Characterisations and properties of good eggs in $\mathrm{PG}(4n-1,q)$, q odd

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Abstract

Using the explicit description given in [9] we prove that a good egg of $\mathrm{PG}(4n-1,q)$, q odd, is isomorphic to an ovoid of $\mathrm{PG}(3,q^n)$ seen over $\mathrm{GF}(q)$ if and only if there exists a $(3n-1)$-space skew from the good element and containing at least 5 egg elements. There are only 3 classes of known examples of eggs in $\mathrm{PG}(4n-1,q)$, which are not elementary. For each of these classes we obtain strong restraints on the intersection of egg elements with $(3n-1)$-spaces spanned by three different egg elements.

1. Introduction and the model

An *egg* $\mathcal{E}(n,m,q)$ in $\mathrm{PG}(2n+m-1,q)$ is a partial $(n-1)$-spread of size q^m+1 such that every 3 egg elements span a $(3n-1)$-space and for every egg element E there exists an $(n+m-1)$-space, denoted by T_E and called the *tangent space of* $\mathcal{E}$ *at* E, containing E but disjoint from every other egg element. The idea of eggs was introduced by Thas in 1971 [13] and it turned out later, see Payne and Thas [11], that the theory of eggs is equivalent to the theory of translation generalized quadrangles (TGQ's). If $n=m$ then $\mathcal{E}(n,m,q)$ is called a *pseudo-oval* and if $2n=m$ then $\mathcal{E}(n,m,q)$ is called a *pseudo-ovoid*. If we take an oval in $\mathrm{PG}(2,q^n)$, respectively an ovoid in $\mathrm{PG}(3,q^n)$, and consider the ambient space over $\mathrm{GF}(q)$, then we obtain

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a pseudo-oval, respectively a pseudo-ovoid, and hence the motivation to use that name. The examples constructed in this way are called *elementary.* In this article we concentrate on the case where q is odd, $2n = m$, and the egg in $\mathrm{PG}(4n-1, q)$ is *good* at some element, which means that every $(3n-1)$-space containing that element and two other egg elements contains exactly $q^n + 1$ egg elements. The following lemma follows from Thas [16].

Lemma 1.1 *If q is odd and $\mathcal{E}$ is a good egg in* $\mathrm{PG}(4n-1, q)$ *then every pseudo-oval contained in $\mathcal{E}$ and containing the good element is elementary.*

Good eggs of $\mathrm{PG}(4n-1, q)$, q odd, can be constructed from semifield flocks of a quadratic cone in $\mathrm{PG}(3, q^n)$ and we will recall the geometric link between these two objects from [7] in section 2. For a recent survey of eggs we refer to Chapter 3 of [7] where we included results of Bader, Ball, Bloemen, Blokhuis, Casse, Kantor, Lunardon, Payne, Penttila, Pinneri, Thas, Van Maldeghem, and Wild. For q even we refer to [2] and [3].

Let $F = \mathrm{GF}(q^n)$, q odd and let $\mathcal{E}$ be a good egg of $\mathrm{PG}(4n-1, q)$. In [9] it was shown that there exist $a_i, b_i, c_i \in F$, for $i \in \{0, \ldots, n-1\}$, such that the elements of $\mathcal{E}$ can be written as

$$E(\gamma) = \{\langle -g_t(\gamma), t, -\gamma t\rangle \| \; t \in F^*\}, \; \forall \gamma \in F^2,$$

$$E(\infty) = \{\langle t, 0, (0,0)\rangle \| \; t \in F^*\},$$

and the tangent spaces can be written as

$$T_E(\gamma) = \{\langle h(\gamma, \delta) + g_t(\gamma), t, \delta\rangle \| \; (t, \delta) \in F \times F^2 \setminus \{(0,0)\}\}, \; \forall \gamma \in F^2,$$

