Classification of 8-dimensional rank two commutative semifields

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Abstract

In this note we classify 8-dimensional rank two commutative semifields (R2CS) over finite fields.

1 Introduction and motivation

A *semifield* is a possibly non-associative algebra with a one and without zero divisors. Finite semifields are well studied objects in combinatorics and finite geometry and have many connections to other interesting geometric structures. They play a central role in the theory of projective planes ([12]), generalised quadrangles ([26]), and polar spaces ([32]), and have applications to perfect nonlinear functions and cryptography ([5]), and maximum rank distance codes ([9]). We refer to the chapter [19] and the references contained therein for background, definitions and more details about these connections.

Of particular interest are commutative semifields which are of rank two over their middle nucleus, so-called *Rank Two Commutative Semifields* (R2CS) (see [8], [3], [19, Section 5]). The property of being commutative implies that these semifields have applications to perfect nonlinear functions (see e.g. [6] for a survey on planar functions and commutative semifields and

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for further references). Moreover, R2CS are equivalent to semifield flocks of a quadratic cone in a 3-dimensional projective space. We refer to the introduction of [21] for an excellent historical overview of the theory of flocks in finite geometry. Consequently, R2CS are also equivalent to translation ovoids of $Q(4,q)$, the parabolic quadric in 4-dimensional projective space. We refer to [11], [22], [14], [13], [2], [15] for further details on these connections. Another, rather remarkable, application of R2CS concerns the theory of eggs and translation generalized quadrangles, see [26, Section 8.7]. As of now the only known examples of eggs in $\mathrm{PG}(4n-1,q)$ are either "elementary", i.e. obtained from an oval or an ovoid by applying the technique of field reduction ([20]), or they are obtained from a R2CS (up to dualising) (see e.g. [13, Chapter 3], [14]).

In this paper we present a computational classification of 8-dimensional rank two commutative finite semifields (that is, 8-dimensional over their centre). This classification relies on the bounds obtained in [1] and [16] on the size of the centre in function of the dimension. Previous classification results have been obtained for 2-dimensional semifields ([10]), for 3-dimensional semifields ([24] and [1]), for 4-dimensional rank two semifields ([7]) and for 6-dimensional rank two semifields with an extra assumption on the size of one of the other nuclei ([23]). Computational classification results have been obtained in [28], and [29]. For an overview and further classification results in the theory of finite semifields we refer to [17, Section 1] and [19, Section 6].

We begin in Section 2 by establishing some basic terminology and giving details on the known examples; we then explain the geometric model we use to search for new examples of rank two commutative semifields. In Section 3 we determine which fields $\mathbb{F}_q$ satisfy a necessary condition to be the centre of an 8-dimensional R2CS, and in Sections 4 and 5 we give the results of our exhaustive search for new examples for the field orders which meet this necessary condition. Finally in Section 6 we give the corresponding existence results for semifield flocks in $\mathrm{PG}(3,q^n)$, translation ovoids of $Q(4,q^n)$, and eggs in $\mathrm{PG}(4n-1,q)$.

2 Preliminaries

We use the notation and terminology from [19]. Given a finite semifield $\mathbb{S}$ with multiplication $(x,y) \mapsto x \circ y$, it is natural to consider the following

substructures. The *left nucleus* $\mathbb{N}(\mathbb{S})$ is the set of elements $x \in \mathbb{S}$ such that for all $y, z \in \mathbb{S}$: $x \circ (y \circ z) = (x \circ y) \circ z$. Analogously, one defines the middle and right nucleus. The intersection of these three subsets of $\mathbb{S}$ is called the nucleus of $\mathbb{S}$ and the intersection of the nucleus of $\mathbb{S}$ with the commutative centre of $\mathbb{S}$ is called the *centre* of $\mathbb{S}$. When we mention the *dimension* of a semifield, we are referring to the dimension over its centre. Restricting the addition and multiplication to any of these substructures one obtains a field. The *rank* of $\mathbb{S}$ is the dimension of $\mathbb{S}$ as a vector space over its middle nucleus. Hence a *rank two commutative semifield* (R2CS) is a semifield with commutative multiplication and which is a two-dimensional vector space over its middle nucleus. A semifield $\mathbb{S}$ is commutative if and only if $\mathbb{S}^d = \mathbb{S}$, and a semifield is called *symplectic* if and only if $[\mathbb{S}^t] = [\mathbb{S}]$.

