Determining the rank of tensors in $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$

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Abstract. Let $\mathbb{F}_q$ be a finite field of order q. This paper uses the classification in [7] of orbits of tensors in $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$ to define two algorithms that take an arbitrary tensor in $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$ and return its orbit, a representative of its orbit, and its rank.

Keywords: Tensor Rank · Rank Distribution · Tensor Decomposition.

1 Introduction and preliminaries

The study of tensors of order at least three has been an active area in recent years, with numerous applications in representation theory, algebraic statistics and complexity theory [6,5]. For example, the problem of determining the complexity of matrix multiplication can be rephrased as the problem of determining the rank of a particular tensor (the matrix multiplication operator). This problem has only been solved for 2×2-matrices (see Strassen and Winograd), and we refer to [6, Chapter 2, Section 4] for more on this topic.

Determining the decomposition of a tensor A is a notoriously hard problem that arises in many other applications such as psychometrics, chemometrics, numerical linear algebra and numerical analysis [5]. In many tensor decomposition problems, the first issue to resolve is to determine the rank of the tensor, which is not always an easy task.

Let $Sym(2)$ denote the symmetric group of order 2 and $V := \mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$, where $\mathbb{F}_q$ is the finite field of order q. Consider then the two natural actions on V of the group G and its subgroup H, where $G \cong \mathrm{GL}(\mathbb{F}_q^2) \times (\mathrm{GL}(\mathbb{F}_q^3) \wr Sym(2))$, as a subgroup of $\mathrm{GL}(V)$ stabilising the set of fundamental tensors in V, and $H \cong \mathrm{GL}(\mathbb{F}_q^2) \times \mathrm{GL}(\mathbb{F}_q^3) \times \mathrm{GL}(\mathbb{F}_q^3)$. In this paper, we study tensors in V under the action of G to present the algorithms *"RankOfTensor"* and *"OrbitOfTensor"*, which take an arbitrary tensor in V and return its orbit, a representative of its orbit, and its rank.

We follow the notation and terminology from [8]. Let A be a tensor in V. The *rank* of A, $Rank(A)$, is defined to be the smallest integer r such that

$$A = \sum_{i=1}^{r} A_i \tag{1}$$

with each A_i, a rank one tensor in V. Recall that the set of rank one tensors (fundamental tensors) in V is the set $\{v_1 \otimes v_2 \otimes v_3 : v_1 \in \mathbb{F}_q^2 \setminus \{0\}\ ,\ v_2, v_3 \in \mathbb{F}_q^3 \setminus \{0\}\}$.

It is clear from this definition that the rank of a tensor is a *projective property* in the vector space V. In other words, the rank of A does not change when A is multiplied by a nonzero scalar. For this reason, to dispose of the unneeded information, it makes sense to consider the problem of rank and decomposition in the projective space $\mathrm{PG}(V)$.

The Segre variety. Projectively, the set of nonzero tensors of rank one corresponds to the set of points on the Segre variety $S_{1,2,2}(\mathbb{F}_q)$, which is the image of the Segre embedding $\sigma_{1,2,2}$ defined as:

$$\begin{gathered}\sigma_{1,2,2}: \mathrm{PG}(\mathbb{F}_q^2) \times \mathrm{PG}(\mathbb{F}_q^3) \times \mathrm{PG}(\mathbb{F}_q^3) \longrightarrow \mathrm{PG}(V)\\ (\ \langle v_1\rangle, \langle v_2\rangle, \langle v_3\rangle\ ) \mapsto \langle v_1 \otimes v_2 \otimes v_3\rangle.\end{gathered}$$

For any projective point in $\mathrm{PG}(V)$, we define its rank to be the rank of any corresponding tensor.

