On eggs and translation generalised quadrangles

Michel Lavrauw
Den Dolech 2
HG 9.53
Technische Universiteit Eindhoven
The Netherlands
lavrauw@win.tue.nl

and

Tim Penttila
Department of mathematics and statistics
University of Western Australia
WA 6907 Australia
penttila@maths.uwa.edu.au

January 14, 2004

Proposed running head: eggs and tgq's.

Abstract

We study eggs in $PG(4n-1,q)$. A new model for eggs is presented in which all known examples are given. We calculate the general form of the dual egg for eggs arising from a semifield flock. Applying this to the egg obtained in L. Bader, G. Lunardon and I. Pinneri [1] from the Penttila-Williams ovoid [11], we obtain the dual egg, which is not isomorphic to any of the previous known examples, see [1]. Furthermore we give a new proof of a conjecture of J.A.Thas [17] using our model, and classify all eggs of $PG(7,2)$ which is equivalent to the classification of all translation generalised quadrangles of order (4,16).

Proofs should be sent to:

Michel Lavrauw
HG 9.53
Den Dolech 2
Technische Universiteit Eindhoven
The Netherlands

lavrauw@win.tue.nl

1. Introduction

Let $PG(n,q)$ denote the projective n-dimensional space over the finite field $GF(q)$ of order q. A *weak egg* $\mathcal{E}$ *of* $PG(2n+m-1,q)$ is a set of q^m+1 $(n-1)$-spaces of $PG(2n+m-1,q)$ such that any three different elements of $\mathcal{E}$ span a $(3n-1)$-space. If each element E of $\mathcal{E}$ is contained in an $(n+m-1)$-dimensional subspace of $PG(2n+m-1,q)$, T_E, which is skew from any element of $\mathcal{E}$ different from E, then $\mathcal{E}$ is called an *egg of* $PG(2n+m-1,q)$. The space T_E is called the *tangent space* of $\mathcal{E}$ at E. The set of tangent spaces of an egg $\mathcal{E}$ is denoted by $T_{\mathcal{E}}$. By projecting the egg from an element onto a $PG(n+m-1,q)$ skew from that element, it is easy to see that the tangent spaces of an egg are uniquely determined by the weak egg. If n=m then an egg $\mathcal{E}$ is called a *pseudo oval* or a *generalised oval*. The only known examples of pseudo ovals are ovals of $PG(2,q^n)$, seen over $GF(q)$. All pseudo ovals of $PG(3n-1,q)$, $q^n \leq 16$ have been classified, see [10]. If $2n=m$ then $\mathcal{E}$ is called a *pseudo ovoid* or a *generalised ovoid*. An ovoid of $PG(3,q^n)$ seen over $GF(q)$ is an example of a pseudo ovoid. Here more examples are known, which will be described later. All known examples of eggs are generalised ovals or generalised ovoids. Following J. A. Thas [16] we call the examples of eggs which are ovals of $PG(2,q^n)$ or ovoids of $PG(3,q^n)$ seen over $GF(q)$ *elementary*. If every four elements of a pseudo ovoid either are contained in a $(3n-1)$-dimensional space or span a $(4n-1)$-dimensional space then the pseudo ovoid is elementary, see [16].

A *generalised quadrangle of order* (s,t) $(GQ(s,t))$, $s>1$, $t>1$, is an incidence structure of points and lines with the properties that any two points are incident with at most one common line, any two lines are incident with at most one common point, every line is incident with $s+1$ points, every point is incident with $t+1$ lines, and given a line l and a point P not incident with l, there is a unique line m and a unique point Q, such that m is incident with P and Q and Q is incident with l. If $s=t$ then we speak of a generalised quadrangle of *order* s $(GQ(s))$. From a $GQ(s,t)$ we get a $GQ(t,s)$ by interchanging the labels point and line, called the *point-line dual* of the generalised quadrangle of order (s,t). For more on generalised quadrangles we refer to [9].

A *translation generalised quadrangle with base point* P (TGQ) is a generalised quadrangle for which there is an abelian group T acting regularly on the points not collinear with P, while fixing every line through P. For more on TGQ's we refer to [9], [17], and [19].

Let $\mathcal{E}$ be an egg of $PG(2n+m-1,q)$. Now embed $PG(2n+m-1,q)$ in a $PG(2n+m,q)$ and construct the incidence structure $T(\mathcal{E})$ as follows. Points are of three types: (i) the points of $PG(2n+m,q) - PG(2n+m-1,q)$; (ii) the $(n+m)$-dimensional subspaces of $PG(2n+m,q)$ which intersect $PG(2n+m-1,q)$ in a tangent space; (iii) the symbol (∞). Lines are of two types: (a) the n-dimensional subspaces of $PG(2n+m,q)$ which intersect $PG(2n+m-1,q)$ in an egg element; (b) the egg elements. Incidence is defined as follows: lines of type (b) are incident with points of type (ii) which contain them and with the point (∞); lines of type (a) are incident with points of type (i) contained in it and with points of type (ii) which contain them.