Semifields are studied up to the isotopism and their Knuth orbit. Two semifields $\mathbb{S}_1$ and $\mathbb{S}_2$ are *isotopic* if there exist non-singular linear maps F, G, H from $\mathbb{S}_1$ to $\mathbb{S}_2$ such that $x^F \circ_2 y^G = (x \circ_1 y)^H$ for all $x, y \in \mathbb{S}_1$. The isotopism class of $\mathbb{S}$ is denoted by $[\mathbb{S}]$. The *Knuth orbit* of a semifield $\mathbb{S}$ is a set of at most six isotopism classes $\mathcal{K}(\mathbb{S}) = \{[\mathbb{S}], [\mathbb{S}^t], [\mathbb{S}^d], [\mathbb{S}^{td}], [\mathbb{S}^{dt}], [\mathbb{S}^{tdt}]\}$, where the operations t and d denote the *transpose* and *dual* operations obtained from the action of the transpositions in the symmetric group S_3 on the indices of the cubical array of structure constants of the semifield.

To fix notation when considering a R2CS $\mathbb{S}$, we will denote the centre by $\mathbb{F}_q$, the finite field with q elements, and we will assume the left nucleus is $\mathbb{F}_{q^n}$. This makes $\mathbb{S}$ into a $2n$-dimensional R2CS of size q^{2n}.

There are only three known examples of R2CS.(Note that by the above definition a finite field is not a R2CS, but in some papers the finite field is also considered as an R2CS.) We give a representation of the corresponding multiplications as binary operations defined on $\mathbb{F}_{q^n} \times \mathbb{F}_{q^n}$ and denoted by juxtaposition $\circ$. Also note that n is necessarily at least 2 since for $n = 1$ one obtains a 2-dimensional semifield which, by Dickson [10], is a field.

The first example goes back to a construction by Dickson in [10] and exists for each odd prime power q and $n \geq 2$:

$$(x, y) \circ (u, v) = (xv + yu, yv + mx^\sigma u^\sigma), \tag{1}$$

where $\sigma \in \mathrm{Aut}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ and m is a non-square in $\mathbb{F}_{q^n}$.

The second family of R2CS was constructed by Cohen and Ganley [8] and exists for $q = 3$ and $n \geq 2$:

$$(x, y) \circ (u, v) = (xv + yu + x^3u^3, yv + \eta x^9u^9 + \eta^{-1}xu), \tag{2}$$

where η is a non-square in $\mathbb{F}_{3^n}$. The third family is the example found by Penttila and Williams [27] for $q = 3$ and $n = 5$ and has multiplication:

$$(x, y) \circ (u, v) = (xv + yu + x^{27}u^{27}, yv + x^9u^9). \tag{3}$$

Cohen and Ganley [8] showed that R2CS in even characteristic don't exist (again note that with our definition the finite field is not an R2CS) and that any R2CS in odd characteristic arises from what we will refer to as a *Cohen–Ganley pair* of functions (f, g): a pair of $\mathbb{F}_q$-linear functions satisfying the property that $g^2(t) - 4tf(t)$ is a non-square for all nonzero $t \in \mathbb{F}_{q^n}$, q odd. Each Cohen–Ganley pair of functions (f, g) gives rise to a semifield $\mathbb{S}(f, g)$ with multiplication

$$(x, y) \circ (u, v) = (xv + yu + f(xu), yv + g(xu)). \tag{4}$$

The condition that $g^2(t) - 4tf(t)$ is a non-square for all nonzero $t \in \mathbb{F}_{q^n}$ is equivalent to the existence of an $\mathbb{F}_q$-linear set $\mathcal{W}$ of rank n whose points have coordinates $(t, f(t), g(t))$, $t \in \mathbb{F}_{q^n}^*$, contained in the set of internal points of the conic with equation $X_2^2 - 4X_0X_1 = 0$.

If $\mathcal{W}$ is contained in a line then the R2CS is of Dickson type. So we are interested in examples where $\mathcal{W}$ contains an $\mathbb{F}_q$-subplane of $\mathrm{PG}(2, q^n)$. Using a computer search, we complete the classification of 8-dimensional R2CSs. This is equivalent to classifying the semifield flocks in $\mathrm{PG}(3, q^4)$ having kernel containing $\mathbb{F}_q$, the translation ovoids of $Q(4, q^4)$ with kernel containing $\mathbb{F}_q$, and good eggs in $\mathrm{PG}(15, q)$. We also classify the 10-dimensional R2CS with center $\mathbb{F}_3$, the semifield flocks in $\mathrm{PG}(3, 3^5)$ with kernel $\mathbb{F}_3$, the translation ovoids in $Q(4, 3^5)$ with kernel $\mathbb{F}_3$, and the good eggs of $\mathrm{PG}(19, 3)$. These applications are detailed in Section 6.

Our work relies on bounds given on the size of the centre, as a function of the dimension, that were first given in [1] and later improved in [16] by showing that in order for an $\mathbb{F}_q$-subplane contained in $\mathcal{I}(\mathcal{C})$ to exist, there must be an $\mathbb{F}_q$-subline contained in an external line of $\mathcal{C}$ and made up entirely of points of $\mathcal{I}(\mathcal{C})$.

