Segre embeddings and finite semifields

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Abstract

Each embedded product space $\mathrm{PG}(n,q)\times\mathrm{PG}(n,q)$ in an (n^2+n-1)-dimensional projective space is obtained by projecting the Segre variety $\mathcal{S}_{n,n,q}$ from an n-subspace δ skew with its first secant variety. On the other hand, when δ is skew with the $(n-1)$-th secant variety, it determines a semifield of order q^{n+1} whose center contains $\mathbb{F}_q$. A relationship arises between a particular class of embeddings of $\mathrm{PG}(n,q)\times\mathrm{PG}(n,q)$ in $\mathrm{PG}(n^2+n-1,q)$ and semifields of the above type. For this reason, such embeddings will be called semifield embeddings. In this paper we show that projectively equivalent semifield embeddings that do not exchange subspaces of different kind are related to isotopic semifields, and conversely. Exchanging the order in the product leads to the transition from a semifield to its transpose.

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1 Introduction

1.1 Preliminaries

A *semifield* is an ordered triple $(\mathbb{S}, +, \circ)$ satisfying all axioms of a field with possibly the exception of the associativity of the product "$\circ$". Here we are interested only in finite semifields. In what follows the semifield $(\mathbb{S}, +, \circ)$ will usually be denoted by $\mathbb{S}$. Before we describe our results, we introduce some of the standard terminology of the theory. For more on the topic, we refer to [8] and [12].

One easily shows that the additive group of a finite semifield is elementary abelian, and the additive order of the elements of $\mathbb{S}$ is called the *characteristic* of $\mathbb{S}$. If the order of $\mathbb{S}$ is q, then the elements of $\mathbb{S}$ are often identified with the elements of the finite field of order q, denoted by $\mathbb{F}_q$. Contained in a semifield are the following important substructures, all of which are isomorphic to a finite field. The *left nucleus* $N_l(\mathbb{S})$, the *middle nucleus* $N_m(\mathbb{S})$, and the right nucleus $N_r(\mathbb{S})$ are defined as follows: the left nucleus is the set of elements $x \in \mathbb{S}$ such that for all elements y and z in $\mathbb{S}$, we have $x \circ (y \circ z) = (x \circ y) \circ z$. In other words, the left nucleus of $\mathbb{S}$ is the set of elements of $\mathbb{S}$ that associate from the left with all other elements of the semifield $\mathbb{S}$. Analogously, the middle nucleus (resp. right nucleus) consists of all elements of $\mathbb{S}$, that associate in the middle (resp. from the right) with all elements of $\mathbb{S}$. The intersection of the *associative center* $N(\mathbb{S})$ (the intersection of the three nuclei) and the commutative center is called the *center* of $\mathbb{S}$ and denoted by $C(\mathbb{S})$. To each semifield $\mathbb{S}$ a corresponding semifield plane $\mathbb{P}(\mathbb{S})$ is related, defined by co-ordinatizing points on $\mathbb{S}$. Albert [1] proved that isomorphisms between semifield planes do not correspond to isomorphisms between semifields, but to so-called isotopisms, where an *isotopism* between two semifields $\mathbb{S}$ and $\hat{\mathbb{S}}$ is a triple of bijective linear mappings (F, G, H) from $\mathbb{S}$ to $\hat{\mathbb{S}}$ such that for all $x, y \in \mathbb{S}$ it holds that $x^F \hat{\circ} y^G = (x \circ y)^H$. It follows that the isotopism classes of semifields correspond to the isomorphism classes of projective planes and, for this reason, semifields are studied up to isotopism instead of up to isomorphism. The isotopism class of a semifield $\mathbb{S}$ is denoted by $[\mathbb{S}]$. In [2] a

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geometric construction is given for finite semifields, starting from particular configuration of two subspaces with respect to a Desarguesian spread; a BEL-configuration. It is also proved that each semifield can be constructed from such a BEL-configuration. This construction was generalized in [11], and it was also shown [10, 11] that these BEL-configurations are linked to linear sets of maximal rank, disjoint from certain secant varieties of a Segre variety. A *linear set* (or $\mathbb{F}_q$-*linear set*) is a set of points in a projective space $\mathrm{PG}(r-1, q^t)$ that corresponds to the set of elements of a Desarguesian $(t-1)$-spread of $\mathrm{PG}(rt-1, q)$, intersecting a subspace U of $\mathrm{PG}(rt-1, q)$. If the subspace U has projective dimension $d-1$, then the linear set is said to have *rank* d. For more on the topic, see [13]. In [11] the group $\mathbb{H}$ of collineations of $\mathrm{PG}(n^2+2n, q^s)$ fixing both families of maximal subspaces contained in the Segre variety $\mathcal{S}_{n,n,q}$ is taken into account. It is then shown that:

Theorem 1.1. [11] *There is a one-to-one correspondence between the isotopism classes of finite semifields of order q^k, with center containing $\mathbb{F}_q$ and left nucleus containing $\mathbb{F}_{q^s}$, $k = (n+1)s$, and the orbits of the action of $\mathbb{H}$ on the $\mathbb{F}_q$-linear sets of rank k in* $\mathrm{PG}(n^2+2n, q^s)$ *skew to the $(n-1)$-th secant variety of $\mathcal{S}_{n,n,q}$.*

Techniques relating isotopism and Segre geometry are present also in [5, 9].

Assume $\mathbb{S}$ is a semifield of order q^n and an algebra over $\mathbb{F}_q$, and let e_1, e_2, $\ldots$, e_n be a basis of $\mathbb{S}$ over $\mathbb{F}_q$. Then the *structure constants* a_{ijk} of $\mathbb{S}$, with respect to that basis, are defined by

$$e_i \circ e_j = \sum_{k=1}^{n} a_{ijk} e_k, \quad i, j = 1, 2, \ldots, n. \tag{1.1}$$

The action of the permutation group S_3 on the indices i, j and k gives rise to at most another five semifields. In particular, exchanging j and k gives rise to the *transpose* $\mathbb{S}^t$ of the semifield $\mathbb{S}$.

