ON THE RANK OF $3 \times 3 \times 3$ -TENSORS

MICHEL LAVRAUW, ANDREA PAVAN AND CORRADO ZANELLA

Abstract. Let U, V and W be finite dimensional vector spaces over the same field. The rank of a tensor τ in $U \otimes V \otimes W$ is the minimum dimension of a subspace of $U \otimes V \otimes W$ containing τ and spanned by fundamental tensors, i.e. tensors of the form $u \otimes v \otimes w$ for some u in U, v in V and w in W. We prove that if U, V and W have dimension three, then the rank of a tensor in $U \otimes V \otimes W$ is at most six, and such a bound cannot be improved in general. Moreover we discuss how the techniques employed in the proof might be extended to prove upper bounds for the rank of a tensor in $U \otimes V \otimes W$ when the dimensions of U, V and W are higher.

1. Introduction

Let U and V be finite dimensional vector spaces over the same field. The *rank* of a tensor τ in $U \otimes V$ is the minimum dimension of a subspace of $U \otimes V$ containing τ and spanned by fundamental tensors, i.e. tensors of the form $u \otimes v$ for some u in U and v in V. The definition makes sense since the fundamental tensors span the whole space, which is finite dimensional. Clearly, a nonzero fundamental tensor has rank one.

The rank of two-fold tensors is well-understood. As an example, it is easy to prove that the maximum rank of a tensor in $U \otimes V$ is precisely the minimum between the dimension m of U and the dimension n of V. Since it only depends on the dimensions, the fact is often expressed saying that the maximum rank of a $m \times n$ -tensor is the minimum between m and n, over any field.

The notion of rank can be extended to any tensor. For example, if W is another finite dimensional vector space over the same field, the rank of a tensor τ in $U \otimes V \otimes W$ is again the minimum dimension of a subspace containing τ and spanned by fundamental tensors. In this context, fundamental tensors are tensors of the form $u \otimes v \otimes w$ with u in U, v in V and w in W.

Here we are interested in the maximum rank of a tensor in a three-fold tensor space. Clearly, such a number only depends on the dimension of the tensor space components and on the ground field; we will focus in particular on the case of an $m \times m \times m$ -tensor.

The maximum rank of a tensor is a natural and elementary question, of interest in various parts of mathematics. For the implications in computational complexity theory, see for instance Chapter 14 of [1] and the references therein. The tensor rank is also of interest in the theory of semifields, due to the relation between bilinear maps and three-fold tensors. In [6] the *tensor rank of a semifield* is defined as the rank of the three-fold tensor which corresponds to the multiplication in the semifield, and it is shown that the tensor rank of a semifield is an invariant of the

The first author acknowledges the support of the Fund for Scientific Research - Flanders (FWO).

isotopism class of the semifield. (Here a *semifield* is a division algebra with an identity element for multiplication, but not necessarily associative.) We will come back to this at the end of Section 2.

To give a precise estimation for the maximum rank of a $m \times m \times m$ -tensor appears to be hard. A trivial upper bound is m^2, as follows from Proposition 1 below. In [2], Atkinson and Stephens proved that the maximum rank over the field of complex numbers is bounded above by $\frac{1}{2}m^2 + O(m)$ and, as far as we know, this is still the best result of its kind.

We concentrate on upper bounds which hold over any field. Elaborating on a technique introduced in [2], in Section 2 we prove that the rank of a $3 \times 3 \times 3$ -tensor is at most six. The bound cannot be improved in general. Note however that, as stated in [2], the maximum rank of a $3 \times 3 \times 3$ -tensor over $\mathbb{C}$ is five. Finally, in Section 3, we discuss how our work relates to [2], and indicate a possible way to extend our results.

2. Main results

In Theorem 1 we prove that the rank of a $3 \times 3 \times 3$ -tensor is at most six over any field. The bound cannot be improved in general. The structure of the proof of the theorem relies on an idea contained in [2], as will be clarified in Section 3.

