Classification of orbits in $K^2 \otimes K^3 \otimes K^r$, $r \geq 1$ and lines in the space of the Veronese surface

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Tensor products have many applications

- Computational complexity theory
- Tensors describe quantum mechanical systems (entanglement)
- Data analysis (chemistry, biology, physics, ...)
- Signal processing, source separation
- ...

Our original motivation:

- Theory of finite semifields: finite non-associative division algebras

Tensor product $\bigotimes_{i \in I} V_i$

Consider m vectorspaces V_i over the field K, $I = \{1, \ldots, m\}$.

- fundamental or pure tensors: $v_1 \otimes \ldots \otimes v_m$, $v_i \in V_i$.
- general element $\tau \in \bigotimes_{i \in I} V_i$

$$\tau = \sum_i v_{1i} \otimes \ldots \otimes v_{mi}$$

- τ defines a multilinear map
- choosing bases for each V_i we obtain a hypercube $(a_{i_1 i_2 \ldots i_m})$

$$\tau = \sum_{i_1, \ldots, i_m} a_{i_1 i_2 \ldots i_m} e_{1 i_1} \otimes \ldots \otimes e_{m i_m}$$

Main issue for applications: "decomposition"

An expression

$$\tau = \sum_{i=1}^{r} v_{1i} \otimes \ldots \otimes v_{mi} \qquad (1)$$

is called a decomposition of $\tau \in V_1 \otimes \ldots \otimes V_m$.

Four important problems:

- Algorithm
- Uniqueness
- Existence: given τ and r, does (1) exist? $\rightarrow$ rank
- Orbits: how many "different" tensors are there?

This talk

"Orbits": how many "different" tensors are there?

Group action

- An element

$$(g_1, g_2, \ldots g_m) \in \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \ldots \times \mathrm{GL}(V_m)$$

acts on the fundamental tensors:

$$v_1 \otimes v_2 \otimes \ldots \otimes v_m \mapsto v_1^{g_1} \otimes v_2^{g_2} \otimes \ldots \otimes v_m^{g_m}.$$

- If $V_i = V = K^n$ for all i, then we also have an action of S_m as follows:

$$\pi : \langle v_1 \otimes v_2 \otimes \ldots \otimes v_m \rangle \mapsto \langle v_{\pi(1)} \otimes v_{\pi(2)} \otimes \ldots \otimes v_{\pi(m)} \rangle.$$

Geometry of tensor spaces

- Segre embedding:

$$\sigma \ : \ \mathrm{PG}(V_1) \times \mathrm{PG}(V_2) \times \ldots \times \mathrm{PG}(V_m) \ \rightarrow \ \mathrm{PG}(\bigotimes_i V_i)$$

$$(\langle v_1 \rangle, \langle v_2 \rangle, \ldots, \langle v_m \rangle) \mapsto \langle v_1 \otimes v_2 \otimes \ldots \otimes v_m \rangle$$

- $S_{n_1,n_2,\ldots,n_m}(K) = Im(\sigma)$ is the Segre variety ($n_i = \dim V_i$)
- The group $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \ldots \times \mathrm{GL}(V_m)$ induces a subgroup G_m of $\mathrm{PGL}(n^m - 1, K)$.
- G_m stabilises $S_{n,\ldots,n}$

Aim

Classify the G_m-orbits on $\mathrm{PG}(\bigotimes_i V_i)$.

Known results

- $m = 1$: trivial
- $m = 2$: $V_1 \otimes V_2 \cong \mathcal{M}(n_1 \times n_2, K)$: $\mathrm{rk}(u) = \mathrm{rk}(M_u)$

