Knuth orbit of semifields.

Concluding remarks

▶ Tensors can be studied in many different ways. We have opted for a geometric/combinatorial approach, because that is what we like, but also because we feel that this allows for a better understanding of the rank and orbits.

▶ This talk is in no sense a complete overview of results on tensors over finite fields. There are many results which I could not mention. I apologise for that. (Including to some of my co-authors.)

▶ Byrne et al. picked up on this tensor approach to study MRD codes. [Byrne, Neri, Ravagnani, Sheekey 2019] *Tensor representation of rank metric codes* (2019) (Semifields are special MRD codes.)

▶ There are still many open problems in the area, and it is my hope that some of you will join us in our endeavours.