Theorem 1.1 *(8.7.1 of S. E. Payne and J. A. Thas [9])*
The incidence structure $T(\mathcal{E})$ is a translation generalised quadrangle (TGQ) of order (q^n, q^m) with base point (∞). Conversely, every TGQ is isomorphic to a $T(\mathcal{E})$ for some egg $\mathcal{E}$ of $PG(2n+m-1, q)$. It follows that the theory of TGQ is equivalent to the theory of eggs.

By the following theorem we know that isomorphic eggs give isomorphic TGQ's and conversely.

Theorem 1.2 *(L. Bader, G. Lunardon, I. Pinneri [1])*
Let $\mathcal{E}_1$, $\mathcal{E}_2$ be two eggs of $PG(2n+m-1, q)$. Then there is an isomorphism from $T(\mathcal{E}_1)$ to $T(\mathcal{E}_2)$, which maps the point (∞) to the point (∞) if and only if there is a collineation of $PG(2n+m-1, q)$ which maps $\mathcal{E}_1$ to $\mathcal{E}_2$.

The next theorem gives some restrictions on the parameters m and n of an egg and states a nice property about the tangent spaces. It is proved using the theory of TGQ.

Theorem 1.3 *(8.7.2 of S. E. Payne and J. A. Thas [9])*
If $\mathcal{E}$ is an egg of $PG(2n+m-1, q)$, then

1. *$n = m$ or $n(a+1) = ma$ with a odd.*
2. *If q is even, then $n = m$ or $m = 2n$.*
3. *If $n \neq m$ (resp., $2n = m$), then each point of $PG(2n+m-1, q)$ which is not contained in an egg element belongs to 0 or $q^{m-n}+1$ (resp., to exactly q^n+1) tangent spaces of $\mathcal{E}$.*
4. *If $n \neq m$ the q^m+1 tangent spaces of $\mathcal{E}$ form an egg $\mathcal{E}^D$ in the dual space of $PG(2n+m-1, q)$, called the dual egg. So in addition to $T(\mathcal{E})$ there arises a TGQ $T(\mathcal{E}^D)$, called the translation dual of $T(\mathcal{E})$.*
5. *If $n \neq m$ (resp., $2n = m$), then each hyperplane of $PG(2n+m-1, q)$ which does not contain a tangent space of $\mathcal{E}$ contains 0 or $q^{m-n}+1$ (resp., contains exactly q^n+1) egg elements.*

2. Eggs, 4-gonal families, and q-clans

In this section we only consider eggs of $PG(4n-1, q)$. This is the case $a = 1$ in Theorem 1.3. There are some examples known of pseudo ovoids which are not elementary. They all arise in a similar way which will be explained here. We start with the connection between 4-gonal families, q-clans, and eggs. Put $F = GF(q^n)$.

Let G be a finite group of order s^2t, $1 < s, 1 < t$, together with a family $J = \{A_i \parallel 0 \leq i \leq t\}$ of $1+t$ subgroups of G, each of order s. Assume furthermore that for each $A_i \in J$, there exists a subgroup A_i^* of G of order st containing A_i.

Put $J^* = \{A_i^* \parallel 0 \leq i \leq t\}$. If (i) $A_iA_j \cap A_k = \{1\}$, for distinct i, j, k, and (ii) $A_i^* \cap A_j = \{1\}$, for $i \neq j$, then j is called a *4-gonal family* for G. It was proved by W. M. Kantor [5] that with every 4-gonal family there corresponds a generalised quadrangle (GQ) of order (s, t).

A q^n*-clan* is a set $\{A_t \| t \in F\}$ of q^n two by two matrices over F, such that the difference of each two is anisotropic, i.e., $\alpha(A_t - A_s)\alpha^T = 0$, implies $\alpha = (0, 0)$ or $s = t$. A q^n-clan is *additive* if $A_t + A_s = A_{t+s}$.

Let $\mathcal{C} = \{A_t \| t \in F\}$ be a q^n-clan, put $K_t = A_t + A_t^T$, and define $g_t(\gamma) = \gamma A_t \gamma^T$ and $\gamma^{\delta_t} = \gamma K_t$ for $\gamma \in F^2$. Let $G = \{(\alpha, c, \beta) \parallel \alpha, \beta \in F^2, c \in F\}$, and define a binary operation on G by:

$$(\alpha, c, \beta) * (\alpha', c', \beta') = (\alpha + \alpha', c + c' + \beta{\alpha'}^T, \beta + \beta').$$

This makes G into a group. Let J be the family of subgroups

$$A(t) = \{(\alpha, g_t(\alpha), \alpha^{\delta_t}) \parallel \alpha \in F\},\ t \in F,$$

and

$$A(\infty) = \{(0, 0, \beta) \parallel \beta \in F^2\}.$$

Let J^* be the family of subgroups

$$A^*(t) = \{(\alpha, c, \alpha^{\delta_t}) \parallel \alpha \in F^2, c \in F\},\ t \in F,$$

and

$$A^*(\infty) = \{(0, c, \beta) \parallel c \in F, \beta \in F^2\}.$$

Then the following theorem is a combination of results of S. E. Payne [7], S. E. Payne [8] and W. M. Kantor [4].