$$T_E(\infty) = \{\langle t, 0, \delta\rangle \| \; (t, \delta) \in F \times F^2 \setminus \{(0,0)\}\},$$

with

$$g_t(a, b) = \sum_{i=0}^{n-1} (a_i a^2 + b_i ab + c_i b^2)^{1/q^i} t^{1/q^i},$$

and

$$h((a, b), (c, d)) = \sum_{i=0}^{n-1} (2a_i ac + b_i(ad + bc) + 2c_i bd)^{1/q^i},$$

and with this notation the egg is good at the element $E(\infty)$. Sometimes we will denote this egg by $\mathcal{E}(\mathbf{a}, \mathbf{b}, \mathbf{c})$, where $\mathbf{a} = (a_0, \ldots, a_{n-1})$, $\mathbf{b} = (b_0, \ldots, b_{n-1})$, $\mathbf{c} = (c_0, \ldots, c_{n-1})$.

Theorem 1.2 *The set $\mathcal{E}$ defined as above forms a good egg of* $\mathrm{PG}(4n-1, q)$ *if and only if $g_t(a, b) = 0$ implies $t = 0$ or $a = b = 0$.*

Proof : If there exist $a, b, t \in F$ with $t \neq 0$ and $(a, b) \neq (0, 0)$ such that $g_t(a, b) = 0$, then $\langle 0, t, -at, -bt\rangle \in T_E(\infty) \cap E(a, b)$, a contradiction. Conversely suppose that the condition $g_t(a, b) = 0$ implies $t = 0$ or $a = b = 0$ is satisfied. It follows from Chapter 3 in [7] (p. 62-66) that the conditions for $\mathcal{E}$ to be an egg of $\mathrm{PG}(4n-1, q)$

are satisfied. Theorem 3.2 in [9] implies that $\mathcal{E}(\infty)$ is a good element of $\mathcal{E}$. This concludes the proof. □

All the known examples of eggs have a good element or the dual egg has a good element.

We now list $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ representing the four classes of known examples of eggs in $\mathrm{PG}(4n-1, q)$.

Elementary pseudo-ovoids
Any ovoid in $\mathrm{PG}(3, q^n)$ gives rise to an egg in $\mathrm{PG}(4n-1, q)$. If the ovoid is an elliptic quadric, which is always the case when q is odd, then we may assume the egg is of the form $\mathcal{E}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ with

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = ((1, 0, \ldots, 0), \mathbf{0}, (-m, 0, \ldots, 0)),$$

where m is a non-square in $\mathrm{GF}(q^n)$.

Pseudo-ovoids of Kantor type

If q is odd, m a non-square in $\mathrm{GF}(q^n)$, and

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = ((1, 0, \ldots, 0), \mathbf{0}, (0, -m, 0, \ldots, 0)),$$

then $\mathcal{E}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is an egg in $\mathrm{PG}(4n-1, q)$. This class of examples is connected with the Kantor-Knuth semifield flock and the corresponding GQ was first discovered by Kantor [6] in 1986.

Pseudo-ovoids of Cohen-Ganley type

If $q = 3$, m a non-square in $\mathrm{GF}(q^n)$, and

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = ((1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), (-m^{-1}, 0, -m, 0, \ldots, 0))$$

then $\mathcal{E}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is an egg in $\mathrm{PG}(4n-1, q)$. This class of examples is connected with the semifields discovered by Cohen and Ganley [5].

Pseudo-ovoids of Penttila-Williams type

If $q = 3$, $n = 5$, and

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = ((1, 0, 0, 0, 0), (0, 0, 0, -1, 0), (0, 1, 0, 0, 0))$$

then $\mathcal{E}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is an egg in $\mathrm{PG}(4n-1, q)$. This sometimes called *sporadic* (see [1]) class of examples is connected with the ovoid of $Q(4, q)$ discovered by Penttila and Williams [12], where $Q(4, q)$ denotes the generalized quadrangle arising from a parabolic quadric in 4-dimensional projective space.

