Good Eggs and Veronese Varieties

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Abstract

We give a new proof of the main theorem of [6] concerning the connection between good eggs in $\mathrm{PG}(4n-1,q)$, q odd, and Veronese varieties, using the model for good eggs in $\mathrm{PG}(4n-1,q)$, q odd, from [2].

Key words: Desarguesian spreads, subgeometries, eggs

1 Introduction

An *egg* $\mathcal{E}$ in $\mathrm{PG}(4n-1,q)$ is a partial $(n-1)$-spread of size $q^{2n}+1$, such that every three egg elements span a $(3n-1)$-space and for every egg element E there exists a $(3n-1)$-space T_E (called the *tangent space of* $\mathcal{E}$ *at* E) which contains E and is skew to the other egg elements. The egg is *good* at an element E if every $(3n-1)$-space which contains E and two other egg elements, contains exactly q^n+1 egg elements. Put $F=\mathrm{GF}(q^n)$, q odd, and let $\mathcal{E}$ be a good egg of $\mathrm{PG}(4n-1,q)$. In [2] it was shown that there exist $a_i,b_i,c_i\in F$, for $i\in\{0,\dots,n-1\}$, such that the elements of $\mathcal{E}$ can be written as

$$E(a,b)=\{\langle -g_t(a,b),t,-at,-bt\rangle \| \; t\in F^*\},\ \forall a,b\in F,$$

$$E(\infty)=\{\langle t,0,0,0\rangle \| \; t\in F^*\},$$

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with

$$g_t(a,b) = \sum_{i=0}^{n-1} (a_i a^2 + b_i ab + c_i b^2)^{1/q^i} t^{1/q^i},$$

and with this notation the egg is good at the element $E(\infty)$. We use the notation $\langle x_1, \ldots, x_d \rangle$ for the projective point corresponding to the vector $(x_1, \ldots, x_d)$ and the egg elements are represented as subsets of $\mathrm{PG}(3, q^n)$. Starting from an ovoid of $\mathrm{PG}(3, q^n)$ one can construct an egg of $\mathrm{PG}(4n-1, q)$, and we call such an egg *elementary* ([5]).

2 A theorem by J. A. Thas

In 1997 J. A. Thas published the paper [6] in which a connection is made between good eggs in $\mathrm{PG}(4n-1, q)$, q odd, and Veronese Varieties in $\mathrm{PG}(5, q^n)$. Here we state the updated version of the main theorem as in [7], but leaving out the connection with translation generalized quadrangles.

Theorem 1 (From Thas [7, Theorem 9.1])
If the egg $\mathcal{E}$ of $\mathrm{PG}(4n-1, q)$, q odd, is good at an element E, then we have one of the following.

(a) There exists a $\mathrm{PG}(3, q^n)$ in the extension $\mathrm{PG}(4n-1, q^n)$ of $\mathrm{PG}(4n-1, q)$ which has exactly one point in common with each of the extensions of the egg elements. The set of these $q^{2n}+1$ points is an elliptic quadric of $\mathrm{PG}(3, q^n)$ and $\mathcal{E}$ is elementary.

(b) We are not in case (a) and there exists a $\mathrm{PG}(4, q^n)$ in $\mathrm{PG}(4n-1, q^n)$ which intersects the extension of E in a line M and which has exactly one point r_i in common with the extension of the other egg elements. Let $\mathcal{W}$ be the set of these intersection points r_i, $i = 1, \ldots, q^{2n}$, and let $\mathcal{M}$ be the set of all common points of M and the conics which contain exactly q^n points of $\mathcal{W}$. Then the set $\mathcal{W} \cup \mathcal{M}$ is the projection of a quadric Veronesean $\mathcal{V}_2^4$ from a point P in a conic plane of $\mathcal{V}_2^4$ onto $\mathrm{PG}(4, q^n)$; the point P is an exterior point of the conic of $\mathcal{V}_2^4$ in the conic plane. In this case the egg $\mathcal{E}$ is isomorphic to the egg of Kantor-type.