The notion of a linear mapping, which is roughly speaking the geometric variant of a semilinear map, was introduced in [3] and further developed in [6]. A *semilinear space* is a non-trivial geometry consisting of points and lines, such that each two distinct lines have at most one point in common. A *linear mapping* between two semilinear spaces is a partial map f such that for the intersection of any line r of the first semilinear space with the domain $D(f)$ and its image one of the following holds: (i) r is contained in $D(f)$ and $f(r)$ is a line such that the restriction of f to r is a bijection with $f(r)$, (ii) there is exactly one point of r not contained in $D(f)$ and f is constant on the remaining points of r, (iii) r has empty intersection with $D(f)$. In the case of semilinear spaces embeddable in projective spaces, linear mappings arise from semilinear maps between the underlying vector spaces, but there are other examples. In [6] the universal property of the Grassmann embedding is proved; more precisely, it is proved that every linear mapping of a Grassmann space in a projective space decomposes as the Grassmann embedding and a linear mapping between projective spaces. The product space of two projective spaces is a semilinear space embeddable in a projective space by means of the Segre embedding. The incidence geometric properties of the Segre embedding have been studied in [14]. The product spaces do not have universal embeddings, but each embedding of a product space S is the composition of (i) an automorphism of S, acting only on the former coordinate, (ii) the Segre embedding and (iii) a linear mapping between projective spaces. This decomposition property, although not as strong as the universal property, is still sufficient to prove that each embedded product space can be obtained by projection of a Segre variety from a subspace skew with its first secant variety. By Theorem 1.1, such subspaces are in some case linked with semifields.

In this paper we point out the above link. The main result is Theorem 3.10, which states a bijection between particular embeddings of $\mathrm{PG}(n,q) \times \mathrm{PG}(n,q)$ in $\mathrm{PG}(n^2+n-1,q)$ and semifields of order q^{n+1} whose center contains $\mathbb{F}_q$.

Concerning the definition of equivalence between embeddings see the definition at p. 5. We stress out that if ι is an embedding, and

$$(A,B)^\xi = (B,A) \quad \text{for } A,B \in \mathrm{PG}(n,q), \tag{1.2}$$

then ι and $\xi\iota$ are not necessarily equivalent embeddings. In fact ι and $\xi\iota$ are associated with transposed semifields (cf. Theorem 3.12.)

1.2 Definitions and tools

If A and B are points of a semilinear space, then AB denotes the joining line, when A and B are distinct and collinear, otherwise $AB = \{A,B\}$. If $\mathcal{A}$ and $\mathcal{B}$ are two sets of points, define

$$\mathcal{A} \vee \mathcal{B} = \mathcal{A} \cup \mathcal{B} \cup \{AB \mid A \in \mathcal{A},\ B \in \mathcal{B}\}.$$

The definition includes the possibility that one or both sets are empty.

If $\mathcal{A}$ and $\mathcal{B}$ are two complementary subspaces in a projective space $\mathbb{P}$, the *projection from* $\mathcal{A}$ *onto* $\mathcal{B}$ is the map $\mathrm{pr}(\mathcal{A},\mathcal{B}) : \mathbb{P} \setminus \mathcal{A} \to \mathcal{B}$ defined by $X^{\mathrm{pr}(\mathcal{A},\mathcal{B})} = (X \vee \mathcal{A}) \cap \mathcal{B}$.

If $\mathbb{P}_1$ and $\mathbb{P}_2$ are two projective spaces, their *product space* $\mathrm{Prod}(\mathbb{P}_1, \mathbb{P}_2)$ is the semilinear space whose point set is $\mathbb{P}_1 \times \mathbb{P}_2$, and whose lines are of type $A_1 \times \ell_2$ or $\ell_1 \times A_2$, where A_i and ℓ_i are a point and a line in $\mathbb{P}_i$ for $i = 1, 2$. The subspace $\mathbb{P}_1 \times A_2$ is called a *maximal subspace of first kind*, and a maximal subspace of second kind is defined analogously. We are only interested in the case in which both projective spaces are finite-dimensional and coordinatized by the same field F, and we will use the shorthand

$$\mathrm{Prod}(n, m, F) = \mathrm{Prod}(\mathrm{PG}(n, F), \mathrm{PG}(m, F)), \quad n, m \geq 0.$$

When $F = \mathbb{F}_q$ we will denote the product space with $\mathrm{Prod}(n, m, q)$.

An *embedding* of $\mathrm{Prod}(n, m, F)$ is a one-to-one map $\iota : \mathrm{Prod}(n, m, F) \to \mathbb{P}$, $\mathbb{P}$ a projective space, such that for any line ℓ of $\mathrm{Prod}(n, m, F)$, ℓ^ι is a (full) line of $\mathbb{P}$. So, if $\max\{n, m\} \geq 2$ and $\mathbb{P}$ is finite-dimensional, also $\mathbb{P}$ can be coordinatized by F. We will always tacitly assume that the image of ι spans $\mathbb{P}$.

An *embedded product space* is a pair $(\mathrm{Prod}(n, m, F), \iota)$ where $\iota : \mathrm{Prod}(n, m, F) \to \mathbb{P}$ is an embedding. Sometimes the embedded product space will be identified with $\mathrm{Prod}(n, m, F)^\iota$. If $\mathbb{P}$ is N-dimensional over F, we will say that this is an embedded product space of *type* (n, m, N, F), or type (n, m, N, q) for $F = \mathbb{F}_q$. Also, ι is an embedding of type (n, m, N, F) or (n, m, N, q). An embedding of type $(n, m, nm + n + m, F)$ is called a *regular embedding.*

Let $\Sigma = \mathrm{Prod}(\mathbb{P}_1, \mathbb{P}_2)$. Two embeddings $\iota : \Sigma \to \mathbb{P}$ and $\iota' : \Sigma \to \mathbb{P}'$ are *equivalent* if a collineation $\Phi : \mathbb{P} \to \mathbb{P}'$ exists such that (i) $\Sigma^{\iota\Phi} = \Sigma^{\iota'}$, and (ii) for any two points A_i in $\mathbb{P}_i$, $i = 1, 2$, points B_i in $\mathbb{P}_i$ exist such that

$$(A_1 \times \mathbb{P}_2)^{\iota\Phi\iota'^{-1}} = B_1 \times \mathbb{P}_2, \quad (\mathbb{P}_1 \times A_2)^{\iota\Phi\iota'^{-1}} = \mathbb{P}_1 \times B_2. \tag{1.3}$$

By (1.3) $\iota\Phi\iota'^{-1}$ is a *kind-preserving automorphism* of Σ.