Contraction spaces. For $A \in V$, we define the *first contraction space of A* to be the following subspace of $\mathbb{F}_q^3 \otimes \mathbb{F}_q^3$:

$$A_1 := \langle u_1^\vee(A) : u_1^\vee \in \mathbb{F}_q^{2\vee} \rangle$$

where $\mathbb{F}_q^{2\vee}$ denotes the dual space of $\mathbb{F}_q^2$, and where the *contraction* $u_1^\vee(A)$ is defined by its action on the fundamental tensors as follows:

$$u_1^\vee(v_1 \otimes v_2 \otimes v_3) = u_1^\vee(v_1)v_2 \otimes v_3. \tag{2}$$

Similarly, the *second* and *third contraction spaces*, A_2 and A_3, can be defined. Note that we are considering in this study the projective subspaces $\mathrm{PG}(A_1)$, $\mathrm{PG}(A_2)$ and $\mathrm{PG}(A_3)$ of $\mathrm{PG}(\mathbb{F}_q^3 \otimes \mathbb{F}_q^3)$, $\mathrm{PG}(\mathbb{F}_q^2 \otimes \mathbb{F}_q^3)$ and $\mathrm{PG}(\mathbb{F}_q^2 \otimes \mathbb{F}_q^3)$, respectively, where we have $\mathrm{PG}(\mathbb{F}_q^3 \otimes \mathbb{F}_q^3) \cong \mathrm{PG}(8,q)$ and $\mathrm{PG}(\mathbb{F}_q^2 \otimes \mathbb{F}_q^3) \cong \mathrm{PG}(5,q)$. Also, remark that the rank of any *contraction* coincides with the usual matrix rank.

Rank distributions. For $1 \leq i \leq 3$, define the *i-th rank distribution of A*, R_i, to be the 3-tuple whose j-th entry is the number of rank j points in the i-th contraction space $\mathrm{PG}(A_i)$. Consider now the canonical basis of F_q^ℓ, $\{e_1, \dots, e_\ell\}$, for $\ell = 2, 3$. We define the canonical basis of V as $\{e_i \otimes e_j \otimes e_k : 1 \leq i \leq 2 \textit{ and } 1 \leq j,k \leq 3\}$. By writing $A \in V$ as $A = \sum A_{i,j,k} e_i \otimes e_j \otimes e_k$, one can view A as a $2 \times 3 \times 3$ rectangular solid whose entries are the $A_{i,j,k}$'s. This solid can be decomposed into slices that completely determine A. For example, we may view A as a collection of 2 size 3×3 matrices: $(A_{1,j,k}), (A_{2,j,k})$, which are called the *horizontal slices of A*, or a collection of 3 matrices $(A_{i,1,k}), (A_{i,2,k}), (A_{i,3,k})$ called the *lateral slices of A*, or a collection of 3 matrices $(A_{i,j,1}), (A_{i,j,2}), (A_{i,j,3})$ called the *frontal slices of A*.

Proposition 1. *(Corollary 2.2 in [8]) Let $G_1 = \mathrm{GL}(\mathbb{F}_q^3) \wr Sym(2)$ and $H_1 := \mathrm{GL}(\mathbb{F}_q^3) \times \mathrm{GL}(\mathbb{F}_q^3)$. Then, two tensors A and C in V are G-equivalent if and only* *if A_1 is G_1-equivalent to C_1, if and only if A is H-equivalent to one of $\{C, C^T\}$, where T is the map on V defined by sending $c_1 \otimes c_2 \otimes c_3$ to $c_1 \otimes c_3 \otimes c_2$ and expanding linearly.*

Theorem 1. *(Theorem 3.10 in [8]) There are 21 H-orbits and 18 G-orbits of tensors in V.*

Note that since we are working projectively, the trivial orbit containing the zero tensor will be ignored.

For the convenience of the reader, we have collected some information from [8] about each G-orbit in V and their contraction spaces including representatives of orbits, the *tensor rank* of each orbit and rank distributions on the webpage [2], to which we will refer as *Table 1*.

2 The algorithms

In this section, we present a GAP function that takes an arbitrary tensor in V and returns its orbit number (see *Table 1*) and a representative of its orbit. The construction of this function is mainly based on the rank distributions of the projective contraction spaces associated with tensors in V, and the fact that tensors of the same orbit have the same rank distributions (see Proposition 1). We follow for this purpose the classification of G-orbits of tensors in V [7] as summarized in [2].

We start with a series of auxiliary functions that will be needed to construct our main function. The calling of most of these functions in GAP requires the usage of the GAP-package *FinInG* [4,3].