Theorem 2.1 ([16]). *There are no $\mathbb{F}_q$-sublines contained in $\ell \cap \mathcal{I}(\mathcal{C})$, where $\mathcal{C}$ is a conic in $\mathrm{PG}(2, q^n)$ and ℓ is an external line to $\mathcal{C}$, for*

$$q \geq 4n^2 - 8n + 2,$$

and for

$$q > 2n^2 - (4 - 2\sqrt{3})n + (3 - 2\sqrt{3})$$

when q is prime.

Corollary 2.2 ([16]). *No $\mathbb{F}_q$-subplane contained in $\mathcal{I}(\mathcal{C})$ exists, where $\mathcal{C}$ is a conic in $\mathrm{PG}(2, q^n)$, for*

$$q \geq 4n^2 - 8n + 2,$$

and for

$$q > 2n^2 - (4 - 2\sqrt{3})n + (3 - 2\sqrt{3})$$

when q is prime.

Let q be odd, and consider the conic $\mathcal{C}$ in $\mathrm{PG}(2, q^n)$ defined by the quadratic form $\mathrm{Q} : X_0X_1 - X_2^2$. Notice that the point $(0, 0, 1)$ lies on the tangent $[1, 0, 0]$, so this point is external. Since $\mathrm{Q}(0, 0, 1) = -1$, we have that the internal points $\mathcal{I}(\mathcal{C})$ are those for which $-\mathrm{Q}(\boldsymbol{v}) \in \boxslash$. The stabilizer $G = \mathrm{PGO}(3, q^n)$ of $\mathcal{C}$ in $\mathrm{PGL}(3, q^n)$ has order $q^n(q^{2n} - 1)$, and contains all matrices of the form

$$\begin{bmatrix} a^2 & b^2 & ab \\ c^2 & d^2 & cd \\ 2ac & abd & ad + bc \end{bmatrix}$$

where $ad - bc \neq 0$ (vector multiplication is from the left).

We have the following, due to Payne [25]:

Theorem 2.3.

1. *G is sharply triply transitive on the points of $\mathcal{C}$;*
2. *G is transitive on $\mathcal{I}(\mathcal{C})$;*
3. *G is transitive on $\mathcal{E}(\mathcal{C})$;*
4. *G is sharply triply transitive on the tangent lines to $\mathcal{C}$;*

5. *G is transitive on the external lines to $\mathcal{C}$;*

6. *G is transitive on the secant lines to $\mathcal{C}$;*

7. *G is transitive on the point-line pairs $(\boldsymbol{p}, \ell)$, where $\boldsymbol{p}$ is an external point on the exterior (resp., secant) line ℓ. The subgroup of G fixing such a pair has order 4.*

8. *G is transitive on the point-line pairs $(\boldsymbol{p}, \ell)$, where $\boldsymbol{p}$ is an internal point on the exterior (resp., secant) line ℓ. The subgroup of G fixing such a pair has order 4.*

3 Existence of sublines contained in $\mathcal{I}(\mathcal{C})$

Our first goal is to determine precisely the values of q for which there exists an $\mathbb{F}_q$-subline contained in an external line to a conic $\mathcal{C}$ in $\mathrm{PG}(2, q^n)$ consisting entirely of internal points of $\mathcal{C}$. To accomplish this we choose η so that $-\eta \in \boxslash$ and $-\eta - 1 \in \square$; then $\boldsymbol{x} = (1, \eta, 0) \in \mathcal{I}(\mathcal{C})$ and $\ell_e = \langle (1, \eta, 0),\ (0, -2\eta, 1) \rangle$ is an external line. The stabilizer in G of $\boldsymbol{x}$ has order $2(q^n + 1)$, and contains the following (normalized) matrices:

$$G_{\boldsymbol{x}} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \right\} \cup \left\{ \begin{bmatrix} a^2 & 1 & a \\ \frac{1}{\eta^2} & a^2 & -\frac{a}{\eta} \\ \pm\frac{2a}{\eta} & \mp 2a & \pm\left(\frac{1}{\eta} - a^2\right) \end{bmatrix} : a \in \mathbb{E} \right\}$$

Now considering the external line $\ell_e = \langle (1, \eta, 0),\ (0, -2\eta, 1) \rangle$ on $\boldsymbol{x}$, the subgroup of G stabilizing both $\boldsymbol{x}$ and ℓ_e has order 4 and is given by

$$G_{\boldsymbol{x}, \ell_e} = \left\langle \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\eta^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{\eta} \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ \frac{1}{\eta^2} & 1 & -\frac{1}{\eta} \\ \frac{2}{\eta} & -2 & \left(\frac{1}{\eta} - 1\right) \end{bmatrix} \right\rangle.$$

Note that the second generator given for this group fixes ℓ_e pointwise.