Let V and W be two vector spaces over the field F, having finite dimensions $n + 1$ and $m + 1$, respectively $(n, m \geq 1)$. Set $N = nm + n + m$. The *Segre variety* in $\mathrm{PG}(V \otimes W) \cong \mathrm{PG}(N, F)$ is

$$\mathcal{S}_{n,m,F} = \mathcal{S}_{n,m} = \{F\mathbf{v} \otimes \mathbf{w} \mid \mathbf{v} \in V \setminus \{\mathbf{0}\}, \mathbf{w} \in W \setminus \{\mathbf{0}\}\}. \tag{1.4}$$

We also use the notation $\mathcal{S}_{n,m,q}$ with obvious meaning. The isomorphism between $\mathrm{PG}(V \otimes W)$ and $\mathrm{PG}(N,F)$ is immaterial; we consider it as fixed.

The Segre variety is an embedded $\mathrm{Prod}(\mathrm{PG}(V), \mathrm{PG}(W))$, by means of the *Segre embedding* b defined by

$$(F\mathbf{v}, F\mathbf{w})^b = F\mathbf{v} \otimes \mathbf{w} \quad \text{for } \mathbf{v} \neq \mathbf{0} \neq \mathbf{w},$$

which is a regular embedding. The maximal subspaces of $\mathrm{PG}(N,F)$ contained in $\mathcal{S}_{n,m}$ are precisely the images of the maximal subspaces of first or second kind in $\mathrm{Prod}(\mathrm{PG}(V), \mathrm{PG}(W))$ [7]. So, they are in the form $\mathrm{PG}(V \otimes \mathbf{w})$ with $\mathbf{w} \in W \setminus \{\mathbf{0}\}$, or $\mathrm{PG}(\mathbf{v} \otimes W)$ with $\mathbf{v} \in V \setminus \{\mathbf{0}\}$.

If $\mathcal{A}$ is any set of points in a projective space, and s is a positive integer, the *s-th secant variety* to $\mathcal{A}$ is the union of the spans of all $(s+1)$-tuples of points of $\mathcal{A}$. If δ is a non-empty subspace of $\mathrm{PG}(N,F)$ and is skew with the first secant variety to $\mathcal{S}_{n,m,F}$, then the product of b and a projection from δ is a non-regular embedding of $\mathrm{Prod}(n,n,F)$. Embeddings of product spaces have been studied in [14]. The main results are the following:

Theorem 1.2. [14] *Let χ be an embedding of $\mathrm{Prod}(\mathbb{P}_1, \mathbb{P}_2)$ into $\mathbb{P}'$, where $\mathbb{P}_1$ and $\mathbb{P}_2$ are finite-dimensional projective spaces with $\dim(\mathbb{P}_1) > 1$ and $\dim(\mathbb{P}_2) \geq 1$. Then there are a collineation τ' of $\mathbb{P}_2$ and a linear mapping ψ such that*

$$\chi = \tau b \psi$$

where $\tau = (\mathbf{1}_{\mathbb{P}_1}, \tau')$.

Theorem 1.3. [14] *Let χ be an embedding of $\mathrm{Prod}(\mathbb{P}_1, \mathbb{P}_2)$, where $\mathbb{P}_1$ and $\mathbb{P}_2$ are finite-dimensional projective spaces with $\dim(\mathbb{P}_1), \dim(\mathbb{P}_2) \geq 1$. If χ is regular, then its image is projectively equivalent to a Segre variety.*

2 Incidence-geometric characterization of Segre varieties

Although the results in this section are folklore, we want to state them in order to settle the notation, and refer to them in the next section.

Let F be a field, and N a positive integer. A *generator $(m+2)$-tuple* $\boldsymbol{\varepsilon} = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{m+1})$ *of* $\mathrm{PG}(N,F)$ is any $(m+2)$-tuple consisting of n-subspaces in $\mathrm{PG}(N,F)$, where $nm+n+m = N$, which are in *general position*, i.e. any $m+1$ of them span $\mathrm{PG}(N,F)$. Notice that a generator $(N+2)$-tuple is a set of $N+2$ points in $\mathrm{PG}(N,F)$ in general position. Such a set is called a *frame* of $\mathrm{PG}(N,F)$. For $A \in \varepsilon_0$ and $j = 1, 2, \ldots, m+1$ define

$$A^{\pi_j} = \left(A \vee \bigvee_{\substack{h=1,\ldots,m+1 \\ h \neq j}} \varepsilon_h \right) \cap \varepsilon_j. \tag{2.1}$$

This π_j is a projective collineation for any $j = 1, 2, \ldots, m+1$.

Proposition 2.1. *Let $\boldsymbol{\varepsilon} = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{m+1})$ be a generator $(m+2)$-tuple of $\mathrm{PG}(N,F)$. For any $A \in \varepsilon_0$ a unique m-subspace $T(A)$ exists, which contains A and intersects all ε_j's for $j = 1, 2, \ldots, m+1$.*

Proof. If a subspace $T(A)$ with the described properties exists, it intersects each of the ε_j's in exactly one point. For $j = 1, 2, \ldots, m+1$,

$$T(A) \bigcap \left(\bigvee_{\substack{h=1,\ldots,m+1 \\ h \neq j}} \varepsilon_h \right)$$

is a hyperplane of $T(A)$. Indeed, the set being intersected with $T(A)$ contains m independent points of $T(A)$, its span is disjoint with ε_j, and does not contain $\varepsilon_j \cap T(A)$. So, $T(A) \cap \varepsilon_j$ is on a line through A and meeting $\bigvee_{\substack{h=1,\ldots,m+1 \\ h \neq j}} \varepsilon_h$. There is exactly one such line, meeting ε_j in A^{π_j}. This gives the uniqueness, since

$$T(A) = A^{\pi_1} \vee A^{\pi_2} \vee \ldots \vee A^{\pi_{m+1}}. \tag{2.2}$$

It remains to show that the subspace $T(A)$ in (2.2) contains A. Define

$$W_j = \left(A \vee \bigvee_{\substack{h=1,\ldots,m+1 \\ h \neq j}} \varepsilon_h \right), \quad \text{and } U_t = \bigcap_{i=1}^{t} W_i, \quad j, t = 1, 2, \ldots, m+1.$$

Clearly for any $t = 1, 2, \ldots, m+1$, $A \in U_t$. Since W_j contains ε_h for $h \neq j$ and also A^{π_j} (by the definition (2.1)), the subspace $T(A)$ is contained in U_t for any

$t = 1, 2, \ldots, m+1$. Further, note that for $t = 1, 2, \ldots, m$, U_t contains ε_{t+1} and U_{t+1} intersects ε_{t+1} precisely in $A^{\pi_{t+1}}$. This implies $\dim(U_{t+1}) \leq \dim(U_t) - n$. So, $\dim(U_t) \leq m(n+1) - (t-1)n$ and this gives $U_{m+1} = T(A)$. □

For a generator $(m+2)$-tuple $\boldsymbol{\varepsilon} = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{m+1})$, the union of all m-subspaces $T(A)$ with $A \in \varepsilon_0$ will be denoted by $\mathcal{S}(\boldsymbol{\varepsilon})$.

