Classification results in Finite Geometry

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Outline of the talk

1. Introduction
2. PART I.
   Some well-known classification problems/results in Finite Geometry
3. PART II.
   Update on the classification of linear systems of conics

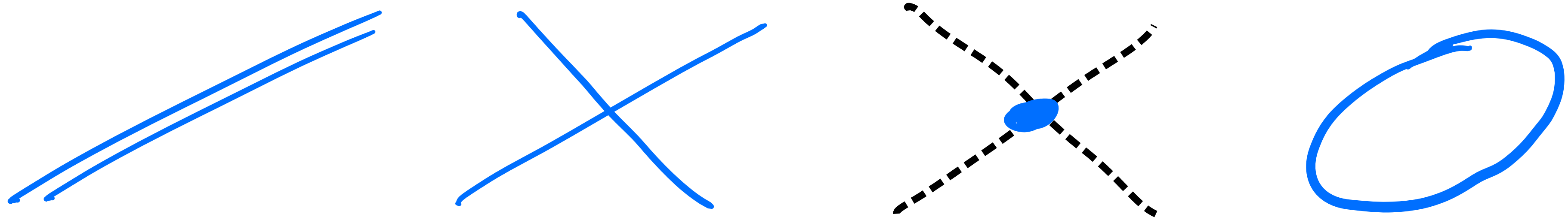
What is a classification?

1. Equivalence classes on a set Ω
   (e.g. isomorphism, isotopism, isogeny, projective equivalence, ...)

2. Orbits of a group action (G, Ω)
   (e.g. classical groups, $\mathrm{P\Gamma L}$, PGL, PSL $\mathrm{PGO}^{\pm}$, PSU, ...)

Examples. 1. Equivalences classes in Ω

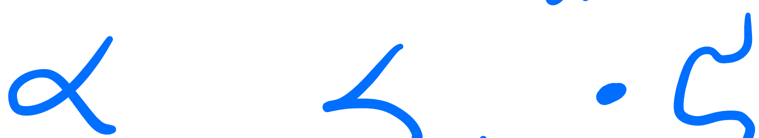
(a) $\Omega =$ set of conics in $\mathrm{PG}(2, q)$ 4 equivalence classes



(b) $\Omega =$ set of cubics in $\mathrm{PG}(2, q)$ Complicated!

→ See chapter 9 of [Hirschfeld-Korchmáros-Torres]
for non-singular cubics (Algebraic curves over finite fields)

→ There are 3 types of irred. sing. cubics



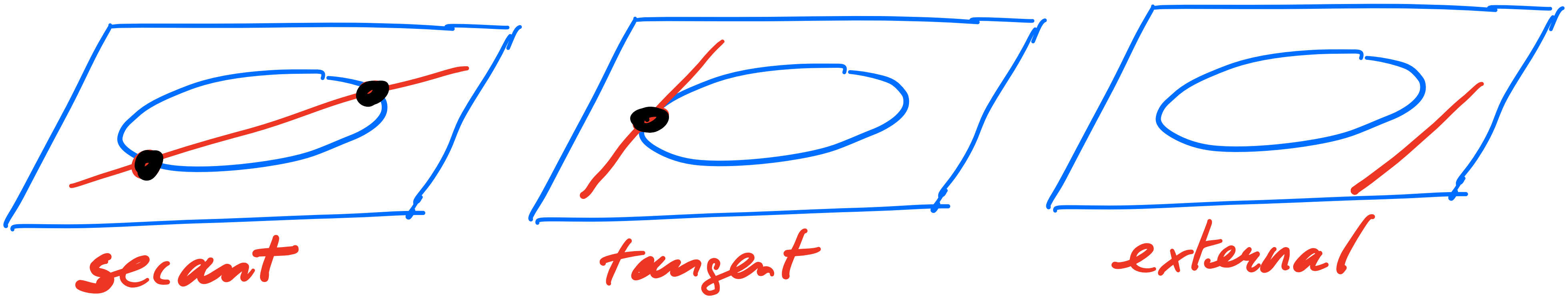
crunode acnode spinode

See Chapter 11 of [Hirschfeld]