Remark 1.3 *In all the examples we see that* $\mathbf{a} = (1, 0, \ldots, 0)$. *This is nothing else than a choice we made in the representation, some kind of normalisation. That we* *are allowed to do that can be seen in many ways. If we consider the subspace spanned by the good element and two other egg elements, let's say* $\langle E(\infty), E(0,0), E(1,0)\rangle$, *there arises an elementary pseudo-oval* $\mathcal{O}$ *(see Lemma 1.1). Since* q *is odd,* $\mathcal{O}$ *arises from a conic* $\mathcal{C}$ *in* $\mathrm{PG}(2,q^n)$. *Applying a collineation of* $\mathrm{PG}(4n-1,q)$ *mapping* $\mathcal{O}$ *to the pseudo-oval arising from the conic with equation* $X_0X_1 + X_2^2 = 0$, *we obtain that* $\mathbf{a} = (1,0,\ldots,0)$.

In Payne [10] the author calculates automorphisms of the translation generalized quadrangles corresponding with the pseudo-ovoids of Cohen-Ganley type. These collineations induce an automorphism of the pseudo-ovoid of Cohen-Ganley type (see Lemma 1 in [1]). Using the model from [9] one can generalise some of these automorphisms to any good egg in $\mathrm{PG}(4n-1,q)$, q odd, and prove the following lemma.

Lemma 1.4 *Let* $\mathcal{E}$ *be a good egg in* $\mathrm{PG}(4n-1,q)$, q *odd. The stabiliser of the good element in the automorphism group of the egg acts transitively on the other egg elements.*

Proof : Consider the map

$$\begin{aligned}\psi_{a,b} : \mathrm{PG}(4n-1,q) &\to \mathrm{PG}(4n-1,q)\\ \langle r,s,t,u\rangle &\mapsto \langle r + h((t,u),(a,b)) - g_s(a,b), s, t-sa, t-sb\rangle.\end{aligned}$$

It is straightforward to check that $\psi_{a,b}$ induces a collineation of $\mathrm{PG}(4n-1,q)$ fixing $E(\infty)$ point-wise and mapping $E(c,d)$ to $E(c+a,d+b)$. □

2. The geometric connection between semifield flocks and good eggs

In this section we give a brief description of the geometric connection between a good egg of $\mathrm{PG}(4n-1,q)$ and a semifield flock of a quadratic cone in $\mathrm{PG}(3,q^n)$, q odd, from [8].

Starting from a semifield flock we obtain every pseudo-oval on the good element contained in the egg and every intersection of the space containing such a pseudo-oval and the tangent space at an egg element not contained in that pseudo-oval in the following way. First we dualise the flock with respect to a duality in $\mathrm{PG}(3,q^n)$. The set of points corresponding with the planes of the flock induce an $(n-1)$-dimensional subspace U over $\mathrm{GF}(q)$ (since the flock is a semifield flock) contained in the set of internal points of the conic (corresponding with the generators of the quadratic cone) in the plane corresponding with the vertex. Dualising the $(3n-1)$-space corresponding to the plane containing the conic, U becomes a $(2n-1)$-space W skew from the elements of an elementary pseudo-oval $\mathcal{O}$. Applying the right collineations in each of these steps (see [8] for details), W is the subspace $T_E(c,d) \cap \pi_{a,b}$, and $\mathcal{O}$ is the elementary pseudo-oval determined by $E(\infty)$, $E(0,0)$, and $E(a,b)$.

Starting from any pseudo-oval contained in a good egg in $\mathrm{PG}(4n-1,q)$ and containing the good element we can reverse each of the steps and obtain a semifield flock of a quadratic cone in $\mathrm{PG}(3,q^n)$. Every two flocks obtained in this way are equivalent (see [8] for details).