(c) We are in case neither (a) nor (b) and there exists a $\mathrm{PG}(5, q^n)$ in $\mathrm{PG}(4n-1, q^n)$ which intersects the extension of E in a plane π, and which has exactly one point r_i in common with the extension of the other egg elements. Let $\mathcal{W}$ be the set of these intersection points r_i, $i = 1, \ldots, q^{2n}$, and let $\mathcal{P}$ be the set of all common points of π and the conics which contain exactly q^n points of $\mathcal{W}$. Then the set $\mathcal{W} \cup \mathcal{P}$ is a quadric Veronesean in $\mathrm{PG}(5, q^n)$

Denote the egg elements by $\{E, E_1, \ldots, E_{q^{2n}}\}$ and let E be the good element. If we project the egg elements from one of its elements onto a $(3n-1)$-space $\mathrm{PG}(3n-1, q)$ skew to that element then we obtain a partial $(n-1)$-spread of size q^{2n}. If we project from E then we can extend this partial $(n-1)$-spread to a Desarguesian $(n-1)$-spread. It was proved by Segre [4] (see also [3]) that this implies that there exists an imaginary plane π in $\mathrm{PG}(4n-1, q^n)$, such that the elements of the Desarguesian spread are the intersections of $\mathrm{PG}(3n-1, q)$ with the subspaces $\langle P, P^\sigma, P^{\sigma^2}, \ldots, P^{\sigma^{n-1}}\rangle$, $P \subset \pi$, where σ is the non-identity collineation of $\mathrm{PG}(4n-1, q^n)$ fixing $\mathrm{PG}(4n-1, q)$ pointwise. Let ρ be the $(n+2)$-space spanned by the good element and π, and let P_i be the intersection of the extension of the egg element E_i with ρ (note that this intersection is indeed a point). Let $\mathcal{W} = \{P_i \| i = 1, \ldots, q^{2n}\}$. Then we will show that one of the following cases occurs.

(a) $\mathcal{W}$ generates a 3-space and then the egg is elementary.

(b) $\mathcal{W}$ generates a 4-space and the egg is of Kantor-type (and $\mathcal{W}$ is the affine part of a projection of a Veronesean of $\mathrm{PG}(5, q^n)$)

(c) $\mathcal{W}$ generates a 5-space and $\mathcal{W}$ is the affine part of a Veronesean in $\mathrm{PG}(5, q^n)$.

3 A new proof

In this section we give a proof of Theorem 1 using the model for good eggs given in the introduction. Let $V(n, q^n)$ denote an n-dimensional vectorspace over $\mathrm{GF}(q^n)$. Let $V(n, q)$ be the vectorspace consisting of vectors of $V(n, q^n)$ with coordinates in $\mathrm{GF}(q)$ with respect to a fixed basis. The egg elements are represented as subsets of $\mathrm{PG}(3, q^n)$. In order to write down the extension of the egg elements to subspaces of $\mathrm{PG}(4n-1, q^n)$, we construct a suitable embedding of $\mathrm{GF}(q^n)$ in $V(n, q^n)$ in the following way. Let X be an element of $\mathrm{G}L(n, q)$ of order $q^n - 1$, and let $\mathbf{v}$ be an eigenvector of X with eigenvalue λ. Then λ is a primitive element of $\mathrm{GF}(q^n)$, and the eigenvalues of X are $\lambda, \lambda^q, \ldots, \lambda^{q^{n-1}}$ with corresponding eigenvectors $\mathbf{v}$, $\mathbf{v}\sigma, \ldots, \mathbf{v}\sigma^{n-1}$, where $\sigma : (x_1, \ldots, x_n) \mapsto (x_1^q, \ldots, x_n^q)$. And so with every $\alpha \in \mathrm{GF}(q^n)^*$ there corresponds a certain power of X, which we denote by $Z(\alpha)$, such that $\alpha\mathbf{v} = \mathbf{v}Z(\alpha)$. Define $\mathbf{e}_{k+1} = \lambda^k\mathbf{v} + \lambda^{kq}\mathbf{v}\sigma + \cdots + \lambda^{kq^{n-1}}\mathbf{v}\sigma^{n-1}$, for $k = 0, \ldots, n-1$. Then $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis of $V(n, q^n)$, consisting of vectors of $V(n, q)$, since $\{\mathbf{v}, \mathbf{v}\sigma, \ldots, \mathbf{v}\sigma^{n-1}\}$ is a basis for $V(n, q^n)$. We define the bijection