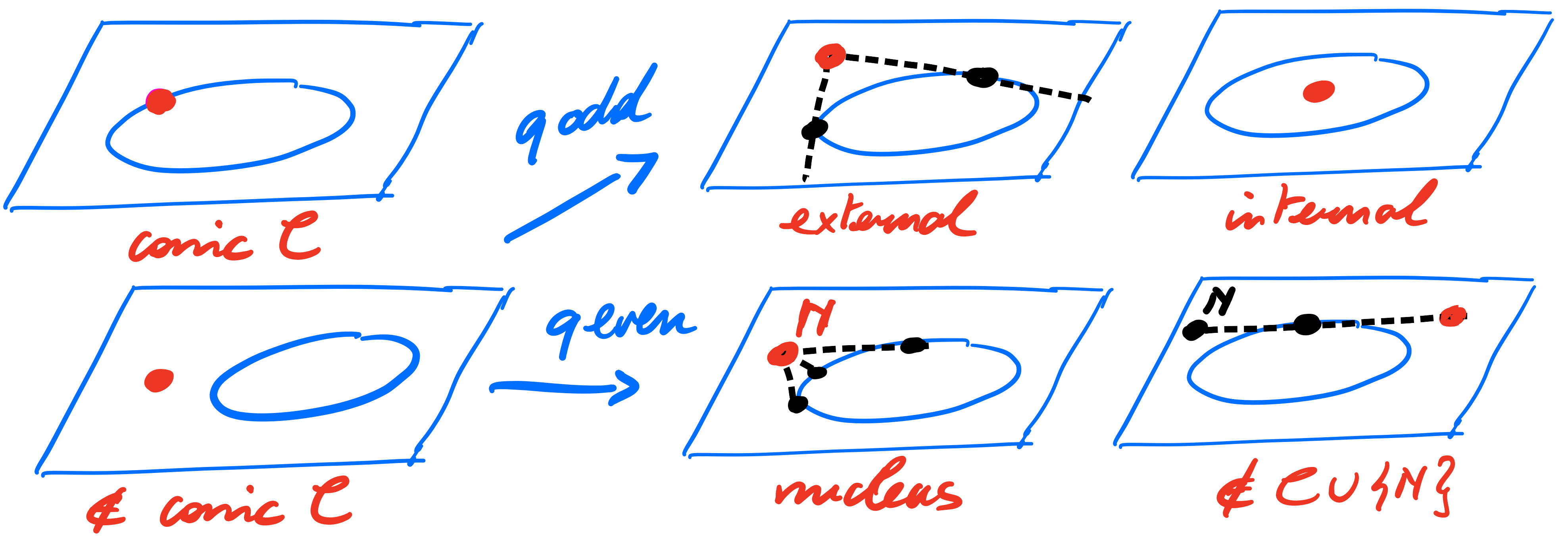
Examples. 2. Orbits of a group action (G, Ω)

G = stabilizer of $\mathcal{Z}(X_0X_1 - X_2)$ in $\mathrm{PGL}(3, q)$.

(c) Ω = set of lines in $\mathrm{PG}(2, q)$,



(d) Ω = set of points in $\mathrm{PG}(2, q)$,



PART I

Some well-known classification problems/results in Finite Geometry

Some well-known classification results.

1. A projective space of dim ≥ 3 is classical. [Veblen-Young 1902]
2. A finite Moufang plane is classical. [Artin-Zorn 1930]
3. An oval in $\mathrm{PG}(2, q)$, q odd, is a conic. [Segre 1954]
4. A finite thick generalized n-gon does not exist for $n \notin \{2, 3, 4, 6, 8\}$. [Feit-Higman 1964]
5. Classification of spherical buildings of rank ≥ 3. [Tits 1974]
6. There is no projective plane of order 10. [Lam-Thiel-Swiercz 1989]
7. The classification of finite simple groups. [... 1983-2004]

   ...

Some longstanding open problems in Finite Geometry

1. Classification of (hyper)ovals in $\mathrm{PG}(2, q)$, q even.
2. Classification of ovoids in $\mathrm{PG}(3, q)$, q even.
3. Classification of unitals in $\mathrm{PG}(2, q^2)$.
4. Classification of $(q + 1)$-arcs in $\mathrm{PG}(n \geq 3, q)$.
5. Classification of projective planes of order 12.

...

Two types of classifications

I. Classifying the geometry

1. A projective space of dim ≥ 3 is classical. [Veblen-Young 1902]
2. A finite Moufang plane is classical. [Artin-Zorn 1930]
3. A finite thick generalized n-gon does not exist for $n \notin \{2, 3, 4, 6, 8\}$. [Feit-Higman 1964]
4. Classification of spherical buildings of rank ≥ 3. [Tits 1974]
5. There is no projective plane of order 10. [Lam-Thiel-Swiercz 1989]
6. The classification of finite simple groups. [... 1983-2004]
7. Classification of projective planes of order 12.

II. Classifying substructures

1. An oval in $\mathrm{PG}(2, q)$, q odd, is a conic. [Segre 1954]
2. Classification of (hyper)ovals in $\mathrm{PG}(2, q)$, q even.
3. Classification of ovoids in $\mathrm{PG}(3, q)$, q even.
4. Classification of unitals in $\mathrm{PG}(2, q^2)$.
5. Classification of $(q + 1)$-arcs in $\mathrm{PG}(n \geq 3, q)$.

Origin for classifying geometries

Foundations of geometry: axiomatic (synthetic) approach to geometry.

From axioms (postulates) to theorems (Euclid, Hilbert, Artin, ...)