3. Two characterisations of elementary eggs

Theorem 3.1 ([7])
Let $\mathcal{E}$ be an egg of $\mathrm{PG(4n-1,q)}$*, q odd, which is good at an element E. Then the following properties are equivalent.*

- *$\mathcal{E}$ is an elementary pseudo-ovoid.*
- *There exists a triple (F, E_1, E_2), where F is not contained in the pseudo-conic $\mathcal{C}$ induced by the elements E_1 and E_2 in the $(3n-1)$-space $\langle E, E_1, E_2\rangle$, such that the tangent space at F contains two elements of the Desarguesian spread induced by $\mathcal{C}$.*
- *All triples (F, E_1, E_2), where F is not contained in the pseudo-conic $\mathcal{C}$ induced by the elements E_1 and E_2 in the $(3n-1)$-space $\langle E, E_1, E_2\rangle$, are such that the tangent space at F contains two elements of the Desarguesian spread induced by $\mathcal{C}$.*

Proof : The proof follows immediately from the geometric connection with semifield flocks explained in the previous section and the fact that all the planes of a linear flock contain a common line. □

Theorem 3.2 *If $\mathcal{E}$ is a non-elementary good egg in* $\mathrm{PG}(4n-1,q)$*, q odd, and π is a $(3n-1)$-space spanned by three egg elements, then π contains the good element and hence contains exactly q^n+1 elements or π contains at most four egg elements.*

Proof : Since $\mathcal{E}$ is a good egg in $\mathrm{PG}(4n-1,q)$, q odd, we may assume its elements can be represented as above. Let π be a $(3n-1)$-space spanned by three egg elements. Since $\mathcal{E}$ is good at $E(\infty)$, we know that if π contains $E(\infty)$, then π contains exactly q^n+1 egg elements. Assume that π does not contain $E(\infty)$. Since the stabiliser of $E(\infty)$ in the automorphism group of the egg is transitive on the other egg elements (Lemma 1.4), we may assume that $\pi = \langle E(0,0), E(a,b), E(c,d)\rangle$, where $ad-bc \neq 0$, since $E(\infty)$ is not contained in π. An egg element $E(e,f)$ is contained in π if and only if $\forall u \in F, \exists r,s,t \in F$:

$$(-g_u(e,f), u, -eu, -fu) = (0,r,0,0)+(-g_s(a,b), s, -as, -bs)+(-g_t(c,d), t, -ct, -dt).$$

Equating coordinates gives us the equations $\forall u \in F, \exists r,s,t \in F$:

$$\begin{aligned} -g_u(e,f) &= -g_s(a,b) - g_t(c,d) \\ u &= r+s+t \\ -eu &= -as-ct \\ -fu &= -bs-dt \end{aligned}$$

Solving for r, s and t gives

$$r = u-s-t,\ \ s = u\left(\frac{de-cf}{ad-bc}\right),\ \ t = u\left(\frac{af-be}{ad-bc}\right).$$

Substituting these values in the first equation we may obtain that the egg element $E(e,f)$ is contained in π if and only if $\forall u \in F$

$$-g_u(e,f) = -g_{u\left(\frac{de-cf}{ad-bc}\right)}(a,b) - g_{u\left(\frac{af-be}{ad-bc}\right)}(c,d).$$

Using the formula for g we get that $\forall u \in F$

$$\sum_{i=0}^{n-1}[(a_i e^2 + b_i ef + c_i f^2)u]^{1/q^i} =$$

$$\sum_{i=0}^{n-1}\left[(a_i a^2 + b_i ab + c_i b^2)u\left(\frac{de-cf}{ad-bc}\right) + (a_i c^2 + b_i cd + c_i d^2)u\left(\frac{af-be}{ad-bc}\right)\right]^{1/q^i},$$