Let U and V be finite dimensional vector spaces over the same field, and let V^* denote the dual of V. There exists a canonical isomorphism from $U \otimes V$ to $\mathrm{Hom}(V^*, U)$, sending $u \otimes v$ to the linear transformation which in turn associates with the form λ on V the vector $\lambda(v)u$. It is not hard to show that the rank of a tensor in $U \otimes V$ is equal to the dimension of the image of the corresponding linear map. The following lemma provides a similar result for three-fold tensors. We define the *rank of a subspace* S of $U \otimes V$ as the minimum dimension of a subspace of $U \otimes V$ containing S and spanned by fundamental tensors. The following proposition is well known (see e.g. [1, p. 355 (14.9)]); we include a proof for the sake of completeness.

Proposition 1. *Let U, V and W be finite dimensional vector spaces over the same field. Also, let τ be a tensor in $U \otimes V \otimes W$ and let us denote by T the linear transformation from W^* to $U \otimes V$ corresponding to τ. Then the rank of τ is equal to the rank of the image of T.*

Proof. Suppose that $\tau = \sum_i u_i \otimes v_i \otimes w_i$, where u_i is in U, v_i is in V and w_i in W. By construction, T sends any form λ on W to $\sum_i \lambda(w_i)\ u_i \otimes v_i$, hence its image is contained in the span of the $u_i \otimes v_i$'s. This shows that the rank of the image of T is less than or equal to the rank of τ.

To prove the converse inequality, assume that the image of T is contained in the span of the $u_i \otimes v_i$'s. Let $w_1, \dots, w_m$ be a basis for W, and let us denote its dual basis by $\lambda_1, \dots, \lambda_m$. Then there exist scalars α_{ij}'s such that

$$T\lambda_j = \sum_i \alpha_{ij}\ u_i \otimes v_i,$$

and τ is equal to

$$\sum_i u_i \otimes v_i \otimes \left(\sum_j \alpha_{ij} w_j \right).$$

□

Proposition 2. *Let U and V be vector spaces over the same field and having the same dimension m. Also, let $u_1, \ldots, u_m$ be a basis of U and let $v_1, \ldots, v_m$ be a basis of V. Finally, let H be a hyperplane of U. If no u_i belongs to H, then $U \otimes V$ is the direct sum of $\langle \{u_i \otimes v_i : i = 1, \ldots, m\} \rangle$ and $H \otimes V$.*

Proof. Put $W = \langle \{u_i \otimes v_i : i = 1, \ldots, m\} \rangle$. For each v_i, the sum of $\langle u_i \rangle \otimes \langle v_i \rangle$ and $H \otimes \langle v_i \rangle$ is contained in $W + H \otimes V$. Since u_i does not belong to H, such a sum is equal to $U \otimes \langle v_i \rangle$. Recalling that the v_i's form a basis for V, it follows that the sum of W and $H \otimes V$ is the whole $U \otimes V$. Finally, the sum of W and $H \otimes V$ is direct since the sum of the dimension of W and of the dimension of $H \otimes V$ is equal to the dimension of $U \otimes V$.

□

Proposition 3. *Let H be a two-dimensional vector space and let V be a three-dimensional vector space, defined over the same field. Any subspace of $H \otimes V$ of dimension at most two has rank at most three.*

Proof. Let M be a subspace of $H \otimes V$ of dimension at most two, and let τ_1 and τ_2 be tensors in $H \otimes V$ whose span contains M. Note that any tensor in $H \otimes V$ has rank at most two. Therefore, if there exists a non-trivial linear combination of τ_1 and τ_2 having rank one, the thesis follows. Otherwise, let us denote by T_i the linear transformation from H^* to V corresponding to τ_i. In our hypothesis, both the image of T_1 and the image of T_2 have dimension two. Since V is three-dimensional, the intersection of the images is not trivial. Therefore there exist λ_1 and λ_2 in H^* and a non-zero v in V such that $v = T_1\lambda_1 = T_2\lambda_2$.