$$\alpha = a_1 + a_2\lambda + \cdots + a_n\lambda^{n-1} \mapsto \bar{\alpha} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n$$

between $\mathrm{GF}(q^n)$ and $V(n,q)$. Since

$$\mathbf{e}_1 Z(a_1 + a_2\lambda + \cdots + a_n\lambda^{n-1}) = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n,$$

we have $\overline{\alpha\beta} = \mathbf{e}_1 Z(\alpha\beta) = \mathbf{e}_1 Z(\beta)Z(\alpha) = \overline{\beta}Z(\alpha)$. This implies that $Z(\alpha)$ is the matrix of the linear transformation in $V(n,q)$ corresponding to multiplying by α in $\mathrm{GF}(q^n)$. The automorphism $\alpha \mapsto \alpha^q$ of $\mathrm{GF}(q^n)$ defines the $\mathrm{GF}(q)-$semilinear map A from $V(n,q)$ in itself, such that $Z(\alpha^q) = A^{-1}Z(\alpha)A$. This implies that $\mathbf{v}A^{-1}Z(\alpha)A = \alpha^q\mathbf{v}$, i.e., $\mathbf{v}A^{-1}$ is an eigenvector of $Z(\alpha)$ with eigenvalue α^q, and it follows that $\mathbf{v}A^{-i}$ is an eigenvector of $Z(\alpha)$ with eigenvalue α^{q^i}. We identify the $\mathrm{GF}(q)-$linear map $t \mapsto g_t(a,b) = \sum_{i=0}^{n-1}(\gamma_i t)^{1/q^i}$ (with $\gamma_i = a_i a^2 + b_i ab + c_i b^2$) in $\mathrm{GF}(q^n)$ with the $\mathrm{GF}(q)-$linear map $L_{a,b}$ in $V(n,q^n)$ defined by: $L_{a,b}(\bar{\alpha}) = \bar{\beta}$ if and only if $g_\alpha(a,b) = \beta$, for all $\alpha, \beta \in \mathrm{GF}(q^n)$. Hence $L_{a,b} : \bar{\alpha} \mapsto \sum_{i=0}^{n-1} \bar{\alpha}A^{-i}Z(\gamma_i^{1/q^i})$, and $L_{a,b}(\mathbf{v}) = \sum_{i=0}^{n-1}\gamma_i\mathbf{v}A^{-i}$. Now we can write down the extension of the egg elements as

$$\bar{E}(a,b) = \{\langle -L_{a,b}(\mathbf{w}), \mathbf{w}, \mathbf{w}Z(-a), \mathbf{w}Z(-b)\rangle \| \mathbf{w} \in V(n,q^n)\},$$

for all $a, b \in \mathrm{GF}(q^n)$. By projecting the egg elements from the good element onto $\mathrm{PG}(3n-1,q) = \{\langle 0,r,s,t\rangle \| (r,s,t) \in (\mathrm{GF}(q^n)^3)^*\}$, the Desarguesian spread obtained this way, corresponds to the imaginary plane π generated by $\langle 0,\mathbf{v},0,0\rangle$, $\langle 0,0,\mathbf{v},0\rangle$, $\langle 0,0,0,\mathbf{v}\rangle$. Eventually we find that the point P_i (with $E_i = E(a,b)$) has coordinates $\langle -L_{a,b}(\mathbf{v}), \mathbf{v}, -a\mathbf{v}, -b\mathbf{v}\rangle$, and hence

$$\mathcal{W} = \{\langle -\sum_{i=0}^{n-1}(a_i a^2 + b_i ab + c_i b^2)\mathbf{v}A^{-i}, \mathbf{v}, -a\mathbf{v}, -b\mathbf{v}\rangle \| a, b \in \mathrm{GF}(q^n)\}.$$