if and only if

$$\sum_{i=0}^{n-1}[(a_i e^2 + b_i ef + c_i f^2)u]^{q^{n-i}}$$

$$= \sum_{i=0}^{n-1}\left[(a_i a^2 + b_i ab + c_i b^2)u\left(\frac{de-cf}{ad-bc}\right) + (a_i c^2 + b_i cd + c_i d^2)u\left(\frac{af-be}{ad-bc}\right)\right]^{q^{n-i}},$$

with $(a_0, a_1, \ldots, a_{n-1}) = (1, 0, \ldots, 0)$. Seen as an equality of two polynomials in u of degree at most q^{n-1} it follows that we must have a polynomial identity. Hence $E(e,f)$ is contained in π if and only if

$$(ad-bc)(a_i e^2 + b_i ef + c_i f^2) = (de-cf)(a_i a^2 + b_i ab + c_i b^2) + (af-be)(a_i c^2 + b_i cd + c_i d^2)$$

for all $i \in \{0, \ldots, n-1\}$, if and only if the point $\langle e, f, 1\rangle$ is a point of the intersection of the conics $\mathcal{C}_i$, $i = 0, \ldots, n-1$, where $\mathcal{C}_i$ is the conic in $\mathrm{PG}(2, q^n)$ with equation

$$(ad-bc)(a_i X^2 + b_i XY + c_i Y^2) =$$

$$(dXZ - cYZ)(a_i a^2 + b_i ab + c_i b^2) + (aYZ - bXZ)(a_i c^2 + b_i cd + c_i d^2),$$

$i \in \{0, \ldots, n-1\}$. Let $\mathcal{I}$ denote the intersection of these conics. Note that $\mathcal{I}$ contains no point on the line $Z = 0$ since that would imply that there exists a pair $(e,f) \neq (0,0)$, with $g_1(e,f) = 0$, in contradiction with Theorem 1.2. Since we may take $(a_0, a_1, \ldots, a_{n-1}) = (1, 0, \ldots, 0)$ (see Remark 1.3), the conic $\mathcal{C}_0$ is different from all other conics and if this is the only non-trivial equation then the egg is elementary. Hence we have at least two different conics with a non-trivial equation. If $\mathcal{C}_0$ is non-degenerate then $\mathcal{I}$ contains at most 4 points. If $\mathcal{C}_0$ is degenerate but has no components in common with all the other conics, then again $\mathcal{I}$ contains at most 4 points. On the other hand if $\mathcal{C}_0$ is degenerate and one of its components is common to all other conics then $\mathcal{I}$ contains a line, and hence contains a point on the line $Z = 0$, a contradiction. This concludes the proof. □

Remark 3.3 *From the above proof it follows that if $E(e,f)$ intersects $\pi = \langle E(0,0), E(a,b), E(c,d)\rangle$, with $ad - bc \neq 0$ in a subspace of dimension $k-1$, then there are exactly q^k values $u \in F$ for which the condition*

$$\sum_{i=0}^{n-1}[(a_i e^2 + b_i ef + c_i f^2)u]^{1/q^i}$$

$$= \sum_{i=0}^{n-1} \left[(a_i a^2 + b_i ab + c_i b^2) u \left(\frac{de - cf}{ad - bc} \right) + (a_i c^2 + b_i cd + c_i d^2) u \left(\frac{af - be}{ad - bc} \right) \right]^{1/q^i} \quad (1)$$

is satisfied.

4. Pseudo-ovoids of Kantor type

Theorem 4.1 ([7])
Let $\mathcal{E}$ be a non-elementary egg of $\mathrm{PG}(4\mathrm{n}-1, \mathrm{q})$*, q odd, which is good at an element E. Then the following properties are equivalent.*