Theorem 2.1 *The set J is a 4-gonal family for G if and only if $\mathcal{C}$ is a q^n-clan.*

In [6] S. E. Payne studies 4-gonal families associated with GQ's whose point-line duals are TGQ's. Starting with a 4-gonal family corresponding with a GQ S, he deduces a 4-gonal family corresponding with the TGQ S^D. The following theorem states the connection between additive q^n-clans and eggs and it is a corollary of the work done by S. E. Payne in [6] and Theorem 2.1.

Theorem 2.2 *The set $\mathcal{C} = \{A_t \parallel t \in F\}$ of two by two matrices over F is an additive q^n-clan if and only if the set $\mathcal{E} = \{E(\gamma) \parallel \gamma \in F^2 \cup \{\infty\}\}$, with*

$$E(\gamma) = \{\langle(t, -g_t(\gamma), -\gamma^{\delta_t})\rangle \parallel t \in F\},\ \forall \gamma \in F^2,$$

$$E(\infty) = \{\langle(0, t, (0, 0))\rangle \parallel t \in F\},$$

together with the set $T_{\mathcal{E}} = \{E^*(\gamma) \parallel \gamma \in F^2 \cup \{\infty\}\}$, with

$$E^*(\gamma) = \{\langle(t, \beta\gamma^T + \gamma^{\delta_t}\gamma^T - g_t(\gamma), \beta)\rangle \parallel t \in F,\ \beta \in F^2\},\ \forall \gamma \in F^2,$$

$$E^*(\infty) = \{\langle(0, t, \beta)\rangle \parallel t \in F,\ \beta \in F^2\},$$

forms an egg of $PG(4n-1, q)$, where $g_t(\gamma) = \gamma A_t \gamma^T$ and $\gamma^{\delta_t} = \gamma(A_t + A_t^T)$.

If we are in the situation of the above theorem then, since A_t is additive, we can write A_t as

$$A_t = \sum_{i=0}^{n-1} \begin{pmatrix} a_i & b_i \\ 0 & c_i \end{pmatrix} t^{q^i},$$

for some $a_i, b_i, c_i \in F$. If an egg $\mathcal{E}$ can be written in this form then we denote the egg as $\mathcal{E}(\overline{a}, \overline{b}, \overline{c})$, where $\overline{a} = (a_0, \ldots, a_{n-1})$, $\overline{b} = (b_0, \ldots, b_{n-1})$, and $\overline{c} = (c_0, \ldots, c_{n-1})$. In this case we can deduce the explicit form of the dual egg in terms of $\overline{a}, \overline{b}, \overline{c}$. We need the following lemma.

Lemma 2.3 *Let tr be the trace map from F to $GF(q)$. Then*

$$tr(\sum_{i=0}^{n-1} A_i t^{q^i}) = 0,$$

for all $t \in F$ if and only if

$$\sum_{i=0}^{n-1} A_i^{q^{n-1-i}} = 0.$$

Proof : Because the trace function is additive and $tr(x) = tr(x^q)$, we get

$$tr\left(\sum_{i=0}^{n-1} A_i t^{q^i}\right) = tr\left[\left(\sum_{i=0}^{n-1} A_i^{q^{n-i-1}}\right) t^{q^{n-1}}\right].$$

Since $tr(ax) = 0,\ \forall x \in F$ implies $a = 0$ the proof is complete. □

Theorem 2.4 *The elements of the dual egg $\mathcal{E}^D(\overline{a}, \overline{b}, \overline{c})$ of an egg $\mathcal{E}(\overline{a}, \overline{b}, \overline{c})$ are given by*

$$\tilde{E}(\gamma) = \{\langle(-\tilde{g}_t(\gamma), t, -\gamma t)\rangle \parallel t \in F\},\ \forall\gamma \in F^2,$$

$$\tilde{E}(\infty) = \{\langle(t, 0, (0,0))\rangle \parallel t \in F\},$$

$$\tilde{E}^*(\gamma) = \{\langle(\tilde{f}(\beta, \gamma) + \tilde{g}_t(\gamma), t, \beta)\rangle \parallel t \in F,\ \beta \in F^2\},\ \forall\gamma \in F^2,$$

$$\tilde{E}^*(\infty) = \{\langle(t, 0, \beta)\ \rangle \parallel t \in F,\ \beta \in F^2\},$$

with

$$\tilde{g}_t(a, b) = \sum_{i=0}^{n-1} (a_i a^2 + b_i ab + c_i b^2)^{1/q^i} t^{1/q^i},$$

and

$$\tilde{f}((a, b), (c, d)) = \sum_{i=0}^{n-1} (2a_i ac + b_i(ad + bc) + 2c_i bd)^{1/q^i}.$$