Since G acts transitively on pairs $(\boldsymbol{p}, \ell)$, where $\boldsymbol{p}$ is an internal point on an external line ℓ, it is sufficient to look for sublines of ℓ_e containing $\boldsymbol{x}$ and contained in $\mathcal{I}(\mathcal{C})$. A subline is determined by three collinear points; so a subline of ℓ_e contained in $\mathcal{I}(\mathcal{C})$ is determined by $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{x} + \mu\boldsymbol{y}$ for

some $\boldsymbol{y} \in (\ell_e \cap \mathcal{I}(\mathcal{C})) \setminus \{\boldsymbol{x}\}$ and $\mu \in \mathbb{F}_{q^n}^*$ satisfying $-\mathrm{Q}(\boldsymbol{x} + \lambda\mu\boldsymbol{y}) \in \boxslash$ for all $\lambda \in \mathbb{F}_q$. The subline determined by these three points is given by $\{\boldsymbol{y}\} \cup \{x + \lambda\mu\boldsymbol{y} \ : \ \lambda \in \mathbb{F}_q\}$.

Using the basis $\{\boldsymbol{v}_1 = (1, \eta, 0),\ \boldsymbol{v}_2 = (0, -2\eta, 1)\}$, we can associate ℓ_e with $\mathrm{PG}(1, q^n)$ having the induced quadratic form $\mathrm{Q}_{\ell_e}(x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2) = \eta x_1^2 - 2\eta x_1 x_2 - x_2^2$ (this form is anisotropic, and is just used to separate the points of ℓ_e into $\mathcal{I}(\mathcal{C})$ and $\mathcal{E}(\mathcal{C})$).

We want to take a point in $(\ell_e \cap \mathcal{I}(\mathcal{C})) \setminus \{\boldsymbol{x}\}$. Since $-\mathrm{Q}(\boldsymbol{v}_2) = 1 \in \square$, $\boldsymbol{v_2} \notin \mathcal{I}(\mathcal{C})$; therefore we will define $\boldsymbol{y}_b = \boldsymbol{v}_1 + b\boldsymbol{v}_2$ with $b \neq 0$. Now we will have $\boldsymbol{y}_b \in \mathcal{I}(\mathcal{C})$ as long as b satisfies $b^2 + 2\eta b - \eta \in \boxslash$. Let $\mathcal{B} = \{s : s \in \mathbb{F}_{q^n} \mid s^2 + 2\eta s - \eta \in \boxslash\}$, then we have that

$$(\ell_e \cap \mathcal{I}(\mathcal{C})) = \{\boldsymbol{x}\} \cup \{\boldsymbol{y}_b : b \in \mathcal{B}\}.$$

Now for our choices of μ, instead of letting μ range over all possible values in $\mathbb{F}_{q^n}^*$, it is sufficient to consider a set $\mathcal{S}$ of representatives of $\mathbb{F}_{q^n}^*/\mathbb{F}_q^*$. Notice that

$$\boldsymbol{x} + \lambda\mu\boldsymbol{y}_b = \boldsymbol{v}_1 + \lambda\mu(\boldsymbol{v}_1 + b\boldsymbol{v}_2) = (1 + \lambda\mu)\boldsymbol{v}_1 + \lambda\mu b\boldsymbol{v}_2;$$

normalizing this vector to $\boldsymbol{v}_1 + \frac{\lambda\mu}{1+\lambda\mu}b\boldsymbol{v}_2$, we see that it is contained in $\mathcal{I}(\mathcal{C})$ if and only if $\frac{\lambda\mu}{1+\lambda\mu}b \in \mathcal{B}$. This is equivalent to having

$$(2b-1)\mu^2\eta\lambda^2 + 2(b-1)\mu\eta\lambda + b^2 - \eta \in \boxslash$$

for all $\lambda \in \mathbb{F}_q$.

To find sublines efficiently, we compute the set $\mathcal{B}$ and, for each value of $\mu \in \mathcal{S}$, the sequence $[\frac{\lambda\mu}{1+\lambda\mu} : \lambda \in \mathbb{F}_q^*]$. Then for each pair $(b, \mu) \in \mathcal{B} \times \mathcal{S}$, we check whether $\frac{\lambda\mu}{1+\lambda\mu}b \in \mathcal{B}$ for all $\lambda \in \mathbb{F}_q$. In this way, we obtain the number of $\mathbb{F}_q$-sublines of ℓ_e containing $\mathbf{x}$ and completely contained in $\mathcal{I}(\mathcal{C})$ for $\mathcal{C}$ a conic in $\mathrm{PG}(2, q^n)$ for $n = 3$ and $n = 4$. We also obtain some partial results for $n = 5$, however it was impossible for us to complete our computations for $q \in \{37, 41, 43, 49\}$ in this case. Our results for $n = 3$ agree with those found in [4]. Notice that from the bounds given by Theorem 2.1, when $n = 3$ we only need to consider $q < 14$; for $n = 4$ we only need to consider $q < 30$; and when $n = 5$ we only need to consider $q < 47$, along with $q = 49$. In the table, a 0 indicates that no subline was found, while a dash indicates that