- *$\mathcal{E}$ is a Kantor pseudo-ovoid.*
- *There exists a triple (F, E_1, E_2), where F is not contained in the pseudo-conic $\mathcal{C}$ induced by the elements E_1 and E_2 in the $(3n-1)$-space $\langle E, E_1, E_2 \rangle$, such that the tangent space at F contains an element of the Desarguesian spread induced by $\mathcal{C}$.*
- *All triples (F, E_1, E_2), where F is not contained in the pseudo-conic $\mathcal{C}$ induced by the elements E_1 and E_2 in the $(3n-1)$-space $\langle E, E_1, E_2 \rangle$, are such that the tangent space at F contains an element of the Desarguesian spread induced by $\mathcal{C}$.*

Proof : The proof follows from the fact that the planes of the corresponding semifield flock all contain a common point (see Thas [15]) and the geometry between semifield flocks and good eggs as explained in Section 2. □

Theorem 4.2 *Let $\mathcal{E}$ be a non-elementary good egg in $\mathrm{PG}(4n-1, q)$, q odd. Then $\mathcal{E}$ is of Kantor type if and only if there exists a subspace π spanned by three egg elements containing exactly four egg elements.*

Proof : Let $\mathcal{E}$ be a pseudo ovoid of Kantor type and π the $(3n-1)$-space generated by $E(0,0)$, $E(a,b)$ and $E(c,d)$, with $ad - bc \neq 0$. From the proof of Theorem 3.2 it follows that $E(e,f)$ will be contained in π if and only if $\langle e, f, 1 \rangle$ is a point of the intersection of the conics $\mathcal{C}_i$, $i = 0, \ldots, n-1$, where $\mathcal{C}_i$ is the conic with equation

$$(ad - bc)(a_i X^2 + b_i XY + c_i Y^2) =$$

$$(dXZ - cYZ)(a_i a^2 + b_i ab + c_i b^2) + (aYZ - bXZ)(a_i c^2 + b_i cd + c_i d^2).$$

Since $\mathcal{E}$ is of Kantor type we may assume $(a_0, a_1, \ldots, a_{n-1}) = (1, 0, \ldots, 0)$, $(b_0, b_1, \ldots, b_{n-1}) = (0, \ldots, 0)$, and $(c_0, \ldots, c_{n-1}) = (0, -m, 0, \ldots, 0)$, where m is a non-square in $\mathrm{GF}(q^n)$. For $i = 0$ we get the conic $\mathcal{C}_0 : (ad - bc)X^2 = (dXZ - cYZ)a^2 + (aYZ - bXZ)c^2$ and for $i = 1$ the conic $\mathcal{C}_1 : (ad - bc)Y^2 = (dXZ - cYZ)b^2 + (aYZ - bXZ)d^2$. If $(c,d) \notin \{(0, 2b), (2a, 0)\}$ then these conics also contain the point

$\langle(a-c)(ad+bc)/(ad-bc), -(b-d)(ad+bc)/(ad-bc), 1\rangle$, and hence π contains the corresponding fourth element.

Conversely suppose that $\mathcal{E}$ is a non-elementary good egg of $\mathrm{PG}(4n-1,q)$, q odd. Let π be a $(3n-1)$-space generated by $E(0,0)$, $E(a,b)$ and $E(c,d)$, with $ad-bc \neq 0$, and suppose that π contains a fourth egg element $E(e,f)$. From the proof of Theorem 3.2 it follows that $\langle e,f,1\rangle$ is a point of the intersection $\mathcal{I}$ of the conics $\mathcal{C}_i$, $i=0,\dots,n-1$, where $\mathcal{C}_i$ is the conic with equation