Proposition 2.2. *Let $\boldsymbol{\varepsilon}$ and $\boldsymbol{\nu}$ be two generator $(m+2)$-tuples of n-subspaces of* $\mathrm{PG}(N, F)$*, $m > 0$, and $\omega_0 : \varepsilon_0 \to \nu_0$ a collineation. Then there is a unique $\omega \in \mathrm{P\Gamma L}(N+1, F)$ such that $\omega|_{\varepsilon_0} = \omega_0$ and $\varepsilon_j^{\omega} = \nu_j$, $j = 1, 2, \ldots, m+1$.*

Proof. For $j = 1, 2, \ldots, m+1$ take π_j as in (2.1) and $\pi'_j : \nu_0 \to \nu_j$ analogously defined. Consider a frame $E_0, E_1, \ldots, E_{n+1}$ for ε_0. Define

$$N_i = E_i^{\omega_0}, \qquad i = 0, 1, \ldots, n+1, \tag{2.3}$$

$$E_{ij} = E_i^{\pi_j},\ N_{ij} = N_i^{\pi'_j}, \qquad i = 0, 1, \ldots, n+1,\ j = 1, 2, \ldots, m+1. \tag{2.4}$$

If a collineation ω with the described properties exists, then for any $i = 0, 1, \ldots, n+1$ and $j = 1, 2, \ldots, m+1$ the image under ω of

$$E_{ij} = \left(E_i \vee \bigvee_{\substack{h = 1, \ldots, m+1 \\ h \neq j}} \varepsilon_h \right) \cap \varepsilon_j \tag{2.5}$$

is

$$\left(N_i \vee \bigvee_{\substack{h = 1, \ldots, m+1 \\ h \neq j}} \nu_h \right) \cap \nu_j = N_{ij}.$$

Take $I \in \{1, 2, \ldots, n+1\}$, $J \in \{1, 2, \ldots, m+1\}$, and consider the set

$$\mathcal{F}_{IJ} = \{E_{ij} \mid i = 1, 2, \ldots, n+1,\ j = 1, 2, \ldots, m+1\} \cup \{E_0\} \setminus \{E_{IJ}\}.$$

By (2.5), $E_{0J} \in \langle \mathcal{F}_{IJ} \rangle$. So, $\langle \mathcal{F}_{IJ} \rangle$ contains the set

$$\{E_{iJ} \mid i = 0, 1, \ldots, n+1,\ i \neq I\} = \{E_i^{\pi_J} \mid i = 0, 1, \ldots, n+1,\ i \neq I\},$$

and $\varepsilon_J \subseteq \langle \mathcal{F}_{IJ} \rangle$. So, $\mathcal{F}_{IJ}$ spans $\mathrm{PG}(N, F)$. This implies that the set

$$\mathcal{F}_{IJ} \cup \{E_{IJ}\} = \{E_{ij} \mid i = 1, 2, \ldots, n+1,\ j = 1, 2, \ldots, m+1\} \cup \{E_0\}$$

is a frame of $\mathrm{PG}(N, F)$. Then the stated uniqueness follows from the uniqueness of the collineation ω of $\mathrm{PG}(N, F)$ related to the same field automorphism of ω_0, and satisfying

$$E_{ij}^{\omega} = N_{ij},\ i = 1, 2, \ldots, n+1,\ j = 1, 2, \ldots, m+1, \quad E_0^{\omega} = N_0. \tag{2.6}$$

It remains to show that $\omega\,|_{\varepsilon_0} = \omega_0$. Note that the m-subspaces

$$S_i = T(E_i) = E_{i1} \vee E_{i2} \vee \ldots \vee E_{i,m+1}, \quad i = 0, 1, \ldots, n+1$$

are in general position, and $E_i \in S_i$. By prop. 2.1, ε_0 is the unique n-subspace containing E_0 and intersecting all S_i's. Since ν_0 is the unique n-subspace through $N_0 = E_0^{\omega}$ intersecting every S_i^{ω}, it holds $\nu_0 = \varepsilon_0^{\omega}$. Finally $E_i^{\omega} = (S_i \cap \varepsilon_0)^{\omega} = N_i$, $i = 0, 1, \ldots, n+1$ implies $\omega\,|_{\varepsilon_0} = \omega_0$. □

Corollary 2.3. *For any couple of generator $(m+2)$-tuples, say $\boldsymbol{\varepsilon}$ and $\boldsymbol{\nu}$, of n-subspaces of $\mathrm{PG}(N, F)$, $\mathcal{S}(\boldsymbol{\varepsilon})$ is projectively equivalent to $\mathcal{S}(\boldsymbol{\nu})$.*

Proposition 2.4. *There is a generator $(m+2)$-tuple $\boldsymbol{\varepsilon}$ such that $\mathcal{S}(\boldsymbol{\varepsilon}) = \mathcal{S}_{n,m}$.*

Proof. Recall that $\mathcal{S}_{n,m}$ is defined in (1.4). Consider a basis $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_m$ of W_F. Let $\mathbf{u} = \sum_{j=0}^{m} \mathbf{e}_j$, and $\varepsilon_j = \mathrm{PG}(V \otimes \mathbf{e}_j)$, $j = 0, 1, \ldots, m$, $\varepsilon_{m+1} = \mathrm{PG}(V \otimes \mathbf{u})$. The $(m+2)$-tuple $\boldsymbol{\varepsilon} = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{m+1})$ is a generator $(m+2)$-tuple. Any point $F\mathbf{v} \otimes \mathbf{w}$ of $\mathcal{S}_{n,m}$ lies on $\mathrm{PG}(\mathbf{v} \otimes W)$ that is an m-subspace having a non-empty intersection with each component of $\boldsymbol{\varepsilon}$. Conversely, for any $A = F\mathbf{u} \otimes \mathbf{e}_0$ in ε_0 it holds $T(A) = \mathrm{PG}(\mathbf{u} \otimes W)$ and this implies $T(A) \subseteq \mathcal{S}_{n,m}$. □

If $\boldsymbol{\varepsilon} = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{m+1})$ is a generator $(m+2)$-tuple, the maximal subspaces in $\mathcal{S}(\boldsymbol{\varepsilon})$ of type $T(A)$ (cf. (2.2)) will be called *transversals*, or *subspaces of second kind of* $\mathcal{S}(\boldsymbol{\varepsilon})$, whereas the remaining ones are *subspaces of first kind.* Among them there are $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{m+1}$.