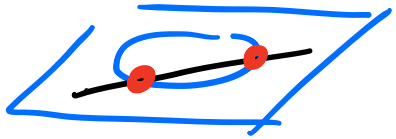
A "coordinate-free" approach to geometry

$\rightarrow$ *incidence geometry* $(\mathcal{E}, \Delta, t, I)$

Origin for the study of substructures

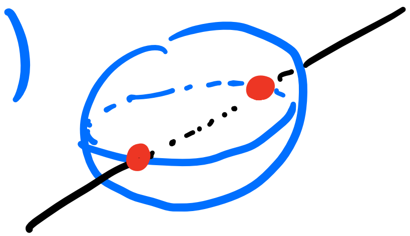
conic $\to$ oval (hyperoval)

$\mathcal{Z}(X_0X_1 - X_2^2)$ $\longrightarrow$ $q+1$ points, no 3 collinear (q even $\to$ nucleus)



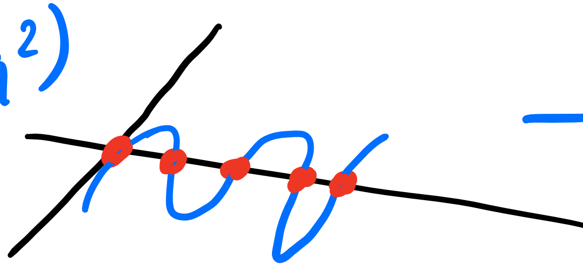
elliptic quadric $\to$ ovoid

$Q^-(3,q)$ $\longrightarrow$ q^2+1 points, no 3 collinear



Hermitian curve $\to$ unital

$\mathcal{H}(2,q^2)$ $\longrightarrow$ q^3+1 points, line intersections: $1, q+1$



normal rational curve $\to$ arc

NRC: $\{(1, t, t^2, \ldots, t^n) : t \in \mathbb{F}_q\}$ in $PG(n,q)$ $(q \geq n)$ $\longrightarrow$ $q+1$ points, each $n+1$ in general position

State of the art

on some longstanding open problems in Finite Geometry

1. Classification of (hyper)ovals in $\mathrm{PG}(2, q)$, q even.

Many examples known, intense computational efforts, classification seems currently out of reach. (Segre, Glynn, Payne, O'Keefe, Penttila, Royle, Pinneri, Abdukhalikov, Vandendriessche)

2. Classification of ovoids in $\mathrm{PG}(3, q)$, q even.

Not much activity since Brown's results [Brown 2000, 2000, 2001], except for the computational classification in $\mathrm{PG}(3, 64)$ [Penttila 2022].

State of the art ... continued

3. Arcs in $\mathrm{PG}(k-1,q)$

Related to MDS conjecture (Maximum Distance Separable codes).

Extensive research (Segre, Tallini, Hirschfeld, Stör, Voloch, Casse, Glynn, Blokhuis, Bruen, Thas, Ball, De Beule, ...)

Still very much alive, but dormant since [Ball-ML 2018], [Ball-ML 2020].

The next two slides:

Status of the MDS conjecture in $\mathrm{PG}(k-1,q)$

Classification of large complete arcs in $\mathrm{PG}(2,q)$

MDS conjecture in $\mathrm{PG}(k-1, q)$

The MDS conjecture is known to be true for the following k. (bounds given only up to first order of magnitude, p is prime.)

$k < \sqrt{q}$, q even [Segre] (1967)

$k < \sqrt{pq}$, $q = p^{2h+1}$ [Voloch] (1991)

$k < q$, $q = p$ [Ball] (2012)

$k < 2\sqrt{q}$, $q = p^2$ [Ball-De Beule] (2012)

$k < \sqrt{q}$, $q = p^{2h}$ [Ball-Lavrauw] (2018)

There are other bounds from Segre, Voloch and Hirschfeld-Korchmáros which are better for smaller q.

Classification of large complete planar arcs

- Hyperovals (size $q+2$) are not classified
- An oval (size $q+1$) in $\mathrm{PG}(2,q)$, q odd, is a conic [Segre 1955]
- An arc of size q is incomplete [Segre 1955] [Tallini 1957]
- In combination with computational results from [Coolsaet and Sticker 2009, 2011] and [Coolsaet 2015], the results from [Ball-ML 2018] complete the classification of complete planar arcs of size $q-1$ and $q-2$.

PART II

Linear systems of conics

Linear systems of algebraic varieties

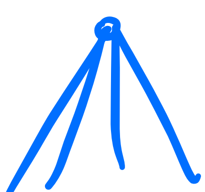
Another classical topic in algebraic geometry.

Fix a field $\mathbb{F}$, a dimension n, and degree d.

Consider the space $\mathbb{P}^N$ of all hypersurfaces $\mathcal{Z}(g)$ in $\mathbb{P}^n$ where $g \in \mathbb{F}[X_0, X_1, \ldots, X_n]$. $(N = \binom{n+d}{d} - 1)$

A *linear system of degree d hypersurfaces in $\mathbb{P}^n$* is a subspace of $\mathbb{P}^N$.

"pencil"

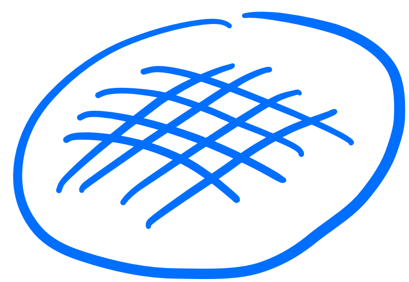


dim=1

"net"

dim=2

"web"



dim=3

...