$$(ad-bc)(a_iX^2+b_iXY+c_iY^2)=$$

$$(dXZ-cYZ)(a_ia^2+b_iab+c_ib^2)+(aYZ-bXZ)(a_ic^2+b_icd+c_id^2).$$

Since we may assume that $(a_1,\dots,a_{n-1})=(1,0,\dots,0)$ (see Remark 1.3) the coefficient of the X^2 term in the equation of the conic $\mathcal{C}_0$ is $ad-bc \neq 0$, while the coefficient of the X^2 term in the equation of the conic $\mathcal{C}_i$, with $i \geq 1$, is zero. Let A be the set of integers $i \neq 0$ for which the conic $\mathcal{C}_i$ has non-trivial equation. Since $\mathcal{E}$ is non-elementary A is non-empty. Moreover, since $\mathcal{I}$ contains the four points $\langle 0,0,1\rangle$, $\langle a,b,1\rangle$, $\langle c,d,1\rangle$ and $\langle e,f,1\rangle$, and every conic $\mathcal{C}_i$, $i \in A$, contains the point $\langle 1,0,0\rangle$, all the conics $\mathcal{C}_i$, $i \in A$, coincide. From the coefficients of the XY term and the Y^2 term it then follows that for every $i,j \in A$ there exists a $\gamma_{ij} \in F$ such that $(b_i,c_i)=\gamma_{ij}(b_j,c_j)$. Consider the elementary pseudo-oval induced by the elements $E(\infty)$, $E(0,0)$, and $E(1,0)$ in the $(3n-1)$-space $\pi_{1,0}$. The elements contained in $\pi \setminus T_E(\infty)$ of the normal spread induced by the elements of the elementary pseudo-oval in $\pi_{1,0}$ are of the form

$$\{\langle st,t,rt,0\rangle \ : \ t \in F^*\},$$

with $r,s \in F$. We will show that the $(2n-1)$-space $T_E(0,1) \cap \pi_{1,0}$

$$=\{\langle\sum_{i=0}^{n-1}[b_is+c_it]^{1/q^i},t,s,0\rangle \ : \ (r,s) \in F^2 \setminus \{(0,0)\}\}$$

contains an element of this normal spread. If there exist an $i \in A$ for which $b_i \neq 0 \neq c_i$ then the above implies that there exists a $\gamma \in F^*$ such that $(b_1,\dots,b_{n-1})=\gamma(c_1,\dots,c_{n-1})$ and then $T_E(0,1) \cap \pi_{1,0}$ contains the normal spread element

$$\{\langle(b_0(-\gamma^{-1})+c_0)t,t,-\gamma^{-1}t,0\rangle \ : \ t \in F^*\}.$$

If $b_i=0$ for all $i \in A$ then $T_E(0,1) \cap \pi_{1,0}$ contains the normal spread element

$$\{\langle b_0s,0,s\rangle \ : \ s \in F^*\}.$$

If $c_i=0$ for all $i \in A$ then $T_E(0,1) \cap \pi_{1,0}$ contains the normal spread element

$$\{\langle c_0t,t,0\rangle \ : \ t \in F^*\}.$$

Applying Theorem 4.1 it follows that the egg is of Kantor type. □

Theorem 4.3 *Let $\mathcal{E}$ be a pseudo-ovoid of Kantor type in* $\mathrm{PG}(4n-1,q)$, *q odd, $n \geq 2$, and let π be the subspace spanned by three egg elements. Then an egg element $E \in \mathcal{E}$ is either contained in π or intersects π in at most a point.*

Proof : Let π be a subspace spanned by three different egg elements. If π contains the good element then π contains exactly q^n+1 egg elements, forming an elementary pseudo-conic in π. Since the secants of a conic in a plane of odd order cover the plane, all other egg elements are skew from π. Now suppose that π does not contain the good element. As before we may assume without loss of generality that π is the $(3n-1)$-space spanned by the $E(0,0)$, $E(a,b)$ and $E(c,d)$, with $ad-bc \neq 0$. Since $\mathcal{E}$ is of Kantor type we may assume that $\mathbf{a} = (1,0,\ldots,0)$, $\mathbf{b} = \mathbf{0}$ and $\mathbf{c} = (0,-m,0,\ldots,0)$, where m is a non-square in F. The condition in Remark 3.3 becomes as follows. If there is an egg element $E(e,f)$ intersecting π in a subspace of dimension $k-1$ then there are exactly q^k values $u \in F$ for which the condition