Proposition 2.5. *Let $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_r$ be n-subspaces of first kind in $\mathcal{S}_{n,m}$. Then $\mathcal{S}_{n,m} \cap (\varepsilon_0 \vee \varepsilon_1 \vee \ldots \vee \varepsilon_r)$ is projectively equivalent to a Segre variety of type $\mathcal{S}_{n,s}$ for some $s \leq r$.*

Proof. It holds $\mathcal{S}_{n,m} = \mathcal{S}(\mathbf{e})$, where $\mathbf{e} = (e_0, e_1, \ldots, e_{n+1})$ is a generator $(n+2)$-tuple of $\mathrm{PG}(N, F)$. For $X \in e_0$, let $\Theta(X) = X^{p_1} \vee X^{p_2} \vee \ldots \vee X^{p_{n+1}}$ be defined (in analogy to (2.2)) to be the unique n-subspace through X meeting each e_i, $i = 1, 2, \ldots, n+1$. Each maximal subspace of the first kind is equal to $\Theta(X)$ for some point X on e_0. So, there exist $r+1$ points $A_0, A_1, \ldots, A_r \in e_0$ such that $\varepsilon_j = \Theta(A_j)$, $j = 0, 1, \ldots, r$. Without loss of generality it may be assumed that $A_0, A_1, \ldots, A_s$ is a basis of $A_0 \vee A_1 \vee \ldots \vee A_r$, and $0 \le s \le r$. Set $J = A_0 \vee A_1 \vee \ldots \vee A_s$. Then J, J^{p_1}, ..., $J^{p_{n+1}}$ are $n+2$ s-subspaces in general position in an $((s+1)(n+1)-1)$-subspace (actually improper) of $\varepsilon_0 \vee \varepsilon_1 \vee \ldots \vee \varepsilon_r$. A Segre variety $\mathcal{S}_{s,n} = \mathcal{S}(J, J^{p_1}, \ldots, J^{p_{n+1}})$ arises. Each transversal in $\mathcal{S}_{s,n}$ is a maximal subspace of the first kind in the starting $\mathcal{S}_{n,m}$. Each point of $\mathcal{S}_{n,m}$ not belonging to $\mathcal{S}_{s,n}$ is on a $\Theta(Y)$, $Y \in e_0 \setminus J$, and $\Theta(Y) \cap (\varepsilon_0 \vee \varepsilon_1 \vee \ldots \vee \varepsilon_r) = \emptyset$. $\square$

A *regulus* of n-subspaces is the set of all n-subspaces of first kind of $\mathcal{S}(\varepsilon_0, \varepsilon_1, \varepsilon_2)$, where ε_0, ε_1, ε_2 are n-subspaces contained in a common $(2n+1)$-subspace, and are pairwise skew. The following proposition is a corollary of prop. 2.5.

Proposition 2.6. *Let ε_0 and ε_1 be two distinct n-subspaces of first kind in $\mathcal{S}_{n,m}$. Then the intersection of $\mathcal{S}_{n,m}$ with $\varepsilon_0 \vee \varepsilon_1$ is a regulus of n-subspaces.*

It should be noted that every $(s+2)$-tuple of n-subspaces of first type of $\mathcal{S}_{n,m}$, which is a generator $(s+2)$-tuple of some subspace of $\langle \mathcal{S}_{n,m} \rangle$, is formed by subspaces in the form $(\mathrm{PG}(n, F) \times B_i)^b$ $(i = 0, 1, \ldots, s+1)$, where b is the Segre embedding, and $B_0, B_1, \ldots, B_{s+1}$ is a frame of an s-subspace of $\mathrm{PG}(m, F)$. Vice versa from a frame of an s-subspace of $\mathrm{PG}(m, F)$ a generator $(s+2)$-tuple arises.

Proposition 2.7. *Let $\boldsymbol{\varepsilon}$ be a generator $(m+2)$-tuple of n-subspaces with $n > 1$. Let S and S' be two $[(t+1)(n+1)-1]$-subspaces of $\mathrm{PG}(N, F)$ $(t \ge 1)$. Also assume that*

$$\mathcal{S}(\boldsymbol{\varepsilon}) \cap S = \mathcal{S}(\varphi_0, \varphi_1, \ldots, \varphi_{t+1}), \qquad \mathcal{S}(\boldsymbol{\varepsilon}) \cap S' = \mathcal{S}(\varphi'_0, \varphi'_1, \ldots, \varphi'_{t+1}),$$

where $\varphi_0, \varphi_1, \ldots, \varphi_{t+1}, \varphi'_0, \varphi'_1, \ldots, \varphi'_{t+1}$ are n-subspaces of first kind in $\mathcal{S}(\boldsymbol{\varepsilon})$. If $\kappa_0 : S \to S'$ is a collineation such that

$$(\mathcal{S}(\boldsymbol{\varepsilon}) \cap S)^{\kappa_0} = \mathcal{S}(\boldsymbol{\varepsilon}) \cap S',$$

then κ_0 can be extended to an $\omega \in \mathrm{P\Gamma L}(N+1, F)$ fixing $\mathcal{S}(\boldsymbol{\varepsilon})$ and preserving the kind of any subspace.