2.1 Auxiliary functions

1. *MatrixOfPoint*: turns a point of a projective space into an $(m \times n)-$matrix containing the coordinates.
2. *RankOfPoint*: returns the rank of *MatrixOfPoint(x,m,n)*.
3. *RankDistribution*: returns the rank distribution of a subspace by considering its points as $m \times n$ matrices using the *RankOfPoint* function.
4. *CubicalArrayFromPointInTensorProductSpace*: returns the horizontal slices of a tensor in $\mathrm{PG}(V)$ where in our case we have $n_1 = 2$, $n_2 = 3$ and $n_3 = 3$. Notice that this function depends on how we choose the coordinates.
5. *ContractionOfPointInTensorProductSpace*: returns the projective contraction $vec^{\vee}(point)$; recall that in our case a point represents a tensor in $\mathrm{PG}(V)$.
6. *SubspaceOfContractions*: returns the projective contraction spaces associated with a projective point in $\mathrm{PG}(V)$.
7. *Rank1PtsOftheContractionSubspace*: returns the set of rank 1 points of $\mathrm{PG}(A_i)$ using the *RankOfPoint* function.
8. *RepO10odd*: returns a represantative of o_{10} if q is odd.
9. *AlternativeRepresentationOfFiniteFieldElements*: gives an alternative way of representing finite fields' elements.

10. *RepO10even*: returns a represantative of o_{10} if q is even.
11. *RepO15odd*: returns a represantative of o_{15} if q is odd.
12. *RepO15even*: returns a represantative of o_{15} if q is even.

2.2 OrbitOfTensor

The *OrbitOfTensor* function takes an arbitrary tensor A in $\mathrm{PG}(V)$ and by using the above auxiliary functions, it calculates the rank distribution of the first contraction space of A, R_1, and compares it with the results in *Table 1* to specify the orbit number containing A. In some cases, R_1 is not enough to distinguish between orbits. For example, orbits o_{10}, o_{11} and o_{12} (resp. o_6 and o_7) have the same R_1. In this case, we calculate R_2 and R_3 to differentiate among them. But since the orbits o_4, o_7 and o_{11} are the only G-orbits of tensors which split under the action of H to o_i and o_i^T [8], we can see that a direct comparison between R_2 and R_3 from *Table 1* will not be enough to distinguish between o_{10}, o_{11} and o_{12} (resp. o_6 and o_7). For this reason, we consider (algorithmically) some extra possible cases of R_2 and R_3 to insure that if $A \in o_j$ then $A^T \in o_j$, where $j \in \{7, 11\}$ [2]. Notice that, we do not have to do a similar work for o_4 since it is completely determined by R_1.

Although rank distributions are sufficient to specify the tensor's orbit in most cases, they are not helpful in distinguishing o_{15} and o_{16} as they have the same rank distributions. For this purpose, we use *Lemma 1* to distinguish between them.

Lemma 1. *Consider the two G-orbits of tensors in V, o_{15} and o_{16}. In both cases $\mathrm{PG}(A_1)$ is a line with rank distribution $[0, 1, q]$. Let x_2 be the unique rank 2 point on $\mathrm{PG}(A_1)$ and x_1 be a point among the q points of rank 3 on $\mathrm{PG}(A_1)$. Then, there exists a unique solid V containing x_2 which intersects $S_{3,3}(\mathbb{F}_q)$ in a subvariety $Q(x_2)$ equivalent to a Segre variety $S_{2,2}(\mathbb{F}_q)$. Furthermore, there is no rank one point in $U \setminus Q(x_2)$ for o_{16} where $U := \langle V, x_1\rangle$, and there is one for o_{15}.*

Proof. The first result is a direct application of [8, Lemma 2.4]. The second one uses the two possible cases of having 2 points y_i, $i = 1, 2$ of rank i such that x_1 is on the line $\langle y_1, y_2\rangle$ and $Q(x_2) = Q(y_2)$ or no such points exist, which were used in [8, Theorem 3.1 case(4)] to define o_{15} and o_{16}, respectively. □

For the same reason, we consider the case $q = 2$ separately. In this case, as R_1 is the same for the orbits o_{10}, o_{12} and o_{14}, we distinguish between o_{10} and o_{14} using R_2. However, as o_{12} and o_{14} have the same rank distributions, we differentiate between them using the geometric description of the second contraction space. In particular, the difference between o_{12} and o_{14} is that for o_{14} the 3 points of rank one in the second contraction space (which is a plane) span the space, while for o_{12} they do not (see *Table 1*).