	Number of sublines on ℓ_e		
q	$n=3$	$n=4$	$n=5$
3	12	120	1200
5	12	600	15072
7	24	912	52080
9	0	1040	91880
11	0	744	115572
13	0	504	102340
17	-	72	≥ 1
19	-	80	≥ 1
23	-	0	≥ 1
25	-	0	≥ 1
27	-	0	≥ 1
29	-	0	≥ 1
31	-	-	≥ 1

the existence is ruled out by the theoretical bound.

Our computational results show the following.

Theorem 3.1. *If there exists an $\mathbb{F}_q$-subline in $\mathrm{PG}(2,q^4)$ contained in $\ell \cap \mathcal{I}(\mathcal{C})$ for some conic $\mathcal{C}$, where ℓ is an external line to $\mathcal{C}$, then $q \leq 19$.*

4 Finding subplanes

Our next goal is to determine, given the existence of the necessary $\mathbb{F}_q$-subline, when there exist $\mathbb{F}_q$-subplanes of $\mathrm{PG}(2,q^n)$ contained in $\mathcal{I}(\mathcal{C})$. An $\mathbb{F}_q$-subplane is completely determined by a quadrangle, so more generally, two $\mathbb{F}_q$-sublines that are not contained in a common line of $\mathrm{PG}(2,q^n)$ will determine an $\mathbb{F}_q$-subplane of $\mathrm{PG}(2,q^n)$.

To determine the existence of $\mathbb{F}_q$-subplanes contained in $\mathcal{I}(\mathcal{C})$, we first fix the point $\boldsymbol{x}$ and then find all of the $\mathbb{F}_q$-sublines containing $\boldsymbol{x}$ which are completely contained in $\mathcal{I}(\mathcal{C})$ (those spanning an external line to $\mathcal{C}$ as well as those spanning a secant line). Then, for each pair of $\mathbb{F}_q$-sublines through $\boldsymbol{x}$ (not

Sublines for 3^4 ((b,μ) pairs) containing $\boldsymbol{x} = (1, \alpha, 0)$
Minimal polynomial of α: $x^4 + 2x^3 + 2$

b	μ	b	μ	b	μ
α^2:	α^{54}, α^{56}	α^5:	α^{27}, α^{48}	α^6:	α
α^{11}:	α^2	α^{13}:	α^{49}	α^{14}:	α^{27}
α^{15}:	α^{30}	α^{16}:	α^{10}	α^{17}:	α^{27}
α^{18}:	α^{27}, α^{72}	α^{20}:	α, α^{37}, α^{48}	α^{21}:	α^5, α^{58}
α^{23}:	α^5, α^{49}	α^{28}:	α^{12}	α^{30}:	α^{23}, α^{54}
α^{32}:	α^{10}	α^{38}:	α^{56}	α^{43}:	α^{30}
α^{44}:	α^{38}	α^{45}:	α^{15}, α^{66}	α^{46}:	α^{13}
α^{47}:	α^{22}	α^{49}:	α^{30}	α^{50}:	α^5
α^{51}:	α^{23}, α^{49}	α^{54}:	α^{27}, α^{75}	α^{55}:	α^{66}
α^{57}:	α^{10}	α^{58}:	α^{30}	α^{60}:	α^{54}
α^{67}:	α^{56}	α^{68}:	α^2, α^{27}, α^{54}	α^{70}:	α^5, α^{10}, α^{73}
α^{72}:	α^{27}	α^{73}:	α^{39}	α^{75}:	α^{75}
α^{76}:	α^{27}				

contained in a common line of $\mathrm{PG}(2, q^n)$), we test whether the $\mathbb{F}_q$-subplane they determine is a subset of $\mathcal{I}(\mathcal{C})$.

In the previous section we give details on finding the $\mathbb{F}_q$-sublines of the external line ℓ_e on $\boldsymbol{x}$ which are contained in $\mathcal{I}(\mathcal{C})$. Once we have these sublines, we compute their images under G_x to get all of the $\mathbb{F}_q$-sublines on $\boldsymbol{x}$ contained in $\mathcal{I}(\mathcal{C})$ generating an external line to $\mathcal{C}$. We then repeat this process beginning with the secant line $\ell_s = \langle (1,0,0), (0,1,0) \rangle$ on $\boldsymbol{x}$. Since all $\mathbb{F}_q$-sublines are assumed to contain $\boldsymbol{x}$, and an $\mathbb{F}_q$-subline is determined by 3 points, we save the sublines as an ordered pair $\{@\boldsymbol{y}, \boldsymbol{z}@\}$ where $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}$ determines the subline.