More precisely, assume that $\nu_0, \nu_1, \ldots, \nu_{m+1}$, ν_∞ are n-subspaces of the first kind of $\mathcal{S}(\boldsymbol{\varepsilon})$ satisfying the following properties: (i) ν_0, ν_1, ..., ν_t, ν_∞

are in $\mathcal{S}(\boldsymbol{\varepsilon}) \cap S$*; (ii)* $\boldsymbol{\nu} = (\nu_0, \nu_1, \ldots, \nu_{m+1})$ *is a generator* $(m+2)$*-tuple; (iii)* $\nu_\infty = (\nu_0 \vee \ldots \vee \nu_t) \cap (\nu_{t+1} \vee \ldots \vee \nu_{m+1})$*. Define* $\nu'_i = \nu_i^{\kappa_0}$ *for* $i = 0, 1, \ldots, t, \infty$*. Furthermore, let* $\nu'_{t+1}, \nu'_{t+2}, \ldots, \nu'_{m+1}$ *be subspaces of the first kind in* $\mathcal{S}(\boldsymbol{\varepsilon})$ *such that* $\boldsymbol{\nu}' = (\nu'_0, \nu'_1, \ldots, \nu'_{m+1})$ *is a generator* $(m+2)$*-tuple, and* $\nu'_\infty = (\nu'_0 \vee \ldots \vee \nu'_t) \cap (\nu'_{t+1} \vee \ldots \vee \nu'_{m+1})$*. Then* κ_0 *can be extended to a unique* $\omega \in \mathrm{P\Gamma L}(N+1, F)$ *such that* $\nu_i^\omega = \nu'_i$ *for* $i = t+1, t+2, \ldots, m+1$*.*

Proof. For $i = 0, 1, \ldots, t, \infty$, $\nu'_i = \nu_i^{\kappa_0}$ is an n-subspace contained in $\mathcal{S}(\boldsymbol{\varepsilon}) \cap S'$. By prop. 2.2 there exists a collineation ω of $\mathrm{PG}(N, F)$ satisfying $\boldsymbol{\nu}^\omega = \boldsymbol{\nu}'$ and $\omega|_{\nu_0} = \kappa_0|_{\nu_0}$, mapping $\mathcal{S}(\boldsymbol{\varepsilon})$ onto itself. Since by construction $\nu_\infty^\omega = \nu'_\infty$, once again applying prop. 2.2 in its uniqueness part gives $\omega|_S = \kappa_0$. □

3 Semifield embeddings

Throughout this section the symbol Σ will be a shorthand for $\mathrm{Prod}(n, n, F)$, $n \geq 2$. A *semifield embedding* of Σ is an embedding ι of type (n, n, n^2+n-1, F), such that for every hyperplane H of $\mathrm{PG}(n, F)$ the restriction of ι to $\mathrm{PG}(n, F) \times H$ is a regular embedding. In the remainder of this paper we only deal with semifield embeddings, although some of the results hold in general. We assume that the Segre embedding $b : \Sigma \to \mathrm{PG}(n^2+2n, F)$ is kind-preserving or, more precisely, that subspaces in the form $(\mathrm{PG}(n, F) \times B)^b$ are of first kind.

The subgroup of $\mathrm{P\Gamma L}((n+1)^2, F)$ of all collineations which fix $\mathcal{S}_{n,n}$ and preserve the kind of any plane in $\mathcal{S}_{n,n}$ will be denoted by $\mathbb{H}$.

A *representation* of the embedded product $\Pi = (\Sigma, \iota)$ is an ordered triple (S, κ, δ) consisting of an (n^2+n-1)-subspace S of $\mathrm{PG}(n^2+2n, F)$, a collineation $\kappa : \langle \Sigma^\iota \rangle \to S$, and an n-subspace δ disjoint from the $(n-1)$-th secant variety to $\mathcal{S}_{n,n}$, such that
(i) $S \cap \mathcal{S}_{n,n} = \mathcal{S}(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)$, where $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)$ is a generator $(n+1)$-tuple of S,
(ii) the embedding $\iota\kappa$ factorizes into an automorphism α of Σ, the Segre embedding b, and the projection c from δ onto S;
(iii) $\iota\kappa$ maps at least one plane of the first kind in Σ into a plane of the first kind in $\mathcal{S}_{n,n}$.

In what follows, when we write $\mathcal{S}(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{t+1})$, we implicitly assume that $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{t+1})$ is a generator $(t+2)$-tuple for some subspace.

Proposition 3.1. *The automorphism* α *in the definition of a representation* *is kind-preserving.*

Proof. Take an n-subspace ν of first kind in $\mathcal{S}(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)$. Both $\nu' = \nu^{(\iota\kappa)^{-1}}$ and $\nu'' = \nu^{(bc)^{-1}} = \nu^{b^{-1}}$ are of the first kind, and $\nu'' = (\nu')^{\alpha}$. $\square$

The following proposition asserts that a representation only depends on the embedded product space and the choice of the kinds.

Proposition 3.2. *Let (T, ι_1) and (T, ι_2) be two embedded products of type (n, n, n^2+n-1, F), such that (a) $T^{\iota_1} = T^{\iota_2}$, (b) $\beta = \iota_2\iota_1^{-1}$ is kind-preserving. If (S, κ, δ) is a representation of (T, ι_1), then (S, κ, δ) also is a representation of (T, ι_2).*

Proof. Clearly, properties (i) and (iii) in the definition of a representation still hold for (T, ι_2). The property (ii) follows from the equation $\iota_2\kappa = \beta\iota_1\kappa = \beta\alpha bc$. $\square$

Proposition 3.3. *If (S, κ, δ) is a representation of the embedded product Π, with semifield embedding, then (i) there is a hyperplane H in $\mathrm{PG}(n, F)$ such that $(\mathrm{PG}(n, F) \times A)^{\iota\kappa}$ is an n-subspace of first kind in $\mathcal{S}_{n,n}$ for any point $A \in H$; (ii) for any proper s-subspace K of $\mathrm{PG}(n, F)$, defining*

$$\mathcal{Q} = (\mathrm{PG}(n, F) \times K)^{\iota\kappa},$$

there is an $\mathcal{S}(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{s+1})$, where $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{s+1}$ are n-subspaces of first kind in $\mathcal{S}_{n,n}$, such that $\mathcal{Q} = \mathcal{S}(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{s+1})^{\mathrm{pr}(\delta, S)}$.