$$e^2u - (mf^2u)^{1/q} =$$

$$a^2\left(\frac{de-cf}{ad-bc}\right)u + c^2\left(\frac{be-af}{ad-bc}\right)u - \left[mb^2\left(\frac{de-cf}{ad-bc}\right)u + md^2\left(\frac{be-af}{ad-bc}\right)u\right]^{1/q}$$

is satisfied. Raising both sides to the exponent q we obtain a polynomial of degree q in u. If there are more than q solutions then the above must be a polynomial identity or in other words if $E(e,f)$ intersects π in more than a point then $E(e,f)$ is contained in π. This concludes the proof. □

5. Pseudo-ovoids of Cohen-Ganley type

Theorem 5.1 *Let $\mathcal{E}$ be a pseudo-ovoid of Cohen-Ganley type in* $\mathrm{PG}(4n-1,3)$, *$n \geq 3$, and let π be the subspace spanned by three egg elements. Then an egg element $E \in \mathcal{E}$ is either contained in π or intersects π in at most a line.*

Proof : The proof is similar as above. In this case we have $q=3$, $\mathbf{a} = (1,0,\ldots,0)$, $\mathbf{b} = (0,1,0,\ldots,0)$ and $\mathbf{c} = (-m^{-1},0,-m,0,\ldots,0)$, and the polynomials in the condition (1) in Remark 3.3 have degree 9, after taking the 9th power. We may conclude that if $E(e,f)$ intersects π in more than a line then $E(e,f)$ is contained in π. □

6. Pseudo-ovoids of Penttila-Williams type

Theorem 6.1 *Let $\mathcal{E}$ be a pseudo-ovoid of Penttila-Williams type, and let π be the subspace spanned by three egg elements. Then an egg element $E \in \mathcal{E}$ is either contained in π or intersects π in at most a line.*

Proof : In this case we have $q=3$, $n=5$, $\mathbf{a} = (1,0,\ldots,0)$, $\mathbf{b} = (0,0,0,-1,0)$ and $\mathbf{c} = (0,1,0,0,0)$. The same argument as in the previous proof turns condition (1) into a condition of the form $\alpha u + \beta u^{1/3} + \gamma u^{1/27} = 0$ for q^k values of $u \in F$ where $k-1$ is the dimension of the intersection of $E(e,f)$ with π. Suppose $\alpha \neq 0 \neq \gamma$. If $\alpha u + \beta u^{1/3} + \gamma u^{1/27} = 0$, then $\alpha^{27}u^{27} + \beta^{27}u^9 + \gamma^{27}u = 0$, and hence $\beta\gamma^{27}u^{1/3} + \gamma^{28}u^{1/27} - \alpha^{28}u^{27} - \alpha\beta^{27}u^9 = 0$. Taking the 9th root we get $\beta^{27}\gamma^3u^9 + \gamma^{30}u - \alpha^{30}u^3 - \alpha^{27}\beta^3u = 0$, for 3^k values of $u \in F$. If $k \geq 3$ then this must be a polynomial identity, which implies that $\alpha = 0$, a contradiction. Hence $\alpha = 0$ or $\gamma = 0$. But then the equation $\alpha u + \beta u^{1/3} + \gamma u^{1/27} = 0$ can be reduced to an equation $\delta u + \epsilon u^{3^i} = 0$, with i at most 2. We may conclude that if $E(e, f)$ intersects π in more than a line then $E(e, f)$ is contained in π. □

Final remark Theorem 3.2 was independently proved by M. R. Brown and J. A. Thas [4] in a completely different way using the connection with Veronesean varieties in $\mathrm{PG}(5, q^n)$ (see Thas [17]). Theorem 4.2 was first obtained by M. R. Brown and J. A. Thas, and after private communications with J. A. Thas, we extended the proof of Theorem 3.2 to prove the result using the model from [9].

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