Examples

(a) $(n, d) = (1, 2)$ $g = aX_0^2 + bX_0X_1 + cX_1^2 \to$ hyperplane in $\mathbb{P}^2$

map $\nu_{1,2} \;:\; \mathbb{P}^1 \to \mathbb{P}^2 \;:\; (x_0, x_1) \mapsto (x_0^2, x_0x_1, x_1^2)$

$\nu_{1,2}(\mathbb{P}^1)$ is $\mathcal{Z}(Y_1^2 - Y_0Y_2)$, a conic

(b) $(n, d) = (1, 3)$ $g = aX_0^3 + bX_0^2X_1 + cX_0X_1^2 + dX_1^3 \to$ hyperplane in $\mathbb{P}^3$

map $\nu_{1,3} \;:\; \mathbb{P}^1 \to \mathbb{P}^3 \;:\; (x_0, x_1) \mapsto (x_0^3, x_0^2x_1, x_0x_1^2, x_1^3)$

$\nu_{1,2}(\mathbb{P}^1)$ is $\mathcal{Z}(Y_0Y_2 - Y_1^2, Y_1Y_3 - Y2^2, Y_0Y_3 - Y_1Y_3)$, a twisted cubic

(c) $(n, d) = (2, 2)$ $g = aX_0^2 + bX_0X_1 + cX_0X_2 + dX_1^2 + eX_1X_2 + fX_2^2 \to \mathbb{P}^5$

map $\nu_{2,3} \;:\; \mathbb{P}^2 \to \mathbb{P}^5 \;:\; (x_0, x_1, x_2) \mapsto (x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2)$

$\nu_{1,2}(\mathbb{P}^1)$ is $\mathcal{Z}(Y_0Y_3 - Y_1^2, \ldots, Y_3Y_5 - Y_4^2)$, a Veronese variety

Linear systems of algebraic varieties over finite fields

A surprisingly long history!

1. Pencils, nets, and webs of conics in $\mathrm{PG}(2, q)$ (*)
   [Dickson 1908], [Wilson 1914], [Campbell 1927], [Campbell 1928], [ML-Popiel 2020], [ML-Popiel-Sheekey 2020], [ML-Popiel-Sheekey 2021], [Alnajjarine-ML-Popiel 2022], [Alnajjarine-ML 2023], [ML-Popiel-Sheekey 2023]
2. Pencils of cubics in $\mathrm{PG}(1, q)$
   [Bruen-Hirschfeld 1977], [Davidov-Marcugini-Pambianco 2021, 2021, 2022, 2023, 2023], [Blokhuis-Pelikaan-Szőnyi 2022], [Günay-ML 2022] [Ceria-Pavese 2023]
3. Pencils of quadrics in $\mathrm{PG}(3, q)$
   [Bruen-Hirschfeld 1988]

(*) For nets of conics over algebraically closed fields of characteristic 0 (or $\neq 2, 3$) see [Abdallah-Emsalem-Iarrobino 2023] (and its 117 references).

Pencils of conics in $\mathrm{PG}(2, q)$

(G, Ω) $\left\{ \begin{array}{l} G \cong PGL(3, q) \\ \Omega = \text{set of pencils of conics} \end{array} \right.$

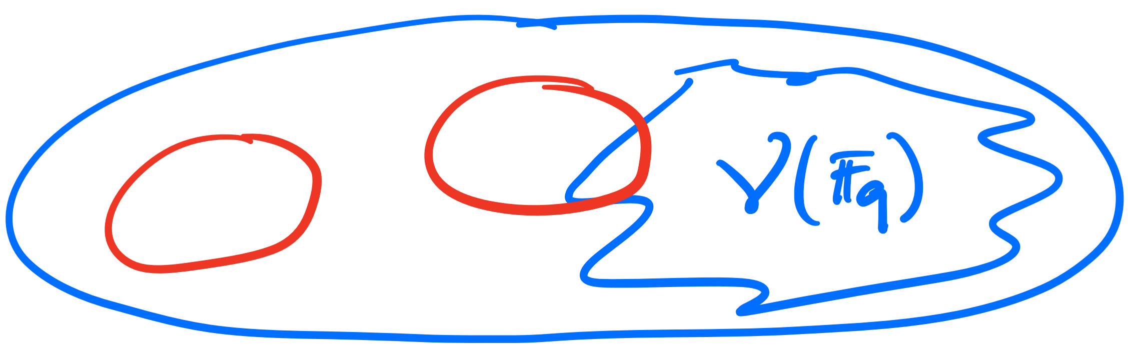
Complete classification

q odd [Dickson 1908] [ML-Popiel 2019]

q even [Alnajjarine-ML-Popiel 2022]

(This completes/corrects [Campbell 1927])

This corresponds to solids in $PG(5, q)$



Theorem (Alnajjarine-ML-Popiel 2022)

There are exactly 15 orbits of solids in $\mathrm{PG}(5, q)$*, q even, under the induced action of* $\mathrm{PGL}(3, q) \leqslant \mathrm{PGL}(6, q)$*.*