If $(a_0, a_1, \ldots, a_{n-1})$ or $(c_0, c_1, \ldots, c_{n-1})$ is $\mathbf{0}$ then one easily sees that the tangent space at $E(\infty)$ intersects one of the other egg elements, contradicting the definition of an egg. If $(b_0, b_1, \ldots, b_{n-1}) \neq \mathbf{0}$, then $\mathcal{W}$ is contained in the subspace spanned by $Q_1 := \langle -\sum_{i=0}^{n-1} a_i\mathbf{v}A^{-i}, 0,0,0\rangle$, $Q_2 := \langle -\sum_{i=0}^{n-1} b_i\mathbf{v}A^{-i}, 0,0,0\rangle$, $Q_3 := \langle -\sum_{i=0}^{n-1} c_i\mathbf{v}A^{-i}, 0,0,0\rangle$, $Q_4 := \langle 0,\mathbf{v},0,0\rangle$, $Q_5 := \langle 0,0,-\mathbf{v},0\rangle$, and $Q_6 := \langle 0,0,0,-\mathbf{v}\rangle$. If the dimension of $U := \langle Q_1,Q_2,Q_3,Q_4,Q_5,Q_6\rangle$ is 5 then by taking $\{Q_1,Q_2,Q_3,Q_4,Q_5,Q_6\}$ as a basis for U the points of $\mathcal{W}$ have coordinates $\langle a^2, ab, b^2, 1, a, b\rangle$, $a, b \in \mathrm{GF}(q^n)$. This is the affine part of the Veronesean $\mathcal{V}_2^4$, and this proves part (c) of the theorem. If U has dimension 4 then, since we may assume $(a_0, a_1, \ldots, a_{n-1}) = (1, 0, \ldots, 0)$ (see e.g. Remark 1.3 in [1]), there exists a $\gamma \in \mathrm{GF}(q^n)^*$ such that $(b_1, b_2, \ldots, b_{n-1}) = \gamma(c_1, c_2, \ldots, c_{n-1})$. But then it is a straightforward calculation to see that the tangent space at $E(0,1)$ contains an element of the Desarguesian spread induced by the q^n+1 egg elements contained in $\langle E(\infty), E(0,0), E(1,0)\rangle$ (see e.g. the proof of Theorem 4.2 in [1]). By [1, Theorem 4.1] we may conclude that the egg is of Kantor type. Choosing $\{Q_1,Q_2,Q_4,Q_5,Q_6\}$ as a basis for U, we see that $\mathcal{W}$ is a projection of the Veronesean from the point $\langle 0,-\gamma,1,0,0,0\rangle$ onto the the hyperplane with equation $X_2 = 0$. This proves part (b). If U is 3-dimensional then again with $(a_0, a_1, \ldots, a_{n-1}) = (1, 0, \ldots, 0)$, $b_i = c_i = 0$ for

$i > 0$. It is easy to see that then the egg is elementary and the set $\mathcal{W}$ is the set of affine points on an elliptic quadric. If $(b_0, b_1, \ldots, b_{n-1}) = \mathbf{0}$, then by using completely the same arguments as above one easily sees that the egg is either of Kantor type or elementary.

References

[1] Michel Lavrauw; Characterisations and properties of good eggs in PG$(4n-1, q)$, q odd, to appear in *Discrete Math.*

[2] Michel Lavrauw, Tim Penttila; On eggs and translation generalised quadrangles. *J. Combin. Theory Ser. A.* **96** (2001), no. 2, 303–315.

[3] G. Lunardon; Insiemi indicatori proiettivi e fibrazioni planari di uno spazio proiettivo finito. *Boll. Un. Mat. Ital.*, (6), **3**-B (1984), 717-735.

[4] B. Segre; Teoria di Galois, fibrazioni proiettive e geometrie non desarguesiane. *Ann. Mat. Pura Appl.* (4) **64** (1964), 1–76.

[5] J. A. Thas; Geometric characterization of the $[n-1]$-ovaloids of the projective space PG$(4n-1, q)$. *Simon Stevin* **47** (1973/74), 97–106.

[6] J. A. Thas; Generalized quadrangles of order (s, s^2). II. *J. Combin. Theory Ser. A* **79** (1997), 223–254.

[7] J. A. Thas; Generalized quadrangles of order (s, s^2). III. *J. Combin. Theory Ser. A* **87** (1999), 247–272.