It turns out that λ_1 and λ_2 form a basis of H. In fact, they are both non-zero. If $\alpha_1\lambda_1 + \alpha_2\lambda_2$ were zero for some scalars α_1 and α_2 not both equal to zero then, setting $\lambda = \alpha_1\lambda_1 = -\alpha_2\lambda_2$, we would have that λ is a non-zero vector in the kernel of $\alpha_2 T_1 + \alpha_1 T_2$, contradicting the assumption that $\alpha_2\tau_1 + \alpha_1\tau_2$ has rank two.

Finally, let us denote by h_1, h_2 the basis of H dual to λ_1, λ_2. Then

$$\tau_1 = (h_1 + h_2) \otimes v + h_2 \otimes \big(T_1\lambda_2 - v\big)$$

and

$$\tau_2 = (h_1 + h_2) \otimes v + h_1 \otimes \big(T_2\lambda_1 - v\big),$$

hence the thesis follows.

□

Proposition 4. *Let U and V be three-dimensional vector spaces over the same field. Also, let u_1, u_2, u_3 be a basis of U and let v_1, v_2, v_3 be a basis of V. For any subspace L of $U \otimes V$ of dimension at most two there exist three fundamental tensors φ_1, φ_2 and φ_3 such that L is contained in the span of the $u_i \otimes v_i$'s and the φ_i's.*

Proof. Let H be a hyperplane of U such that no u_i belongs to it. By Proposition 2 it makes sense to consider the projection M of L onto $H \otimes V$ along $\langle \{u_i \otimes v_i : i = 1, \ldots, 3\} \rangle$. Note that M is a subspace of $H \otimes V$ of dimension at most two, and that it is enough to show that M has rank at most three. Therefore the thesis follows by Proposition 3.

□

The following proposition is a restatement in terms of tensors of a well known result for matrices.

Proposition 5. *Let U and V be vector spaces over the same field and having the same dimension m. For any tensor σ in $U \otimes V$ there exist basis $u_1, \dots, u_m$ of U and $v_1, \dots, v_m$ of V such that σ is in the span of the $u_i \otimes v_i$'s.*

Theorem 1. *The rank of a $3 \times 3 \times 3$ -tensor is at most six over any field.*

Proof. Let U, V and W be three-dimensional vector spaces over the same field, and let τ be a tensor in $U \otimes V \otimes W$. Let us denote by N the image of the linear transformation from W^* to $U \otimes V$ corresponding to τ. By Proposition 1, we have to show that the rank of N is at most six.

The dimension of N is at most three. Therefore we can choose a tensor σ of $U \otimes V$ and a subspace L of $U \otimes V$ of dimension at most two in such a way that N is contained in $\langle \sigma, L \rangle$. By Proposition 5, there exist basis u_1, u_2, u_3 of U and v_1, v_2, v_3 of V such that σ belongs to $\langle \{u_i \otimes v_i : i = 1, \dots, 3\} \rangle$. Now the thesis follows by Proposition 4.

□

Let $\mathbb{F}_q$ be a field of size q, and let $\mathbb{F}_{q^m}$ be a field extension of $\mathbb{F}_q$ degree m. Multiplication of $\mathbb{F}_{q^m}$ is a bilinear map over $\mathbb{F}_q$, which corresponds to a tensor τ in $\mathbb{F}_{q^m} \otimes \mathbb{F}^*_{q^m} \otimes \mathbb{F}^*_{q^m}$, the tensor products being over $\mathbb{F}_q$. It is well-known that the rank of τ is at least $2m$ if $q < 2m - 2$; see for instance Remark 17.30 of [1]. Therefore the rank of τ is six when $q = 2, 3$ and $m = 3$.

We remark that we have also verified that the tensor rank of all nonassociative semifields of order 27 is equal to six. These semifields are classified and the only examples are the so-called twisted fields. Following the notation of [6], the multiplication in such a semifield $\mathbb{S}$ corresponds to a tensor $T_{\mathbb{S}}$ in $\mathbb{F}_3^3 \otimes \mathbb{F}_3^3 \otimes \mathbb{F}_3^3$. After an exhaustive search we found no five-dimensional subspace of $\mathbb{F}_3^3 \otimes \mathbb{F}_3^3 \otimes \mathbb{F}_3^3$ containing $T_{\mathbb{S}}$ and generated by its fundamental tensors. In order to make the search feasible, we used the computer-algebra system GAP [3] and its package FinInG [4], dedicated to finite incidence geometry.