Orbit	**Point OD**	**Hyperplane OD**	**Stabiliser**	**Orbit size**
Ω_1	$[1, q+1, 2q^2-1, q^3-q^2]$	$[1, q/2, q/2, 0]$	$E_q^2 \rtimes (E_q \times C_{q-1})$	$(q^3-1)(q+1)$
Ω_2	$[q+1, q+1, 2q^2-q-1, q^3-q^2]$	$[1, q, 0, 0]$	$E_q^{1+2} \rtimes C_{q-1}^2$	$(q^2+q+1)(q+1)$
Ω_3	$[1, q^2+q+1, q^2-1, q^3-q^2]$	$[q+1, 0, 0, 0]$	$E_q^2 \rtimes \mathrm{GL}(2, q)$	q^2+q+1
Ω_4	$[q+2, 1, 2q^2-2, q^3-q^2]$	$[0, q+1, 0, 0]$	$\mathrm{GL}(2, q)$	$q^2(q^2+q+1)$
Ω_5	$[1, q+1, q^2-1, q^3]$	$[1, 0, 0, q]$	$E_q^2 \rtimes C_{q-1}$	$q(q^3-1)(q+1)$
Ω_6	$[2, q+1, q^2+q-2, q^3-q]$	$[1, 1, 0, q-1]$	$C_{q-1}^2 \rtimes C_2$	$\frac{1}{2}q^3(q^2+q+1)(q+1)$
Ω_7	$[0, q+1, q^2+q, q^3-q]$	$[1, 0, 1, q-1]$	$D_{2(q+1)} \times C_{q-1}$	$\frac{1}{2}q^3(q^3-1)$
Ω_8	$[3, 1, q^2+2q-3, q^3-q]$	$[0, 2, 0, q-1]$	$C_{q-1} \times C_2$	$\frac{1}{2}q^3(q^3-1)(q+1)$
Ω_9	$[4, 1, q^2+3q-4, q^3-2q]$	$[0, 3, 0, q-2]$	Sym_4	$\frac{1}{24}q^3(q^3-1)(q^2-1)$
Ω_{10}	$[1, 1, q^2+2q-1, q^3-q]$	$[0, 1, 1, q-1]$	$C_{q-1} \times C_2$	$\frac{1}{2}q^3(q^3-1)(q+1)$
Ω_{11}	$[2, 1, q^2+q-2, q^3]$	$[0, 1, 0, q]$	$E_q \rtimes C_{q-1}$	$q^2(q^3-1)(q+1)$
Ω_{12}	$[2, 1, q^2+q-2, q^3]$	$[0, 1, 0, q]$	C_2^2	$\frac{1}{4}q^3(q^3-1)(q^2-1)$
Ω_{13}	$[0, 1, q^2+3q, q^3-2q]$	$[0, 1, 2, q-2]$	$C_2^2 \rtimes C_2$	$\frac{1}{8}q^3(q^3-1)(q^2-1)$
Ω_{14}	$[0, 1, q^2+q, q^3]$	$[0, 0, 1, q]$	C_4	$\frac{1}{4}q^3(q^3-1)(q^2-1)$
Ω_{15}	$[1, 1, q^2-1, q^3+q]$	$[0, 0, 0, q+1]$	C_3	$\frac{1}{3}q^3(q^3-1)(q^2-1)$

This table is taken from [ML-Popiel 2019], [Alnajjarine-ML-Popiel 2022]

Webs of conics in $\mathrm{PG}(2,q)$

Complete classification: 15 orbits for any q.

Theorem (ML - Popiel 2019)

Classification of orbits of lines in $\langle \mathcal{V}_3(\mathbb{F}_q)\rangle$.

- 3 of the 14 tensor-orbits from [ML-Sheekey 2015] do not have a symmetric representative (o_4, o_7, o_{11})
- q odd: *tensor*-orbits o_8, o_{13}, o_{14}, o_{15} split into two orbits
- q even: *tensor*-orbits o_8, o_{12}, o_{13}, o_{16} split into two orbits
- in total 15 orbits of lines in $\langle \mathcal{V}_3(\mathbb{F}_q)\rangle$
- unique orbit of constant rank 3 lines

Nets of conics in $\mathrm{PG}(2, q)$

- No complete classification yet.
- No common treatment for q odd and even (existence of nucleus plane, cubics, Hessian, inflexion points, ...)
- Net $\mathcal{N} \to$ plane $\pi_{\mathcal{N}}$ in $\mathrm{PG}(5, q)$
- $\pi_{\mathcal{N}} \cap \mathcal{V}^{(2)} \to$ planar cubic curve $\mathcal{Z}(\Delta_{\mathcal{N}})$

$\swarrow$ $\searrow$

q odd q even

Nets of conics in $\mathrm{PG}(2, q)$, q odd

- No complete classification yet.
- Planes meeting the Veronese variety are classified.
- They correspond to nets of rank one.