The real computationally intensive aspect of our work concerns the determination of whether two sublines form a compatible pair, that is, if the two sublines determine a rank 3 $\mathbb{F}_q$-linear set which is contained in $\mathcal{I}(\mathcal{C})$. For two $\mathbb{F}_q$ sublines ℓ_1 and ℓ_2 generated by $\{\boldsymbol{x}, \boldsymbol{y}_1, \boldsymbol{z}_1\}$ and $\{\boldsymbol{x}, \boldsymbol{y}_2, \boldsymbol{z}_2\}$, respectively, we first compute values μ_1 and μ_2 so that the $\mathbb{F}_q$-subplane spanned by these two lines is given by $\langle \boldsymbol{x}, \mu_1 \boldsymbol{y}_1, \mu_2 \boldsymbol{y}_2 \rangle_q$. Then we test that $\lambda \mu_1 \boldsymbol{y}_1 + \boldsymbol{y}_2 \in \mathcal{I}(\mathcal{C})$ for all $\lambda \in \mathbb{F}_q^*$, and that $\boldsymbol{x} + \lambda_1 \mu_1 \boldsymbol{y}_1 + \lambda_2 \mu_2 \boldsymbol{y}_2 \in \mathcal{I}(\mathcal{C})$ for all $\lambda_1, \lambda_2 \in \mathbb{F}_q^*$. If these conditions are satisfied, then ℓ_1 and ℓ_2 generate an $\mathbb{F}_q$-subplane contained in $\mathcal{I}(\mathcal{C})$.

Our computational work proves the following.

Theorem 4.1. *No $\mathbb{F}_q$-subplane contained in $\mathcal{I}(\mathcal{C})$ exists, where $\mathcal{C}$ is a conic in $\mathrm{PG}(2, q^4)$, unless $q = 3$.*

With $n = 4$ and $q = 3$ we find 13 $\mathbb{F}_3$-subplanes (up to conjugacy in $\mathrm{P\Gamma L}(3, 3^4)$) contained in $\mathcal{I}(\mathcal{C})$, 10 of which can be embedded in the linear set associated with the Cohen–Ganley semifield.

5 Linear sets of higher rank

To put together rank 4 $\mathbb{F}_q$-linear sets, we first need to find *all* the rank 3 linear sets (not just the subplanes). It is fairly easy to find the examples that are contained in a line. Each rank 3 linear set is saved as an ordered pair of $\mathbb{F}_q$-linear lines containing $\boldsymbol{x}$. Then, for each $\mathbb{F}_q$-subline contained in either ℓ_e or ℓ_s, we compile the set Π_ℓ of rank 3 $\mathbb{F}_q$-linear sets whose first generating line is ℓ. We form a graph Γ_ℓ on Π_ℓ, where two planes π_1, $\pi_2 \in \Pi_\ell$ are adjacent if their second generating lines together generate a rank 3 $\mathbb{F}_q$-linear set contained in $\mathcal{I}(\mathcal{C})$. Then a clique of size $q(q+1)$ in Γ_ℓ corresponds to a rank 4 $\mathbb{F}_q$-linear set contained in $\mathcal{I}(\mathcal{C})$.

Running this algorithm using the rank 3 linear sets found in $\mathrm{PG}(2, 3^4)$, we find 174 rank 4 $\mathbb{F}_q$-linear sets contained in $\mathcal{I}(\mathcal{C})$ that contain an $\mathbb{F}_q$-subplane. They are all equivalent up to isomorphism, corresponding to a semifield of Cohen–Ganley type. We are also able to run this algorithm in $\mathrm{PG}(2, 3^5)$, using an increased clique size to look for rank 5 $\mathbb{F}_q$-linear sets; here all examples found correspond to a semifield of Cohen–Ganley type or else to the example due to Penttila and Williams.

Theorem 5.1. *An 8-dimensional R2CS is either a Dickson semifield, or of Cohen–Ganley type (with center $\mathbb{F}_3$).*

Theorem 5.2. *A 10-dimensional R2CS with center $\mathbb{F}_3$ is either a Dickson semifield, of Cohen–Ganley type, or Penttila–Williams.*

6 Implications of our results

There are many connections between R2CS and various geometric objects. Here we give details on some of these connections, and state the implications

of our classification of R2CS in these other settings.