Proof : To find $\tilde{E}(\gamma)$, respectively $\tilde{E}^*(\gamma)$, we calculate the vector space dual of $E^*(\gamma)$, respectively $E(\gamma)$ in $V(4n,q)$ with respect to the inproduct:

$$((x,y,z,w),(x',y',z',w')) \mapsto tr(xx'+yy'+zz'+ww'),$$

where tr is the trace map from $F \to GF(q)$. If (x,y,z,w) is in the vector space dual of $E^*(\gamma)$ then $tr\left[xt + y(\beta\gamma^T + \gamma^{\delta_t}\gamma^T - g_t(\gamma)) + (z,w)\beta^T\right] = 0$, for all $t \in F$, for all $\beta \in F^2$. With $\gamma = (a,b)$ and $\beta = (c,d)$, this is

$$tr\left[xt + y(ac + bd + \gamma^{\delta_t}\gamma^T - g_t(\gamma)) + zc + wd\right] = 0,$$

for all $c,d,t \in F$. For $t = 0$, this equation is satisfied if $w = -by$ and $z = -ay$. Substituting this back into the equation we get that $tr\left[xt + y(\gamma^{\delta_t}\gamma^T - g_t(\gamma))\right] = 0$, for all $t \in F$. Using the formula for g_t and δ_t this is equivalent with

$$tr\left[(x + y(a_0a^2 + b_0ab + c_0b^2))t + \sum_{i=1}^{n-1}(a_ia^2 + b_iab + c_ib^2)t^{q^i}\right] = 0,$$

for all $t \in F$. Using the above lemma, it follows that (x,y,z,w) is of the form

$$(-\sum_{i=0}^{n-1}(a_ia^2 + b_iab + c_ib^2)^{1/q^i}t^{1/q^i}, t, -at, -bt),$$

for some $t \in F$. This proves the form of the elements $\tilde{E}(\gamma)$ of the dual egg. The tangent spaces are obtained in the same way. □

Using the same notation we now present all the known examples and their duals up to isomorphism. We only write down a typical egg element $E(\gamma)$ and dual egg element $\tilde{E}(\gamma)$. They all arise from the theory of TGQ. We exclude the elementary pseudo ovoids.

1. Arising from the Kantor-Knuth semifield flock. The corresponding GQ was first discovered by Kantor [4] in (1986). Here the eggs are self dual as proven by S. E. Payne [6]. They exist for q odd. Let $\sigma \in Aut(F)$, and s a nonsquare in F, then

$$E(\gamma) = \{\langle(t, -a^2t + sb^2t^\sigma, -2at, 2bst^\sigma)\rangle \parallel t \in F\},$$

$$\tilde{E}(\gamma) = \{\langle(-a^2t + (sb^2)^{\sigma^{-1}}t^{\sigma^{-1}}, t, -at, -bt)\rangle \parallel t \in F\}.$$

2. Arising from the Ganley translation plane (1981) [2]. They first appeared in [3]. They are not self dual; here the translation dual was called the *Roman* GQ by S. E. Payne [6]. They exist for $q = 3$. Let s be a nonsquare in F, then

$$E(\gamma) = \{\langle(t, -(a^2-b^2s)t - abt^3 + b^2s^{-1}t^9, at - bt^3, -at^3 - b(st + s^{-1}t^9))\rangle \parallel t \in F\},$$

$$\tilde{E}(\gamma) = \{\langle(-(a^2 - b^2s)t - (ab)^{1/3}t^{1/3} + (b^2s^{-1})^{1/9}t^{1/9}, t, -at, -bt)\rangle \parallel t \in F\}.$$

3. Arising from the Penttila-Williams ovoid of $Q(4,q)$ (1999) [11]. They are not self dual, see L. Bader, G. Lunardon, I. Pinneri [1]. Here $q = 3$ and $n = 5$.

$$E(\gamma) = \{\langle(t, -a^2t - abt^3 + b^2t^{27}, at - bt^3, -at^3 - bt^{27})\rangle \parallel t \in F\}$$

$$\tilde{E}(\gamma) = \{\langle(-a^2t - (ab)^{1/3}t^{1/3} + (b^2)^{1/27}t^{1/27}, t, -at, -bt)\rangle \parallel t \in F\}$$

In the last example the dual egg is new, as was proven in [1].