Theorem (ML-Popiel-Sheekey 2020)

There are 15 nets of conics of rank one in $\mathrm{PG}(2, q)$*,* q *odd.*

- Stabilisers
- Combinatorial invariants for q odd
  all orbit distributions are determined: point-orbit, line-orbit, solid-orbit, and hyperplane-orbit distributions
  [ML-Popiel-Sheekey 2021]

Line-orbit distributions

Orbit	q (3)	o_5	o_6	$o_{8,1}$	$o_{8,2}$	o_9	o_{10}	o_{12}	$o_{13,1}$	$o_{13,2}$	$o_{14,1}$	$o_{14,2}$	$o_{15,1}$	$o_{15,2}$	o_{16}	o_{17}
Σ_1		$\frac{q(q+1)}{2}$	$q+1$				$\frac{q(q-1)}{2}$									
Σ_2		3		$\frac{3(q-1)}{2}$	$\frac{3(q-1)}{2}$						$\frac{(q-1)^2}{4}$	$\frac{3(q-1)^2}{4}$				
Σ_3		1	1	q		$q-1$			$\frac{q(q-1)}{2}$	$\frac{q(q-1)}{2}$						
Σ_4		1		$2q$				1	$\frac{q(q-3)}{2}$	$\frac{q(q-1)}{2}$					$q-1$	
Σ_5		1		$q-1$	$q-1$	2			$\frac{q-1}{2}$	$\frac{q-1}{2}$	$\frac{(q-1)(q-3)}{8}$	$\frac{(q-1)(3q-5)}{8}$	$\frac{(q-1)^2}{4}$	$\frac{(q+1)(q-1)}{4}$		
Σ_6				$\frac{q+1}{2}$	$\frac{q+1}{2}$		1						$\frac{(q+1)(q-1)}{2}$	$\frac{(q+1)(q-1)}{2}$		
Σ_7			$q+1$					q^2								
Σ_8			1	q				1	$q(q-1)$						$q-1$	
Σ_9			1	q					q		$\frac{q(q-1)}{2}$		$\frac{q(q-1)}{2}$			
Σ_{10}			1		q					q		$\frac{q(q-1)}{2}$	$\frac{q(q-1)}{2}$			
Σ_{11}	0			q		1			q		$\frac{q(q-3)}{6}$		$\frac{q(q-1)}{2}$			$\frac{q^2}{3}$
	$\not\equiv 0$			q		1			$q-1$		$\frac{(q-1)(q-2)}{6}$		$\frac{q(q-1)}{2}$		1	$\frac{(q+1)(q-1)}{3}$
Σ_{12}	1			$\frac{q-1}{2}$	$\frac{q-1}{2}$	2			$\frac{q-7}{2}$	$\frac{q-1}{2}$	$\frac{(q-1)(q-7)}{24}+1$	$\frac{(q-1)(q-3)}{8}$	$\frac{(q-1)^2}{4}$	$\frac{(q+1)(q-1)}{4}$	3	$\frac{(q-1)(q+2)}{3}$
	$\not\equiv 1$			$\frac{q-1}{2}$	$\frac{q-1}{2}$	2			$\frac{q-3}{2}$	$\frac{q-1}{2}$	$\frac{(q-3)(q-5)}{24}$	$\frac{(q-1)(q-3)}{8}$	$\frac{(q-1)^2}{4}$	$\frac{(q+1)(q-1)}{4}$	1	$\frac{q(q+1)}{3}$
Σ_{13}	-1			$\frac{q+1}{2}$	$\frac{q+1}{2}$				$\frac{q-5}{2}$	$\frac{q+1}{2}$	$\frac{(q+1)(q-5)}{24}+1$	$\frac{(q+1)(q-1)}{8}$	$\frac{(q+1)(q-3)}{4}$	$\frac{(q+1)(q-1)}{4}$	3	$\frac{(q+1)(q-2)}{3}$
	$\not\equiv -1$			$\frac{q+1}{2}$	$\frac{q+1}{2}$				$\frac{q-1}{2}$	$\frac{q+1}{2}$	$\frac{(q-1)(q-3)}{24}$	$\frac{(q+1)(q-1)}{8}$	$\frac{(q+1)(q-3)}{4}$	$\frac{(q+1)(q-1)}{4}$	1	$\frac{q(q-1)}{3}$
Σ_{14}	1			$\frac{q-1}{2}$	$\frac{q-1}{2}$	2			$\frac{q-1}{2}$	$\frac{q-1}{2}$	$\frac{(q-1)(q-7)}{24}$	$\frac{(q-1)(q-3)}{8}$	$\frac{(q-1)^2}{4}$	$\frac{(q+1)(q-1)}{4}$		$\frac{(q-1)^2}{3}+q$
	-1			$\frac{q+1}{2}$	$\frac{q+1}{2}$				$\frac{q+1}{2}$	$\frac{q+1}{2}$	$\frac{(q+1)(q-5)}{24}$	$\frac{(q+1)(q-1)}{8}$	$\frac{(q-1)(q-3)}{4}$	$\frac{(q+1)(q-1)}{4}$		$\frac{(q-1)^2}{3}-q$
Σ'_{14}	0			q		1					$\frac{q(q-1)}{6}$		$\frac{q(q-1)}{2}$		q	$\frac{q(q-1)}{3}$
Σ_{15}			1			q									q^2	

TABLE 1. Line-orbit distributions of planes in $\mathrm{PG}(5,q)$, q odd, that intersect the quadric Veronesean. Where applicable, the second column indicates the congruence class(es) of q modulo 3.