  $\Rightarrow$ one orbit for each rank

Known results for $m = 3$

- $\mathbb{F}_p^2 \otimes \mathbb{F}_p^3 \otimes \mathbb{F}_p^3$ [Brahana (1933)] + [Thrall (1938)]
- $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ [Thrall-Chanler (1938)]
- $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ has a finite number of orbits only if $a \leq 2$, $b \leq 3$ [Kac (1980)] [Kraśkiewicz-Weyman (2009)]
- $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ [Nurmiev (2000)]
- $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ [Parfenov (2001)]
- $\mathbb{F}_2^2 \otimes \mathbb{F}_2^2 \otimes \mathbb{F}_2^2$ [Glynn et al. (2006)]
- $\mathbb{F}_2^2 \otimes \mathbb{F}_2^2 \otimes \mathbb{F}_2^2$ [Havlicek et al. (2012)]
- $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^c$ [Buczyński-Landsberg (2013)]
- computational [Bremner-Stavrou (2013)] for small p
- geometric $K^2 \otimes K^2 \otimes K^2$ [ML - J. Sheekey (2014)]

Aim

Classification of the G_3-orbits on $\mathrm{PG}(K^a \otimes K^b \otimes K^c)$

Preferably with a proof which is

- Comprehensive
- Geometric
- Independent
- Elementary
- Insightful

Orbits in $K^2 \otimes K^3 \otimes K^r$, $r \geq 1$

Theorem (ML-J. Sheekey 2016)

Classification of orbits in $K^2 \otimes K^3 \otimes K^r$, $\forall r \geq 1$, *for* $K = \mathbb{F}_q$, *for* K *an algebraically closed field, and for* $K = \mathbb{R}$.

Proof

- contraction spaces of $A \in K^2 \otimes K^3 \otimes K^3$
- $\mathrm{PG}(A_1)$'s in $\langle S_{3,3}(K) \rangle$
- rank distribution $[a, b, c]$ of $\mathrm{PG}(A_1)$
- classification of lines in $\langle S_{3,3}(K) \rangle$: 14 orbits
- classification of orbits in $K^2 \otimes K^3 \otimes K^3$
- $\mathrm{PG}(A_3)$'s in $\langle S_{2,3}(K) \rangle$
- all subspaces in $\langle S_{2,3}(K) \rangle$
- classification in $K^2 \otimes K^3 \otimes K^r$, $\forall r \geq 1$.

The orbits in $K^2 \otimes K^3 \otimes K^3$

Theorem (ML-J. Sheekey 2016)

Orbits in $K^2 \otimes K^3 \otimes K^3$ for $K = \mathbb{F}_q$:

Orbit	Canonical form	Condition	$r_1(A)$
o_0	0		$[0, 0, 0]$
o_1	$e_1 \otimes e_1 \otimes e_1$		$[1, 0, 0]$
o_2	$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2)$		$[0, 1, 0]$
o_3	$e_1 \otimes e$		$[0, 0, 1]$
o_4	$e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_1 \otimes e_2$		$[q+1, 0, 0]$
o_5	$e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2$		$[2, q-1, 0]$
o_6	$e_1 \otimes e_1 \otimes e_1 + e_2 \otimes (e_1 \otimes e_2 + e_2 \otimes e_1)$		$[1, q, 0]$
o_7	$e_1 \otimes e_1 \otimes e_3 + e_2 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2)$		$[1, q, 0]$
o_8	$e_1 \otimes e_1 \otimes e_1 + e_2 \otimes (e_2 \otimes e_2 + e_3 \otimes e_3)$		$[1, 1, q-1]$
o_9	$e_1 \otimes e_3 \otimes e_1 + e_2 \otimes e$		$[1, 0, q]$
o_{10}	$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2 + ue_1 \otimes e_2) + e_2 \otimes (e_1 \otimes e_2 + ve_2 \otimes e_1)$	$(*)$	$[0, q+1, 0]$
o_{11}	$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2) + e_2 \otimes (e_1 \otimes e_2 + e_2 \otimes e_3)$		$[0, q+1, 0]$
o_{12}	$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2) + e_2 \otimes (e_1 \otimes e_3 + e_3 \otimes e_2)$		$[0, q+1, 0]$
o_{13}	$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2) + e_2 \otimes (e_1 \otimes e_2 + e_3 \otimes e_3)$		$[0, 2, q-1]$
o_{14}	$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2) + e_2 \otimes (e_2 \otimes e_2 + e_3 \otimes e_3)$		$[0, 3, q-2]$
o_{15}	$e_1 \otimes (e + ue_1 \otimes e_2) + e_2 \otimes (e_1 \otimes e_2 + ve_2 \otimes e_1)$	$(*)$	$[0, 1, q]$
o_{16}	$e_1 \otimes e + e_2 \otimes (e_1 \otimes e_2 + e_2 \otimes e_3)$		$[0, 1, q]$
o_{17}	$e_1 \otimes e + e_2 \otimes (e_1 \otimes e_2 + e_2 \otimes e_3 + e_3 \otimes (\alpha e_1 + \beta e_2 + \gamma e_3))$	$(**)$	$[0, 0, q+1]$