In most of the cases, except for o_{10}, o_{15} and o_{17}, the orbit representative is directly deduced from *Table 1* and it is defined by its two horizontal slices. For example, a representative of o_{11} is $e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2) + e_2 \otimes (e_1 \otimes e_2 + e_2 \otimes e_3)$ (see *Table 1*), and this can be represented by its horizontal slices as

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}.$$

Representative for o_{17}. We know that the orbit o_{17} has representatives of the form $e_1 \otimes (e) + e_2 \otimes (e_1 \otimes e_2 + e_2 \otimes e_3 + e_3 \otimes (\alpha e_1 + \beta e_2 + \gamma e_3))$ where $\lambda^3 + \gamma\lambda^2 - \beta\lambda + \alpha \neq 0$ for all $\lambda \in \mathbb{F}_q$ and where $e = e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3$ (see *Table 1*). Instead of computing these parameters for every q (which would become computationally infeasible for very large q), we will give an explicit construction which does not require any computation at all. First, observe that o_{17} is the only orbit of lines in $\mathrm{PG}(\mathbb{F}_q^3 \otimes \mathbb{F}_q^3) \cong \mathrm{PG}(8, q)$ consisting entirely of points of rank 3 (see [7]). Thus, to obtain a representative for the orbit o_{17} it suffices to construct such a line of *constant rank* 3. In order to do so, consider the cubic extension $\mathbb{F}_{q^3}$ of $\mathbb{F}_q$ as an $\mathbb{F}_q$-vector space W and the set $U = \{M_\alpha : \alpha \in \mathbb{F}_{q^3}\}$ where M_α is the matrix representative of the linear operator on W defined by: $x \to \alpha x$. Clearly, U is a 3-dimensional $\mathbb{F}_q$-vector space consisting of the zero matrix and $q^3 - 1$ matrices of rank 3. Any 2-dimensional $\mathbb{F}_q$-subspace of U will give us a representative of o_{17}. Furthermore, a basis of this subspace gives us the two horizontal slices of the representative. In particular, we consider the 2-dimensional $\mathbb{F}_q$-subspace generated by the identity matrix and the companion matrix of the minimal polynomial of a primitive element w of the cubic extension.

Representatives for o_{10} and o_{15}. By *Table 1*, we can see that $e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2 + ue_1 \otimes e_2) + e_2 \otimes (e_1 \otimes e_2 + ve_2 \otimes e_1)$ and $e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 + ue_1 \otimes e_2) + e_2 \otimes (e_1 \otimes e_2 + ve_2 \otimes e_1)$ are representatives of o_{10} and o_{15} respectively, where $v\lambda^2 + uv\lambda - 1 \neq 0$ for all $\lambda \in \mathbb{F}_q$ and $u, v \in \mathbb{F}_q^*$. Similar to the previous case, we give an explicit construction of o_{10}, which does not require any computations. It follows from [9] that o_{10} has a representative line of constant 2-rank 2×2-matrices, which is an external line to a conic in $\mathcal{V}_3(\mathbb{F}_q)$, where $\mathcal{V}_3(\mathbb{F}_q)$ is the image of the map $\nu_3 : \mathrm{PG}(2, q) \to \mathrm{PG}(5, q)$ induced by the mapping sending $v \in \mathbb{F}_q^3$ to $v \otimes v$. Thus, by constructing such a line and taking any 2 points on it, we obtain the required representative. First, recall that interior points of the conic $(C) : X_0X_2 - X_1^2 = 0$ in $\mathrm{PG}(2, q)$ are (x, y, z) where $xz - y^2$ are non-squares. Hence, if q is odd, one can start with a primitive root in $\mathbb{F}_q$ (which is a non-square in $\mathbb{F}_q$). Then, by considering its image under the polarity α associated to (C), we obtain an external line to (C) in $\mathrm{PG}(2, q)$. This line can be seen in $\mathrm{PG}(8, q)$ by embedding $\mathrm{PG}(2, q)$ as the set of points with last column and last row equal to zero. If q is even, a similar argument works. In this case, we can start with the minimal polynomial of a generator of the multiplicative group of $\mathbb{F}_{q^2}$ to obtain an irreducible quadratic polynomial over $\mathbb{F}_q$. The coefficients can then be used as the dual coordinates of a line in $\mathrm{PG}(2, q)$ disjoint from the conic consisting of the points (a^2, ab, b^2) with $(a, b) \in \mathrm{PG}(1, q)$. Once we have that line, we can map it to a line in $\mathrm{PG}(8, q)$ by embedding

$\mathrm{PG}(2,q)$ as the set of points with last column and last row equal to zero. Now, by using a representative of o_{10}, we can find the above u and v, which gives us directly a representative of o_{15}.