3. A model for eggs of $PG(4n-1,q)$

Motivated by the previous section and in the spirit of the model for skew translation generalised quadrangles (STGQ) presented in [9], we present a model for a weak egg $\mathcal{E}$ of $PG(4n-1,q)$.

Let $\mathcal{E}(g,\delta)$ be the set $\{E(\gamma) \parallel \gamma \in F^2 \cup \{\infty\}\}$, with

$$E(\gamma) = \{\langle(t, -g_t(\gamma), -\gamma^{\delta_t})\rangle \parallel t \in F\}, \textit{and}$$

$$E(\infty) = \{\langle(0,t,0,0)\rangle \parallel t \in F\},$$

where $g : t \to g_t$, $g_t : F^2 \to F$, and $\delta : t \to \delta_t$, $\delta_t : F^2 \to F^2$.

Working out the conditions for a weak egg we obtain the following theorem.

Theorem 3.1 *The set $\mathcal{E}(g,\delta)$ forms a weak egg of $PG(4n-1,q)$ if and only if*
(i) the functions g and δ are linear in t over $GF(q)$,
(ii) δ_t is a bijection for $t \neq 0$,
(iii) $g_{t_1}(\gamma_1) + g_{t_2}(\gamma_2) \neq g_{t_1+t_2}((\gamma_1^{\delta_{t_1}} + \gamma_2^{\delta_{t_2}})^{\delta^{-1}_{t_1+t_2}})$, for all $t_1 \neq 0$, $t_2 \neq 0$, $t_1 + t_2 \neq 0$, $\gamma_1 \neq \gamma_2$ and $\gamma_1 \neq (\gamma_1^{\delta_{t_1}} + \gamma_2^{\delta_{t_2}})^{\delta^{-1}_{t_1+t_2}} \neq \gamma_2$.

We say that a (weak) egg $\mathcal{E}$ of $PG(4n-1,q)$ is *good at an element* E if every $(3n-1)$-space containing E and at least two other (weak) egg elements, contains exactly q^n+1 (weak) egg elements, see [19]. So an egg of $PG(4n-1,q)$ which is good at an element induces an egg of $PG(3n-1,q)$ in every $(3n-1)$-space containing the good element and at least two other elements of the egg.

Theorem 3.2 *The weak egg $\mathcal{E}$ is good at an element if and only if $\mathcal{E}$ is isomorphic to a weak egg $\mathcal{E}(g,\delta)$ with $\delta_t : \gamma \mapsto \gamma t$.*

Proof : Suppose $\mathcal{E}(g,\delta)$ is a weak egg with $\delta_t : \gamma \mapsto \gamma t$. Projecting $\mathcal{E}(g,\delta)$ from $E(\infty)$ shows that $\mathcal{E}(g,\delta)$ is good at $E(\infty)$. Conversely suppose that $\mathcal{E}$ is a weak egg which is good at an element. Without loss of generality we may assume that $\mathcal{E}$ is of the form $\mathcal{E}(g,\delta)$ and good at $E(\infty)$. Projecting from $E(\infty)$ onto $W = \{\langle(r,0,s,t)\rangle \parallel r,s,t \in F\}$, gives a partial $(n-1)$-spread $\mathcal{P}$ of W. Since $\mathcal{E}(g,\delta)$ is good at $E(\infty)$, every $(2n-1)$-space of W spanned by two elements of $\mathcal{P}$, contains exactly q^n elements of $\mathcal{P}$. If $\mathcal{B}$ is the set of $(2n-1)$-spaces spanned by two elements of $\mathcal{P}$, then with respect to inclusion, the elements of $\mathcal{P}$ and $\mathcal{B}$ form the points and lines of an affine plane $\mathcal{A}$ of order q^n. Let T be the set of points of W, not contained in an element of $\mathcal{P}$. Every two elements of $\mathcal{B}$ necessarily meet in an $(n-1)$-space of W. Two elements of $\mathcal{B}$ which correspond with two parallel lines of $\mathcal{A}$, meet in an $(n-1)$-space contained in T. It follows that all lines belonging to the same parallel class of $\mathcal{A}$, intersect T in a common $(n-1)$-space. Let $\mathcal{L}$ be the set of all these $(n-1)$-spaces of T. Any two elements of $\mathcal{L}$ are disjoint since two non-parallel lines of $\mathcal{A}$ meet in a point of $\mathcal{A}$, i.e., an element of $\mathcal{P}$. This shows that $\mathcal{L}$ partitions the set T. We completed the partial spread $\mathcal{P}$ to a normal spread of W. By a theorem of Segre [13] it follows that the affine plane $\mathcal{A}$ is Desarguesian. This implies that the set $\mathcal{P}$ is isomorphic with the set $\{\{\langle(t,0,-at,-bt)\rangle \parallel t\in F\} \parallel a,b\in F\}$, under a collineation of W. Extending this collineation to a collineation of $PG(4n-1,q)$, the result follows. $\square$

In [17], J. A. Thas proves that, for q odd, every sub GQ that you get from a $(3n-1)$-space on the good element is isomorphic to $Q(4,q^n)$. Together with Theorem 1.2, this implies the following lemma.