Proof. (i) The assertion follows from the equation $\iota\kappa = \alpha bc$. Indeed, $(S \cap \mathcal{S}_{n,n})^c = S \cap \mathcal{S}_{n,n} = \mathcal{S}(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)$ and a hyperplane H exists such that $\mathcal{S}(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n) = (\mathrm{PG}(n, F) \times H)^b$. Hence

$$(\mathrm{PG}(n, F) \times H)^{\alpha^{-1}\iota\kappa} = \mathcal{S}(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n).$$

The automorphism α^{-1} is kind-preserving and the proof is complete.

(ii) Just set $\mathcal{S}(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{s+1}) = (\mathrm{PG}(2, F) \times K)^{\alpha b}$. $\square$

Proposition 3.4. *Let S be an $(n^2 + n - 1)$-subspace of $\mathrm{PG}(n^2 + 2n, F)$ intersecting $\mathcal{S}_{n,n}$ in $\mathcal{S}(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)$, where $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)$ is a generator $(n+1)$-tuple of S whose elements are n-subspaces of first kind in $\mathcal{S}_{n,n}$. Then for any embedded product Π, with semifield embedding, there is a representation in the form (S, κ, δ).*

Proof. By Theorem 1.2, there exists a triple (S', κ', δ) (not necessarily a representation) such that $\iota\kappa' = \alpha bc$, where $\alpha \in \mathrm{Aut}(\Sigma)$ is kind-preserving, and $c = \mathrm{pr}(\delta, S')$. Assume that δ is not disjoint from the $(n-1)$-th secant variety of $\mathcal{S}_{n,n}$. Then there is a point $X \in \delta$ that belongs to the join of n points of type $(P_i, Q_i)^b$, $i = 0, 1, \dots, n-1$. This implies that, for $H = Q_0 \vee \dots \vee Q_{n-1}$, the dimension of the span of $(\mathrm{PG}(n, F) \times H)^c$ is less than n^2+n-1, and ι is not a semifield embedding, a contradiction. So, δ is disjoint from the $(n-1)$-secant variety. Since S is contained in such a secant variety, $\delta \cap S = \emptyset$. Consider the collineation $\kappa'' = \mathrm{pr}(\delta, S)\,|_{S'}$. Setting $\kappa = \kappa'\kappa''$ completes the proof. $\square$

Proposition 3.5. *Let (S, κ, δ) be a representation of an embedded product Π, with semifield embedding, and T an (n^2+n-1)-subspace of $\mathrm{PG}(n^2+2n, F)$, intersecting $\mathcal{S}_{n,n}$ in $\mathcal{S}(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n)$, where $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n)$ is a generator $(n+1)$-tuple of T whose elements are n-subspaces of first kind in $\mathcal{S}_{n,n}$. Then $\delta \cap T = \emptyset$. If $\theta = \kappa\,[\mathrm{pr}(\delta, T)\,|_S]$, then (T, θ, δ) is also a representation of Π.*

Proof. We prove (iii) for (T, θ, δ), the remaining properties being straightforward. Since $n \geq 2$, S and T share a subspace $\varphi = \psi^{\iota\kappa}$ of first kind. The assertion follows from $\psi^{\iota\kappa} = \psi^{\iota\theta}$. $\square$

Theorem 3.6. *Let (S, κ, δ) and (S', κ', δ') be representations of two embedded products $\Pi = (\Sigma, \iota)$ and $\Pi' = (\Sigma, \iota')$, both of type (n, n, n^2+n-1, F), with semifield embeddings. If ι and ι' are equivalent embeddings, then δ and δ' are in the same orbit of planes under the action of $\mathbb{H}$.*

Proof. By assumption a kind-preserving automorphism σ_0 of Σ exists such that $\iota^{-1}\sigma_0\iota'$ can be extended to a $\sigma \in \mathrm{P\Gamma L}(n^2+n, F)$. Take a hyperplane H in $\mathrm{PG}(n, F)$. By prop. 3.3 (ii), two generator $(n+1)$-tuples for (n^2+n-1)-subspaces, say $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n)$ and $(\varepsilon'_0, \varepsilon'_1, \dots, \varepsilon'_n)$, formed by subspaces of first kind in $\mathcal{S}_{n,n}$ exist such that

$$\begin{aligned}(\mathrm{PG}(n, F) \times H)^{\iota\kappa} &= \mathcal{S}(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n)^{\mathrm{pr}(\delta, S)},\\ (\mathrm{PG}(n, F) \times H)^{\iota\sigma\kappa'} &= (\mathrm{PG}(n, F) \times H)^{\sigma_0\iota'\kappa'} = \mathcal{S}(\varepsilon'_0, \varepsilon'_1, \dots, \varepsilon'_n)^{\mathrm{pr}(\delta', S')}.\end{aligned}$$

Denote by T and T' the spans of $\mathcal{S}(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n)$ and $\mathcal{S}(\varepsilon'_0, \varepsilon'_1, \dots, \varepsilon'_n)$, respectively. By prop. 3.5 there are also representations (T, θ, δ) and (T', θ', δ') of Π and Π', respectively. Taking into account that the restriction of $\mathrm{pr}(\delta, S)\,\mathrm{pr}(\delta, T)$ to T is the identity, and defining

$$\kappa_0 = \theta^{-1}\sigma\theta' : T \to T',$$

$\mathcal{S}(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)^{\kappa_0} = \mathcal{S}(\varepsilon'_0, \varepsilon'_1, \ldots, \varepsilon'_n)$ holds. In $\mathcal{S}_{n,n}$ there exist two n-subspaces ν_1 and ν_2 of first kind such that $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \nu_1, \nu_2)$ is a generator $(n+2)$-tuple, and $\varepsilon_0 = T \cap (\nu_1 \vee \nu_2)$. As a matter of fact, T corresponds to a hyperplane of the latter factor in Σ, and $\nu_1 \vee \nu_2$ corresponds to a line, meeting that hyperplane in the point corresponding to ε_0. Next, define $\varepsilon''_i = \varepsilon_i^{\kappa_0}$ for $i = 0, 1, \ldots, n$, and $\varphi_j = \nu_j^{\mathrm{pr}(\delta,T)}$ for $j = 1, 2$. The n-subspaces ε_0, φ_1 and φ_2 are in a regulus $\mathcal{Q}$ which is contained in T. Two lines m and m' in $\mathrm{PG}(n, F)$ exist such that $\mathcal{Q} = (\mathrm{PG}(n, F) \times m)^{\iota\theta}$, and $\mathcal{Q}^{\kappa_0} = (\mathrm{PG}(n, F) \times m')^{\iota'\theta'}$. Three n-subspaces on a regulus of first kind of $\mathcal{S}_{n,n}$ exist whose images through $\mathrm{pr}(\delta', T')$ are $\varepsilon''_0 = \varepsilon_0^{\kappa_0}$, $\varphi_1^{\kappa_0}$ and $\varphi_2^{\kappa_0}$. The first one is ε''_0, and let us denote the remaining ones with ν'_1 and ν'_2. By prop. 2.7 an $\omega \in \mathbb{H}$ exists such that $\omega\,|_T = \kappa_0$ and $\nu_i^\omega = \nu'_i$, $i = 1, 2$. The equations