This table is taken from [ML-Popiel-Sheekey 2021]

Line-orbit distributions

Orbit	$q\ (3)$	o_5	o_6	$o_{8,1}$	$o_{8,2}$	o_9	o_{10}	o_{12}	$o_{13,1}$	$o_{13,2}$
Σ_1		$\frac{q(q+1)}{2}$	$q+1$				$\frac{q(q-1)}{2}$			
Σ_2		3		$\frac{3(q-1)}{2}$	$\frac{3(q-1)}{2}$					
Σ_3		1	1	q		$q-1$			$\frac{q(q-1)}{2}$	$\frac{q(q-1)}{2}$
Σ_4		1		$2q$				1	$\frac{q(q-3)}{2}$	$\frac{q(q-1)}{2}$
Σ_5		1		$q-1$	$q-1$	2			$\frac{q-1}{2}$	$\frac{q-1}{2}$
Σ_6				$\frac{q+1}{2}$	$\frac{q+1}{2}$		1			
Σ_7			$q+1$					q^2		
Σ_8			1	q				1	$q(q-1)$	
Σ_9			1	q					q	
Σ_{10}			1		q					q
Σ_{11}	0			q		1			q	
	$\not\equiv 0$			q		1			$q-1$	
Σ_{12}	1			$\frac{q-1}{2}$	$\frac{q-1}{2}$	2			$\frac{q-7}{2}$	$\frac{q-1}{2}$
	$\not\equiv 1$			$\frac{q-1}{2}$	$\frac{q-1}{2}$	2			$\frac{q-3}{2}$	$\frac{q-1}{2}$
Σ_{13}	-1			$\frac{q+1}{2}$	$\frac{q+1}{2}$				$\frac{q-5}{2}$	$\frac{q+1}{2}$
	$\not\equiv -1$			$\frac{q+1}{2}$	$\frac{q+1}{2}$				$\frac{q-1}{2}$	$\frac{q+1}{2}$
Σ_{14}	1			$\frac{q-1}{2}$	$\frac{q-1}{2}$	2			$\frac{q-1}{2}$	$\frac{q-1}{2}$
	-1			$\frac{q+1}{2}$	$\frac{q+1}{2}$				$\frac{q+1}{2}$	$\frac{q+1}{2}$
Σ'_{14}	0			q		1				
Σ_{15}			1			q				

Nets of conics in $\mathrm{PG}(2, q)$, q even

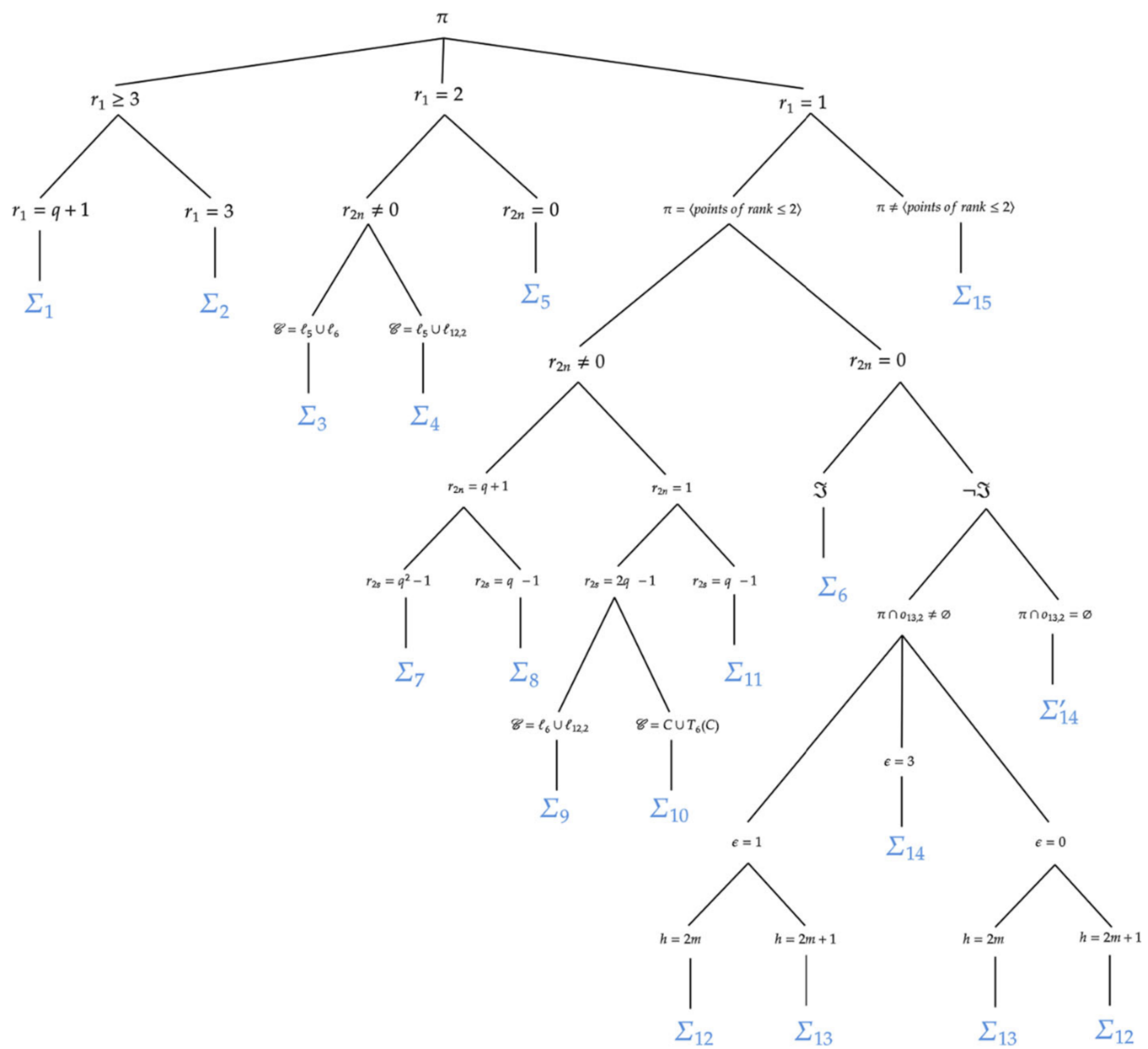
- No complete classification yet.
- Planes meeting the Veronese variety are classified.
  [Alnajjarine-ML-Popiel 2023]
  They correspond to nets with non-empty base.