6.1 Semifield flocks

A flock of a quadratic cone $\mathcal{K}$ of $\mathrm{PG}(3, q^n)$ having vertex v is a partition of $\mathcal{K} \setminus \{v\}$ into q^n conics. We let $v = (0, 0, 0, 1)$ and let the conic $\mathcal{C}$ in the plane $\pi = [0, 0, 0, 1]$ be the base of the cone. Then the planes of the flock can be written as

$$\{\pi_t : tX_0 + f(t)X_1 + g(t)X_2 + X_3 = 0 \mid t \in \mathbb{F}_{q^n}\}$$

for some $f, g : \mathbb{F}_{q^n} \to \mathbb{F}_{q^n}$; we denote such a flock by $\mathcal{F}(f, g)$.

A flock corresponds to a set

$$\mathcal{W} = \{(t, f(t), g(t)) \mid t \in \mathbb{F}_{q^n}\}$$

of interior points of a conic $\mathcal{C}'$ in $\mathrm{PG}(2, q^n)$ (see [30]). If f and g are linear over a subfield of $\mathbb{F}_{q^n}$ (i.e. if (f, g) is a Cohen–Ganley pair) then we say $\mathcal{F}(f, g)$ is a *linear* flock. The maximal subfield of $\mathbb{F}_{q^n}$ for which f and g are linear is called the *kernel* of the semifield flock. Notice that if $\mathcal{F}(f, g)$ is a semifield flock of $\mathrm{PG}(3, q^n)$ with kernel $\mathbb{F}_q$ then $\mathcal{W}$ is a rank n $\mathbb{F}_q$-linear set contained in the set of interior points of a conic in $\mathrm{PG}(2, q^n)$, so such a semifield flock is equivalent to a $2n$-dimensional R2CS with center $\mathbb{F}_q$. Furthermore if $\mathcal{W}$ is contained in a line then $\mathcal{F}$ is of Kantor–Knuth type [30]; this corresponds to a R2CS of Dickson type.

Corollary 6.1. *A semifield flock of* $\mathrm{PG}(3, q^4)$ *with kernel* $\mathbb{F}_q$ *is of Kantor–Knuth type or of Cohen–Ganley type (with kernel* $\mathbb{F}_3$*).*

Corollary 6.2. *A semifield flock of* $\mathrm{PG}(3, 3^5)$ *with kernel* $\mathbb{F}_3$ *is of Kantor–Knuth type, of Cohen–Ganley type, or of Penttila–Williams type.*

6.2 Ovoids of the parabolic quadric in 4-dimensional projective space

The parabolic quadric $Q(4, s)$ is the incidence structure of points and lines of a nondegenerate quadric in $\mathrm{PG}(4, s)$. The quadric $Q(4, s)$ is also an example

of a generalized quadrangle, and is known to be isomorphic to the example $T_2(\mathcal{C})$ constructed from a conic $\mathcal{C}$ in $\mathrm{PG}(2,s)$, see [26].

A set of s^2+1 points $\mathcal{O}$ of $Q(4,s)$ is called an *ovoid* if no two points of $\mathcal{O}$ are collinear in $Q(4,s)$. An ovoid $\mathcal{O}$ in $Q(4,s)$ is a *translation ovoid* if there is a point $\boldsymbol{p} \in \mathcal{O}$ and a group G of collineations of $Q(4,s)$ stabilizing $\mathcal{O}$, fixing $\boldsymbol{p}$, and acting regularly on the points of $\mathcal{O} \setminus \{\boldsymbol{p}\}$. This group G is necessarily elementary abelian, and hence is a vector space over some subfield $\mathbb{F}_q$ of $\mathbb{F}_s$; the largest such subfield is called the *kernel* of the translation ovoid. Put $n = [\mathbb{F}_s : \mathbb{F}_q]$, so $s = q^n$.

By [16, Section 3.2], the classification result from Theorem 3.1 has the following applications to ovoids of $Q(4,q^4)$. Given an ovoid $\mathcal{O}$ of $Q(4,q^n)$, for each point $\boldsymbol{p} \in \mathcal{O}$, fix some conic $\mathcal{C}_{\boldsymbol{p}}$ contained in the cone $\boldsymbol{p}^\perp \cap Q(4,q^n)$; we will denote the plane containing $\mathcal{C}_{\boldsymbol{p}}$ by $\pi_{\boldsymbol{p}}$. Then we can consider $Q(4,q^n) \simeq T_2(\mathcal{C}_{\boldsymbol{p}})$. In this model, $\boldsymbol{p}$ corresponds to the point (∞), and the points of $\mathcal{O} \setminus \{\boldsymbol{p}\}$ correspond to a set $\mathcal{V}_{\boldsymbol{p}}$ of q^{2n} affine points. Each two points of $\mathcal{V}_{\boldsymbol{p}}$ span a line intersecting the plane $\pi_{\boldsymbol{p}}$ in a point not on $\mathcal{C}_{\boldsymbol{p}}$. Define

$$\mathcal{U}_{\boldsymbol{p}} = \{\langle x,y\rangle \cap \pi_{\boldsymbol{p}} \mid x,y \in \mathcal{V}_{\boldsymbol{p}}\}.$$

If the set $\mathcal{U}_{\boldsymbol{p}}$ contains a dual $\mathbb{F}_q$-subline on an internal point with respect to $\mathcal{C}_{\boldsymbol{p}}$, then dualising over $\mathbb{F}_q$, we have an $\mathbb{F}_q$-subline spanning an external line with respect to $\mathcal{C}_{\boldsymbol{p}}$. This gives us the following.