$$\delta = (\varphi_1 \vee \nu_1) \cap (\varphi_2 \vee \nu_2), \quad \delta' = (\varphi_1^{\kappa_0} \vee \nu'_1) \cap (\varphi_2^{\kappa_0} \vee \nu'_2)$$

imply $\delta^\omega = \delta'$. □

Theorem 3.6 with $\Pi = \Pi'$ and $\sigma = \mathbf{1}_{\mathrm{PG}(n^2+n-1,F)}$ gives

Corollary 3.7. *For every embedded product Π of type $(n, n, n^2 + n - 1, F)$, with semifield embedding, there is precisely one orbit $\Omega(\Pi)$ of n-subspaces under the action of $\mathbb{H}$, such that for any representation (S, κ, δ) of Π it holds $\delta \in \Omega(\Pi)$.*

The following is a simple remark and is needed in order to prove the converse of Theorem 3.6:

Proposition 3.8. *If (S, κ, δ) is a representation of the embedded product Π, then $(S^h, \kappa h, \delta^h)$ is a representation of Π for any $h \in \mathbb{H}$.*

Proposition 3.9. *Let (S, κ, δ) and (S', κ', δ') be representations of two embedded products $\Pi = (\Sigma, \iota)$ and $\Pi' = (\Sigma, \iota')$, respectively, with semifield embeddings. If an $h \in \mathbb{H}$ exists such that $\delta^h = \delta'$, then ι and ι' are equivalent embeddings.*

Proof. It has be to shown that there is a collineation σ of $\mathrm{PG}(n^2 + n - 1, F)$ such that $\Pi^\sigma = \Pi'$, and $\sigma_0 = \iota\sigma(\iota')^{-1}$ is kind-preserving. The triple $(S^h, \kappa h, \delta')$ is a representation of Π. By prop. 3.5 another representation of Π is of type (S', θ, δ'). Hence the maps $\iota\theta$ and $\iota'\kappa'$ have the same image. The map $\sigma = \theta(\kappa')^{-1}$ is an element of $\mathrm{P\Gamma L}(n^2 + n, F)$ that sends Π onto Π'. The last statement follows by considering that σ_0 is kind-preserving for at least one n-subspace. □

By Theorem 1.1 with $s = 1$, each n-subspace external to the $(n-1)$-th secant variety of the Segre variety $\mathcal{S}_{n,n,q}$ gives rise to a semifield of order q^{n+1}, and two such n-subspaces give rise to isotopic semifields if and only if they belong to the same orbit of the group $\mathbb{H}$. Furthermore, each semifield of order q^{n+1} with center containing $\mathbb{F}_q$ is associated to such an n-subspace.

Theorem 3.10. *Let n be a positive integer, and q a power of a prime. There is a one-to-one correspondence between equivalence classes of semifield embeddings of* $\mathrm{Prod}(n,n,q)$ *and isotopism classes of semifields of order q^{n+1}, whose centers contain $\mathbb{F}_q$.*

Proof. For $n = 1$ this follows from the uniqueness of the semifield of order q^2 with center containing $\mathbb{F}_q$ [4], and the transitive action of $\mathbb{H}$ on the set of lines external to $Q^+(3,q)$. For $n > 1$, the assertion is a consequence of Theorem 3.6 and prop. 3.9. $\square$

Remark. As is clear from prop. 3.2, the correspondence indicated by Theorem 3.10 can be understood as correspondence between isotopism classes of semifields and pairs formed by an embedded product space and a choice of the kinds of its maximal subspaces.

Every embedding of type $(2,2,5,q)$ is a semifield embedding. So we have

Corollary 3.11. *There is a one-to-one correspondence between equivalence classes of embeddings of type* $(2,2,5,q)$ *and isotopism classes of semifields of order q^3, whose centers contain $\mathbb{F}_q$.*

We are also interested on the algebraic impact of type exchanging by embedding. The correspondence indicated by Theorem 1.1 is not unique. Here we refer to that described in [12]. Suppose that the points of $\mathrm{PG}(n^2+2n,q)$ are coordinatized by $(n+1)\times(n+1)$ matrices over $\mathbb{F}_q$, and that the points of $\mathcal{S}_{n,n}$ correspond to rank one matrices. By means of this coordinatization, the n-subspace δ associated to a semifield $\mathbb{S}$ of order q^{n+1} and center containing $\mathbb{F}_q$, whose structure constants are a_{ijk}, $i,j,k = 1,2,\ldots,n+1$ with respect to a given basis of $\mathbb{F}_{q^{n+1}}$ over $\mathbb{F}_q$, is the span of the matrices $M_i = (c^{(i)}_{jk})$, $i = 1,2,\ldots,n+1$, defined by $c^{(i)}_{jk} = a_{ijk}$ for all i, j and k. So, δ is associated with the subspace of $\mathrm{End}(V(n+1,q))$ of the left multiplications by all elements of $\mathbb{S}$.

Theorem 3.12. *Let ι be a semifield embedding of* $\mathrm{Prod}(n,n,q)$ *and let ξ be the map defined in* (1.2)*. Then the semifields associated with ι and $\xi\iota$ are, up to isotopism, transpose of each other.*

Proof. The automorphism ξ of $\mathrm{Prod}(n,n,q)$ defined in (1.2) induces the transposition Ξ in $\mathrm{PG}(n^2+n,q)$, and clearly δ^Ξ is the $(n+1)$-subspace associated with the semifield $\mathbb{S}^t$. If (S,κ,δ) is a representation of (Σ,ι), then $(S^\Xi,\kappa\Xi,\delta^\Xi)$ is a representation of $(\Sigma,\xi\iota)$. □

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