Orbits in $K^2 \otimes K^3 \otimes K^r$ $r \geq 1$

Theorem (ML-J. Sheekey 2016)

The number of H-orbits of tensors in $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^r$ *is as listed in the following table:*

r	1	2	3	4	5	≥ 6
$\#H-orbits$	3	10	21	28	30	31

Theorem (ML-J. Sheekey 2016)

The number of G-orbits of tensors in $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^r$ *is as listed in the following table:*

r	1	2	3	≥ 4
$\#G-orbits$	3	9	18	$\#H-orbits$

Orbits in $K^2 \otimes K^3 \otimes K^r$, $r \geq 1$

Theorem (ML-J. Sheekey 2016)

If $\mathbb{F}$ is an algebraically closed field or the field of real numbers, then the number of H-orbits and G-orbits of tensors in $\mathbb{F}^2 \otimes \mathbb{F}^3 \otimes \mathbb{F}^r$ is as listed in the following tables.

r	1	2	3	4	5	≥ 6	
$\#H-orbits$	3	9	18	24	26	27	$\mathbb{F}$ *algebraically closed*
$\#H-orbits$	3	10	20	27	29	30	$\mathbb{F} = \mathbb{R}$

r	1	2	3	≥ 4	
$\#G-orbits$	3	8	15	$\#H-orbits$	$\mathbb{F}$ *algebraically closed*
$\#G-orbits$	3	9	17	$\#H-orbits$	$\mathbb{F} = \mathbb{R}$

Line orbits in $\langle \mathcal{V}_3(\mathbb{F}_q) \rangle$

- $\mathcal{V}_3(\mathbb{F}_q)$ is the Veronese surface in $\mathrm{PG}(5, q)$.
- $H_3 = Aut(\mathcal{V}_3(\mathbb{F}_q)) \leq \mathrm{PGL}(6, q)$, $H_3 \cong \mathrm{PGL}(3, q)$.

Questions:

- What are the H_3-orbits of lines in $\langle \mathcal{V}_3(\mathbb{F}_q) \rangle$?
- Which G_2-orbits of lines in $\mathrm{PG}(\mathbb{F}_q^3 \otimes \mathbb{F}_q^3) \cong \mathrm{PG}(8, q)$ are represented in the space of the Veronese surface $\langle \mathcal{V}_3(\mathbb{F}_q) \rangle$?
- Which G_2-orbits split under the group H_3?

Line orbits in $\langle \mathcal{V}_3(\mathbb{F}_q) \rangle$

Theorem (ML - Tomasz Popiel)

Classification of orbits of lines in $\langle \mathcal{V}_3(\mathbb{F}_q) \rangle$.

- 3 of the 14 G_2-orbits are not represented (o_4, o_7, o_{11})
- q odd: G_2-orbits o_8, o_{13}, o_{14}, o_{15} split into two H_3-orbits
- q even: G_2-orbits o_8, o_{12}, o_{13}, o_{16} split into two H_3-orbits
- in total 15 H_3-orbits of lines in $\langle \mathcal{V}_3(\mathbb{F}_q) \rangle$
- unique H_3-orbit of constant rank 3 lines

Thank you for your attention!