3. Generalizations

The following statement depends on the choice of the ground field $\mathbb{F}$.

Statement 1. *Let U and V be vector spaces of dimension m over $\mathbb{F}$. Also, let $u_1, \dots, u_m$ be a basis for U and let $v_1, \dots, v_m$ be a basis for V. For any subspace L of $U \otimes V$ of dimension at most two, there exist m fundamental tensors $\varphi_1, \dots, \varphi_m$ such that L is in the span of the $u_i \otimes v_i$'s and the φ_i's.*

It was shown in [2] that the statement is true when the ground field is the field of the complex numbers. The proof exploits in a crucial way both the separability properties and the algebraic closeness of $\mathbb{C}$. The interest in Statement 1 is mainly justified by the following proposition.

Proposition 6. *If Statement 1 holds over a field $\mathbb{F}$ then the rank of a $m \times m \times m$ -tensor over $\mathbb{F}$ is at most*

$$\left(\left\lceil \frac{1}{2}(m-1) \right\rceil + 1 \right) m.$$

Proof. Let U, V and W be m-dimensional vector spaces over $\mathbb{F}$, and let τ be a tensor in $U \otimes V \otimes W$. By Proposition 1, the rank of τ is equal to the rank of the image N of the linear transformation from W^* to $U \otimes V$ corresponding to τ.

Since N has dimension at most m, it is contained in some $\langle \sigma, L_1, \ldots, L_k \rangle$, where σ is a tensor in $U \otimes V$, the L_i's are subspaces of $U \otimes V$ of dimension at most two, and $k \leq \lceil \frac{1}{2}(m-1) \rceil$. By Proposition 5, there exist basis $u_1, \ldots, u_m$ of U and $v_1, \ldots, v_m$ of V such that σ is in the span of the $u_i \otimes v_i$'s. Now the thesis follows applying Statement 1 to each of the L_i's.

□

Proposition 6 and its proof are essentially contained in [2]. Proposition 4 says that Statement 1 is true when $m = 3$. Therefore Theorem 1 is a special case of Proposition 6.

To some extent, the proof of Proposition 4 generalizes to a proof of Statement 1, over any ground field. More precisely, we can choose a hyperplane H of U containing no u_i and, by virtue of Proposition 2, consider the projection M of L onto $H \otimes V$ along $\langle \{u_i \otimes v_i : i = 1, \ldots, m\} \rangle$. By Proposition 3, when $m = 3$ any subspace of $H \otimes V$ of dimension at most two has rank at most m. This is not true when $m > 3$. Indeed, the rank of a two-dimensional subspace of a space of $(m-1) \times m$ -tensors over any field can be as big as $\lfloor \frac{3}{2}m - 1 \rfloor$. This is essentially proved in [5]; see in particular Theorem 3.5. However, the discussion shows that Statement 1 for a field $\mathbb{F}$ would be a consequence of the following one.

Statement 2. *Let U and V be vector spaces of dimension m over $\mathbb{F}$. Also, let $u_1, \ldots, u_m$ be a basis for U and let $v_1, \ldots, v_m$ be a basis for V. For any subspace L of $U \otimes V$ of dimension at most two, there exists a hyperplane H of U such that* (i) *no u_i belongs to H and that* (ii) *the projection of L onto $H \otimes V$ along $\langle \{u_i \otimes v_i : i = 1, \ldots, m\} \rangle$ has rank at most m.*

It appears to be unknown whether Statement 2 is true or false. We conclude with a sufficient condition for establishing whether a subspace of dimension at most two in a space of $(m-1) \times m$ -tensors has rank at most m.