Lemma 3.3 *If $\mathcal{E}$ is an egg of $PG(4n-1,q)$, q odd, is good at an element E, then every pseudo oval on E is elementary*

The next theorem proves a conjecture of J. A. Thas [19]. The conjecture was first proved by J. A. Thas in [20], as a corollary of a more general result. Here a shorter direct proof of the conjecture is given.

Theorem 3.4 *An egg $\mathcal{E}$ of $PG(4n-1,q)$, q odd, is good at an element if and only if $T(\mathcal{E})$ is the translation dual of the point-line dual of a semifield flock GQ.*

Proof : Starting from a semifield flock GQ, it follows from Theorem 2.4 and Theorem 3.2 that the egg is good at an element. Conversely, let $\mathcal{E}$ be an egg of $PG(4n-1,q)$ which is good at an element. Without loss of generality we may assume that $\mathcal{E}$ is of the form $\mathcal{E}(g,\delta)$ and good at $E(\infty)$. From Theorem 3.2 it follows that we may assume that $\delta_t:\gamma\mapsto\gamma t$. Define the following $(3n-1)$-spaces:

$$V_a=\{\langle(r,s,-ar,t)\rangle \parallel r,s,t\in F\}, \forall a\in F,$$

$$W_b=\{\langle(r,s,t,-br)\rangle \parallel r,s,t\in F\}, \forall b\in F,$$

$$U=\{\langle(r,s,-at,-bt)\rangle \parallel r,s,t\in F\}.$$

Then every one of these $(3n-1)$-spaces contains exactly q^n+1 egg elements. Hence they intersect $\mathcal{E}(g,\delta)$ in a pseudo oval on $E(\infty)$. Now fix $b\in F$ and consider the pseudo oval $\mathcal{C}_b$ lying in W_b. By the above lemma $\mathcal{C}_b$ is isomorphic to an oval of $PG(2,q^n)$, seen over $GF(q)$. Since q is odd this oval is a conic C (see B. Segre [14]). So we can write the points of C as $\langle(1,f_1x^2+f_2x+f_3,x)\rangle$, for some $f_1,f_2,f_3\in F$. If we look at the points of C as $(n-1)$-spaces over $GF(q)$, then we may write them as

$\{\langle(t,(f_1x^2+f_2x+f_3)t,xt)\rangle \parallel t\in F\}$. The set of these $(n-1)$-spaces is a pseudo oval of $PG(3n-1,q)$. We denote this pseudo oval with $\mathcal{C}$. So there exists a collineation of W_b mapping $\mathcal{C}$ to $\mathcal{C}_b$. The elements of $\mathcal{C}_b$ are of the form $\{\langle(t,-g_t(a,b),-at)\rangle \parallel t\in F\}$, where we omit the last coordinate, since it is fixed in W_b. Without loss of generality we may assume that there exists a collineation $(A,\sigma)\in P\Gamma L(3n,q)$, which maps the $(n-1)$-space $\{\langle(t,(f_1a^2+f_2a+f_3)t,-at)\rangle \parallel t\in F\}$ to $\{\langle(t,-g_t(a,b),-at)\rangle \parallel t\in F\}$, such that

$$A(t,((f_1a^2+f_2a+f_3)t,-at)^T)^\sigma=(t,-g_t(a,b),-at)^T.$$

This implies that $\sigma=1$ and A is of the form

$$\begin{pmatrix} I_n & 0 & 0\\ A_1 & A_2 & A_3\\ 0 & 0 & I_n\end{pmatrix},$$

where I_n is the $(n\times n)$ identity matrix, and A_1,A_2,A_3 are $(n\times n)$ matrices over $GF(q)$. Since every linear operator on F over $GF(q)$ can be represented by a unique q-polynomial over F (see Theorem 9.4.4 in [12]), there exist $\alpha_i,\beta_i,\gamma_i\in F$ such that

$$-g_t(a,b)=\sum_{i=0}^{n-1}\alpha_i t^{q^i}+\sum_{i=0}^{n-1}\beta_i((f_1a^2+f_2a+f_3)t)^{q^i}+\sum_{i=0}^{n-1}\gamma_i(-at)^{q^i}.$$