Corollary 6.3. *If $\mathcal{O}$ is an ovoid in $Q(4,q^4)$, q odd, and $\mathcal{U}_{\boldsymbol{p}}$ contains a dual $\mathbb{F}_q$-subline on an internal point of $\mathcal{C}_{\boldsymbol{p}}$, for some point $\boldsymbol{p} \in \mathcal{O}$, then $q \leq 19$.*

If $\mathcal{O}$ is a translation ovoid of $Q(4,q^n)$ with respect to the point $\boldsymbol{p}$ having kernel $\mathbb{F}_q$, then the set $\mathcal{U}_{\boldsymbol{p}}$ is a rank $2n$ $\mathbb{F}_q$-linear set, and its dual is a rank n $\mathbb{F}_q$-linear set contained in $\mathcal{I}(\mathcal{C}_{\boldsymbol{p}})$.

Corollary 6.4. *A translation ovoid in $Q(4,q^4)$ with kernel $\mathbb{F}_q$ is either a Kantor ovoid, or a Thas–Payne ovoid (with $q=3$).*

Corollary 6.5. *A translation ovoid in $Q(4,3^5)$ with kernel $\mathbb{F}_3$ is either a Kantor ovoid, a Thas–Payne ovoid, or Penttila–Williams ovoid.*

6.3 Eggs

We define an *egg* $\mathcal{E}$ in $\mathrm{PG}(4n-1,q)$ to be a partial $(n-1)$-spread of size $q^{2n}+1$ such that every 3 elements of $\mathcal{E}$ span a $(3n-1)$-space and, for every element

$E \in \mathcal{E}$, there exists a $(3n-1)$-space, denoted T_E and called the *tangent space of $\mathcal{E}$ at E*, containing E and disjoint from every other egg element. An egg is called *good at an element $E \in \mathcal{E}$* if every $(3n-1)$-space containing E and at least two other elements of $\mathcal{E}$ contains exactly q^n+1 elements of $\mathcal{E}$. We say that an egg $\mathcal{E}$ of $\mathrm{PG}(4n-1,q)$ is a *good egg* if there exists an element $E \in \mathcal{E}$ for which $\mathcal{E}$ is good at E. The standard example of an egg in $\mathrm{PG}(4n-1,q)$ is obtained by applying field reduction to an ovoid of $\mathrm{PG}(3,q^n)$; an egg that can be obtained in this way is called *elementary*.

It is shown in [31] (see [18] for a shorter direct proof) that good eggs of $\mathrm{PG}(4n-1,q)$, q odd, are equivalent to semifield flocks of $\mathrm{PG}(3,q^n)$ with kernel containing $\mathbb{F}_q$. This gives us the following result.

Corollary 6.6. *If $\mathcal{E}$ is a good egg of $\mathrm{PG}(15,q)$ with kernel $\mathbb{F}_q$, q odd, then $\mathcal{E}$ is either elementary, of Kantor–Knuth type, or Cohen–Ganley.*

Even if we do not assume that the egg $\mathcal{E}$ has a good element, it is shown in [16] that an egg with certain properties implies the existence of an $\mathbb{F}_q$-subline contained in the set of interior points of a conic $\mathcal{C}$ in $\mathrm{PG}(2,q^n)$ which spans an external line with respect to $\mathcal{C}$, giving the following result.

Corollary 6.7. *Let $\mathcal{E}$ be an egg of $\mathrm{PG}(15,q)$, q odd. If there exists an 11-space ρ containing an elementary pseudo-oval $\mathcal{O}_q$ contained in $\mathcal{E}$ corresponding to a conic $\mathcal{C}$ of $\mathrm{PG}(2,q^4)$, and there is a tangent space intersecting ρ in a 7-space $\mathcal{U}$ whose associated $\mathbb{F}_q$-linear set in $\langle\mathcal{C}\rangle \simeq \mathrm{PG}(2,q^n)$ contains a dual $\mathbb{F}_q$-subline on an internal point w.r.t. $\mathcal{C}$, then $q \leq 19$.*

Corollary 6.8. *If $\mathcal{E}$ is a good egg of $\mathrm{PG}(19,3)$ with kernel $\mathbb{F}_3$, then $\mathcal{E}$ is either elementary, Kantor–Knuth, Cohen–Ganley, or Penttila–Williams.*

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