Proposition 7. *Let H be an $(m-1)$-dimensional vector space and V be an m-dimensional vector space over $\mathbb{F}$. Also, let σ and τ be tensors in $H \otimes V$ and suppose that the corresponding linear transformations S and T from H^* to V have the property that there exists a basis $\lambda_1, \ldots, \lambda_{m-1}$ of H^* such that $S\lambda_{i-1} = T\lambda_i$ for every $i = 2, \ldots, m-1$. If the size of $\mathbb{F}$ is at least $m-1$, then the rank of $\langle \sigma, \tau \rangle$ is at most m.*

Proof. By hypothesis $\mathbb{F}$ is big enough to ensure the existence of scalars α_i and β_i for each index i between 1 and m, such that $\alpha_i \beta_j \neq \alpha_j \beta_i$ for every $j \neq i$. Moreover, let us choose non zero scalars $c_1, \ldots, c_m$.

For every $i = 1, \ldots, m-1$, denote by h_i the vector in H defined by

$$\lambda_j(h_i) = \alpha_i^{j-1} \beta_i^{m-j-1} c_i$$

for every λ_j. Also, put A equal to

$$\begin{array}{cc} & j \\ i & \begin{bmatrix} & \vdots & \\ \cdots & \alpha_j^{i-1} \beta_j^{m-i} c_j & \cdots \\ & \vdots & \end{bmatrix} \end{array}$$

where i and j are between 1 and m. Its determinant is

$$c_1 \cdots c_m \prod_{i<j} \left(\alpha_i\beta_j - \alpha_j\beta_i\right),$$

hence it is invertible by construction. Put $w_1 = T\lambda_1$, $w_i = T\lambda_i = S\lambda_{i-1}$ for $i = 2, \dots, m-1$ and $w_m = S\lambda_{m-1}$. Finally, set

$$v_i = \sum_j \omega_{ij} w_j,$$

where i and j are between 1 and m, and ω_{ij} denotes the component on the i-th row and j-th column of A^{-1}. Then

$$\sigma = \sum_i \alpha_i \ h_i \otimes v_i \quad \text{and} \quad \tau = \sum_i \beta_i \ h_i \otimes v_i,$$

and the thesis follows.

□

The restriction on the ground field that appears in Proposition 7 cannot be avoided. Using the same notations, let $h_1, \dots, h_{m-1}$ be a basis of H, let $v_1, \dots, v_m$ be a basis of V, and put

$$\sigma = \sum_{i=1}^{m-1} h_i \otimes v_{i+1} \quad \text{and} \quad \tau = \sum_{i=1}^{m-1} h_i \otimes v_i.$$

Denote by S and T the linear transformations corresponding to σ and τ, and by $\lambda_1, \dots, \lambda_{m-1}$ the dual basis of $h_1, \dots, h_{m-1}$. We have that $S\lambda_{i-1} = T\lambda_i$ for every $i = 2, \dots, m-1$. However, the span of σ and τ has rank bigger than m as soon as the ground field has less than $m-1$ elements. This can be shown reversing the arguments employed in the proof of Proposition 7.

Acknowledgement. The authors thank John Sheekey and Rod Gow for interesting conversations on the topic of the paper during their research visit to Vicenza.

References

[1] P. Bürgisser, M. Clausen, M. A. Shokrollahi. *Algebraic Complexity Theory.* Springer, 1997.
[2] M.D. Atkinson, N.M. Stephens. *On the maximal multiplicative complexity of a family of bilinear forms.* Linear Alg. and Appl. 27 (1979), 1-8.
[3] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.12; 2008. (http://www.gap-system.org)
[4] FinInG: a GAP package for Finite Incidence Geometry, 2012. (http://cage.ugent.be/geometry/fining)
[5] J. Ja' Ja'. *Optimal evaluation of pairs of bilinear forms.* Proceedings of the tenth annual ACM symposium on Theory of computing, p.173-183, May 01-03, 1978, San Diego, California, US.
[6] M. Lavrauw. Finite semifields and nonsingular tensors. Preprint.