Simplifying this expression we get that there exist $a_i,b_i,c_i\in F$ such that

$$g_t(a,b)=\sum_{i=0}^{n-1}(a_ia^2+b_ia+c_i)^{q^i}t^{q^i}.$$

This was for a fixed $b\in F$, so the coefficients may depend on b. Repeating the same argument for all $b\in F$, we get that there exist maps a_i,b_i,c_i from F to F such that

$$g_t(a,b)=\sum_{i=0}^{n-1}(a_i(b)a^2+b_i(b)a+c_i(b))^{q^i}t^{q^i}.$$

We can apply the same reasoning to the pseudo ovals contained in the $(3n-1)$-spaces V_a, for all $a\in F$, and for U. So there exist maps d_i,e_i,f_i from F to F, and constants, $u_i,v_i,w_i\in F$, such that

$$g_t(a,b)=\sum_{i=0}^{n-1}(d_i(a)+e_i(a)b+f_i(a)b^2)^{q^i}t^{q^i},$$

and

$$g_t(a,a)=\sum_{i=0}^{n-1}(u_ia^2+v_ia+w_i)^{q^i}t^{q^i}.$$

Consider the pseudo ovals in W_0 and V_0. Their elements are of the form $\{\langle(t,-g_t(a,0),-at,0)\rangle \parallel t\in F\}$ and $\{\langle(t,-g_t(0,b),0,-bt)\rangle \parallel t\in F\}$, respectively. Using a coordinate transformation involving only the first $3n$ coordinates we can get rid of the linear terms (terms with a in) and the constant terms (terms without

a) in $g_t(a,0)$. This only adds constant terms or linear terms to $g_t(a,b)$. Using a coordinate transformation involving the second n and the last n coordinates, we can get rid of the linear terms in $g_t(0,b)$. Again this only adds linear terms to $g_t(a,b)$. We use the same notation for the possible new $g_t(a,b)$. It follows that $g_t(0,0)=0$, which implies that

$$\sum_{i=0}^{n-1}(w_i t)^{q^i} = \sum_{i=0}^{n-1}(c_i(0)t)^{q^i} = \sum_{i=0}^{n-1}(d_i(0)t)^{q^i} = 0.$$

The form of $g_t(a,0)$ and $g_t(0,b)$ implies that

$$\sum_{i=0}^{n-1}(b_i(0)at)^{q^i} = \sum_{i=0}^{n-1}(e_i(0)bt)^{q^i} = 0,$$

and therefore

$$\sum_{i=0}^{n-1}(c_i(b)t)^{q^i} = \sum_{i=0}^{n-1}(f_i(0)b^2 t)^{q^i},$$

and

$$\sum_{i=0}^{n-1}(a_i(0)a^2 t)^{q^i} = \sum_{i=0}^{n-1}(d_i(a)t)^{q^i}.$$

It also follows that the total degree in a and b must be 2 (up to the exponents q_i). This implies that we obtained the following formula for $g_t(a,b)$:

$$g_t(a,b) = \sum_{i=0}^{n-1}(a_i(0)a^2 + b_i(b)a + f_i(0)b^2)^{q^i} t^{q^i},$$

and

$$g_t(a,b) = \sum_{i=0}^{n-1}(a_i(0)a^2 + e_i(a)b + f_i(0)b^2)^{q^i} t^{q^i}.$$

From $g_t(a,a)$ it then follows that we can also replace $b_i(b)a$ and $e_i(a)b$ by a constant times ab. We have shown that there exist constants $a_i, b_i, c_i \in F$ such that $g_t(a,b)$ can be written as

$$g_t(a,b) = \sum_{i=1}^{n-1}(a_i a^2 + b_i ab + c_i b^2)^{q^i} t^{q^i}.$$

Theorem 2.4 implies that $\mathcal{E}(g,\delta)$ is the dual of an egg $\mathcal{E}^D$, such that $T(\mathcal{E}^D)$ is the point-line dual of a semifield flock GQ. □

4. Classification of eggs in $PG(7,2)$

In this section, we show by computer that there is a unique translation generalised quadrangle of order $(4,16)$. This is based on the classification of eggs in $PG(7,2)$. The fundamental, underlying, computer-based result is the following lemma.

Lemma 4.1 *Let E be an egg of $PG(7,2)$, and H be a hyperplane containing no tangent space to E. Then H contains 5 elements of E which span a 5-space.*

Proof : By Theorem 1.3 (5), H contains 5 elements of E. We must show that these span a 5-space. The stabiliser of H in $PGL(8,2)$ is transitive on unordered triples of lines of H spanning a 5-space. It has two orbits on unordered quadruples of lines of H, any triple of which span a 5-space, namely, those for which span a 5-space and those which span H. It has at most three orbits on ordered quintuples of lines of H, any triple of which span a 5-space, and such that the quintuple spans H. We must show that such quintuples cannot be the set of lines of E in H. We use the tangent spaces to E to do this. Since H contains no tangent space to E, each tangent space to E must meet H in a 4-space. Considering each of the three possible quintuples of lines in turn, we find that each has one line lying on four 4-spaces disjoint from the remaining 4 lines, and the other 4 lines lie on two 4-spaces disjoint from the other lines. Thus there are 64 possible tangent structures on intersection with H. Using the fact that the remaining 12 lines of E meet H in a point on no tangent space to E,and on no transversal line to a pair of elements of the quintuple, we can rule out all but one of the quintuples, and only 6 possible tangent structures survive for that quintuple, each of which leaves 14 possible candidates for the 12 points. Finally, these 6 possibilities can be eliminated by noting that there must be no lines joining 2 of the 12 points and meeting an element of the quintuple.