Theorem (Alnajjarine-ML-Popiel 2023)

There are 15 nets of conics with non-empty base in $\mathrm{PG}(2, q)$, q even.

- Stabilisers
- Point-orbit distribution is determined.

Decision tree [Alnajjarine - ML 2023]



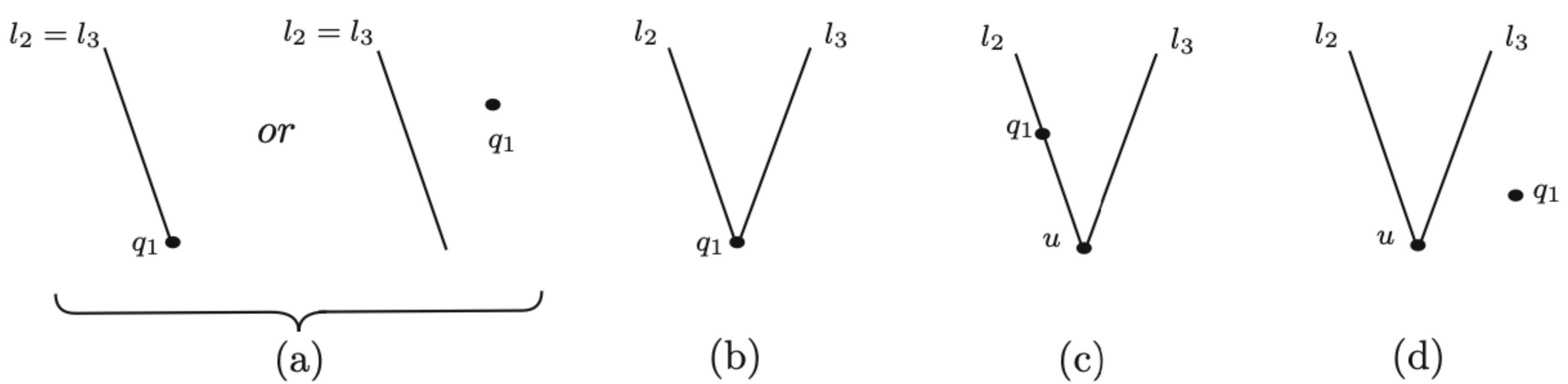
Example subcase of [Alnajjarine - ML 2023]

Planes containing one rank-1 point and spanned by points of rank at most 2

$\pi_{\mathcal{N}} = \langle Q_1, Q_2, Q_3 \rangle$, $rank(Q_1) = 1$, and $rank(Q_2) = rank(Q_3) = 2$,

$\rightarrow$ conics $\mathcal{C}(Q_2)$ and $\mathcal{C}(Q_3)$ on $\mathcal{V}(\mathbb{F}_q)$.

Denote by q_1, l_2 and l_3 the respective preimages of Q_1, $\mathcal{C}(Q_2)$ and $\mathcal{C}(Q_3)$ under the Veronese embedding. By symmetry, we are left with the following possibilities:



Concluding remarks

- There are still many open classification problems to work on, and it is my hope that some of you will join us in our endeavours to make progress.
- G-orbit classifications often lead to new constructions of desirable objects (e.g. Tits ovoids from Suzuki group, quasi-Hermitian varieties, BLT-sets, ovoids of polar spaces, hemisystems, ...)
- Ideas appearing in the work on nets of conics can be applied to the study of linear systems of other algebraic varieties.
- We should be careful with claims about classification results. It is easy to make mistakes (and there are many in the literature). A complete classification should only be accepted when it is accompanied by a detailed description of the representatives, and a complete (verifiable) proof dealing with every single case.

Thank you for your attention!