2.3 RankOfTensor

The *RankOfTensor* function takes an arbitrary tensor A in $\mathrm{PG}(V)$ and uses the *OrbitOfTensor* function to specify the G-orbit of the tensor and returns the tensor's rank. The code of all of these functions can be found in [1].

3 Computations and Summary

Example 1.
```
gap> q:=397; sv:=SegreVariety([PG(1,q),PG(2,q),PG(2,q)]);
397
Segre Variety in ProjectiveSpace(17, 397)
gap> m:=Size(Points(sv));
9936552395502
gap> pg:=AmbientSpace(sv);
ProjectiveSpace(17, 397)
gap> n:=Size(Points(pg));
151542321438098147995655901146938756967526078
gap> A:=VectorSpaceToElement(pg,[Z(397)^0,Z(397)^336,Z(397)^339,
Z(397)^37,Z(397)^233,Z(397)^56,Z(397)^268,Z(397)^363,Z(397)^342,
Z(397)^297,Z(397)^146,Z(397)^71,Z(397)^57,Z(397)^84,Z(397)^33,
Z(397)^203,Z(397)^229,Z(397)^191]);
gap> OrbitOfTensor(A)[1]; time;
14
94
gap> RankOfTensor(A);
3
gap> time;
141
gap> NrCombinations([1..m], 3);
163514371865202881474954561407873423500
```

Summary. The *RankOfTensor* is an efficient tool to compute tensor ranks of points in $\mathrm{PG}(V)$. Without this algorithm, it is computationally infeasible to do this. For example, consider q, sv, pg and A from *Example 1.* The space pg has n points. Among these we have m points of rank 1, which gives a 38-order of magnitude number of possible 3-combinations of points of rank 1, which might generate a plane containing A. This reflects how hard it would be to compute the rank without this algorithm.

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References

1. Alnajjarine, N., Lavrauw, M.: Determining the rank of tensors in $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$, http://people.sabanciuniv.edu/mlavrauw/T233/T233_paper.html.
2. Alnajjarine, N., Lavrauw, M. Projective description and properties of the G-orbits of $(2 \times 3 \times 3)$-tensors. http://people.sabanciuniv.edu/mlavrauw/T233/table1.html.
3. FinInG: Finite Incidence Geometry: FinInG – a GAP package. http://www.fining.org Bamberg, J., Betten, A., Cara, Ph., Beule, J. De., Lavrauw, M., Neunhöffer, M. Accessed 31 Mar 2018.
4. The GAP Group: GAP Groups, Algorithms, and Programming, https://www.gap-system.org. Last accessed 19 June 2019
5. Kolda, T. G., Bader, B. W.: Tensor Decompositions and Applications. SIAM REVIEW **51**(3), 455-500 (2009)
6. Landsberg, J. M.: Tensors: Geometry and Applications: Geometry and Applications. 2nd edn. American Mathematical Society, USA (2011)
7. Lavrauw, M., Sheekey, J.: Classification of subspaces in $\mathbb{F}^2 \otimes \mathbb{F}^3$ and orbits in $\mathbb{F}^2 \otimes \mathbb{F}^3 \otimes \mathbb{F}^r$. Journal of Geometry **108**(1), 5-23 (2017)
8. Lavrauw, M., Sheekey, J.: Canonical forms of $2\times$ $3\times$ 3 tensors over the real field, algebraically closed fields, and finite fields. Linear Algebra and its Applications **476**, 133-47 (2015)
9. Lavrauw, M., Tomasz, P.: The symmetric representation of lines in $\mathrm{PG}(\mathbb{F}_q^3 \otimes \mathbb{F}_q^3)$. To appear in Discrete Mathematics.