□

Theorem 4.2 *There is a unique translation generalised quadrangle of order* $(4,16)$*, namely, the classical generalised quadrangle* $Q(5,4)$*.*

Proof : By [9], 8.7.1, each translation generalised quadrangle of order $(4,16)$ arises from an egg E of $PG(7,2)$. By the Lemma above, each $PG(5,2)$ containing 3 elements of E contains exactly 5 elements of E. By [9], 8.7.4, it follows that the generalised quadrangle is isomorphic to $T_3(O)$, for some ovoid O of $PG(3,4)$. By a theorem of Barlotti, O is an elliptic quadric. By [9], 3.2.4, the generalised quadrangle is isomorphic to $Q(5,4)$. □

Acknowledgements
The first author thanks the Department of Mathematics and Statistics at the University of Western Australia for an enjoyable stay whilst visiting the second author from February to June, 2000. We would like to thank Maska Law for helpful discussions during that visit.

References

[1] Bader, L.; Lunardon, G.; Pinneri, I. A new semifield flock. *J. Combin. Theory Ser. A* **86** (1999), no. 1, 49–62.

[2] Ganley, Michael J. Central weak nucleus semifields. *European J. Combin.* **2** (1981), no. 4, 339–347.

[3] Gevaert, Hans; Johnson, Norman L. Flocks of quadrativ cones, generalized quadrangles and translation planes. *Geom. Dedicata* **27** (1988), no. 1, 95–98.

[4] Kantor, William M. Some generalized quadrangles with parameters q^2, q. *Math. Z. i* **192** (1986), no. 1, 45–50.

[5] Kantor, W. M. Generalized quadrangles associated with $G_2(q)$ *J. Combin. Theory Ser. A* **29** (1980), no. 2, 212–219.

[6] Payne, S. E. An essay on skew translation generalised quadrangles. *Geom. Dedicata* **32** (1989), no. 1, 93–118.

[7] Payne, S. E. Generalized quadrangles as group coset geometries. *Proceedings of the Eleventh Southeastern Conference on Combinatorics, Graph Theory and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1980), Vol. II. Congr. Numer.* **29** (1980), 717–734.

[8] Payne, S. E. A new infinite family of generalized quadrangles. *Proceedings of the sixteenth Southeastern international conference on combinatorics, graph theory and computing (Boca Raton, Fla., 1985). Congr. Numer.* **49** (1985), 115–128.

[9] Payne, S. E.; Thas, J. A. Finite generalized quadrangles. Research Notes in Mathematics, 110. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1984. vi+312 pp. ISBN:0-273-08655-3.

[10] Penttila, Tim Translation generalised quadrangles and elation Laguerre planes of order 16. European J. Combin., to appear.

[11] Penttila, Tim; Williams, Blair Ovoids in parabolic spaces, Geom. Dedicata, to appear.

[12] Roman, S. Field theory. Graduate Texts in Mathematics, 158. Springer-Verlag, New York, 1995. xii+272 pp. ISBN: 0-387-94407-9; 0-387-94408-7.

[13] Segre, B. Teoria di Galois, fibrazioni proiettive e geometrie non desarguesiane. *Ann. Mat. Pura Appl. (4)* **64** (1964) 1–76.

[14] Segre, B. Ovals in a finite projective plane. *Canad. J. Math.* **7** (1955). 414–416.

[15] Thas, J. A. The m-dimensional projective space $S_m(M_n(\mathrm{GF}(q)))$ over the total matrix algebra $M_n(\mathrm{GF}(q))$ of the $n \times n$-matrices with elements in the Galois field $\mathrm{GF}(q)$. *Rend. Mat.* (6) **4** (1971), 459–532.

[16] Thas, J. A. Geometric characterization of the $[n-1]$-ovaloids of the projective space $\mathrm{PG}(4n-1, q)$. *Simon Stevin* **47** (1974), 97–106.

[17] Thas, J. A. Generalized quadrangles of order (s, s^2). I. *J. Combin. Theory Ser. A* **67** (1994), no. 2, 140–160.

[18] Thas, J. A. Generalized polygons, *Handbook of incidence geometry*, 383–431, *North-Holland*, *Amsterdam*, 1995.

[19] Thas, J. A. Generalized quadrangles of order (s, s^2). II. *J. Combin. Theory Ser. A* **79** (1997), 223–254.

[20] Thas, J. A. Generalized quadrangles of order (s, s^2). III. *J. Combin. Theory Ser. A* **87** (1999), 247–272.