Eggs in $\mathrm{PG}(4n-1,q)$, q even, containing a pseudo pointed conic

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Abstract

An ovoid of $\mathrm{PG}(3,q)$ can be defined as a set of q^2+1 points with the property that every three points span a plane and at every point there is a unique tangent plane. In 2000 M. R. Brown ([5]) proved that if an ovoid of $\mathrm{PG}(3,q)$, q even, contains a pointed conic, then either $q=4$ and the ovoid is an elliptic quadric, or $q=8$ and the ovoid is a Tits ovoid. Generalising the definition of an ovoid to a set of $(n-1)$-spaces of $\mathrm{PG}(4n-1,q)$ J. A. Thas [24] introduced the notion of pseudo-ovoids or eggs: a set of $q^{2n}+1$ $(n-1)$-spaces in $\mathrm{PG}(4n-1,q)$, with the property that any three egg elements span a $(3n-1)$-space and at every egg element there is a unique tangent $(3n-1)$-space. We prove that an egg in $\mathrm{PG}(4n-1,q)$, q even, contains a pseudo pointed conic, that is, a pseudo-oval arising from a pointed conic of $\mathrm{PG}(2,q^n)$, q even, if and only if the egg is elementary and the ovoid is either an elliptic quadric in $\mathrm{PG}(3,4)$ or a Tits ovoid in $\mathrm{PG}(3,8)$.

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1. Introduction and preliminaries

An *oval* $\mathcal{O}$ of $\mathrm{PG}(2,q)$ is a set of $q+1$ points no three collinear. In 1954 it was shown by B. Segre [23] that if q is odd then an oval in $\mathrm{PG}(2,q)$ is a conic. For q even, many ovals are known which are not conics (see [6] for a recent survey). If q is even, then the tangents to $\mathcal{O}$ are coincident in a fixed point called the *nucleus* of $\mathcal{O}$ (see [13, Lemma 8.6]). A *pointed conic* of $\mathrm{PG}(2,q)$, q even, is any oval constructed by taking a conic, removing any point and then including the nucleus of the conic. As the group of the conic is transitive on the points of the conic, all pointed conics are projectively equivalent to the oval $\{(1,t,\sqrt{t}) \,:\, t \in \mathrm{GF}(q)\} \cup \{(0,0,1)\}$. We note that a pointed conic of $\mathrm{PG}(2,q)$ is a conic if and only if $q=4$.

An *ovoid* of $\mathrm{PG}(3,q)$ is a set of q^2+1 points such that every three points span a plane. If we exclude $\mathrm{PG}(3,2)$, that is, assuming $q>2$, then q^2+1 is the maximal cardinality of a set of points satisfying this property. Moreover all the tangent lines to an ovoid at a certain point lie in a plane ([2], [20]); the *tangent plane* at that point. In 1955 A. Barlotti [2] and G. Panella [20] independently proved that an ovoid in $\mathrm{PG}(3,q)$, q odd, is an elliptic quadric. For q even, one other example of an ovoid is known; called the Tits ovoid, which exists for $q=2^{2e+1}$, $e \geq 1$. For results characterising the elliptic quadric and the Tits ovoid we refer to the survey [6].

A result fundamental to the proof of the main result of this paper is the following characterisation of an ovoid containing a pointed conic.

Theorem 1.1 (M. R. Brown [5]). *Let $\mathcal{O}$ be an ovoid of* $\mathrm{PG}(3,q)$*, q even, and π a plane of* $\mathrm{PG}(3,q)$ *such that $\pi \cap \mathcal{O}$ is a pointed conic. Then either $q=4$ and $\mathcal{O}$ is an elliptic quadric or $q=8$ and $\mathcal{O}$ is a Tits ovoid.*

An $(n-1)$*-spread* (*partial $(n-1)$-spread*) $\mathcal{S}$ of $\mathrm{PG}(rn-1,q)$ is a set of $(n-1)$-spaces such that any point of $\mathrm{PG}(rn-1,q)$ is contained in exactly (at most) one element of $\mathcal{S}$ (also called a *spread* if the dimension of the elements of $\mathcal{S}$ is understood). A spread $\mathcal{S}$ is called *Desarguesian* if the incidence geometry defined by taking the elements of $\mathcal{S}$ as points, the subspaces spanned by two different elements of $\mathcal{S}$ as lines, and the natural incidence relation (symmetric containment), is isomorphic to a Desarguesian projective space.

An *egg* $\mathcal{E}$ in $\mathrm{PG}(4n-1,q)$ (or *pseudo-ovoid*) is a partial $(n-1)$-spread of size $q^{2n}+1$, such that every three egg elements span a $(3n-1)$-space and for every egg element E there exists a $(3n-1)$-space T_E (called the *tangent space of $\mathcal{E}$ at E*) which contains E and is skew from the other egg elements. A pseudo-oval (or an *egg in* $\mathrm{PG}(3n-1,q)$) is a partial $(n-1)$-spread of size q^n+1, such that every three elements of the pseudo-oval span $\mathrm{PG}(3n-1,q)$. The notion of eggs was introduced by J. A. Thas in 1971 ([24]). An egg $\mathcal{E}$ in $\mathrm{PG}(4n-1,q)$ is called *a good egg* if there is exists an egg element E such that every $(3n-1)$-space containing E and two other egg elements contains exactly q^n+1 egg elements. In that case E is called a *good element* of $\mathcal{E}$. If the elements of a pseudo-ovoid, respectively pseudo-oval, belong to a Desarguesian $(n-1)$-spread of $\mathrm{PG}(4n-1,q)$, respectively $\mathrm{PG}(3n-1,q)$, then the pseudo-ovoid, respectively pseudo-oval, is called *elementary.* It follows that an elementary pseudo-oval arises from an oval of $\mathrm{PG}(2,q^n)$ and an elementary pseudo-ovoid arises from an ovoid of $\mathrm{PG}(3,q^n)$. If the oval is a conic, then we say that the elementary pseudo-oval is a *pseudo-conic* or a *classical pseudo-oval,* while if the oval is a pointed conic we say that the elementary pseudo-oval is a *pseudo pointed conic.* If the ovoid is an elliptic quadric, then we call the pseudo-ovoid a *classical pseudo-ovoid.* In 1974 J. A. Thas proved that if every four egg elements span $\mathrm{PG}(4n-1,q)$ or are contained in a $(3n-1)$-dimensional space, then the egg is elementary ([25]).

The only known examples of pseudo-ovals are elementary and pseudo-ovals have been classified by computer for $q^n \leq 16$ ([22]). More examples are known for pseudo-ovoids, all of them over a field of odd characteristic and they are connected to certain semifields (see Chapter 3 of [15] for a survey and [16] for recent results for the case when q is odd).

In this article we are concerned about pseudo-ovoids in the case when q is even. All known examples of eggs in $\mathrm{PG}(4n-1, q)$, q even, are elementary. Pseudo-ovoids have been classified by computer for $q^n \leq 4$ ([17]). In [10] M. R. Brown and M. Lavrauw prove the following theorem.

Theorem 1.2. *An egg $\mathcal{E}$ in $\mathrm{PG}(4n-1, q)$, q even, contains a pseudo-conic if an only if the egg is classical, that is arising from an elliptic quadric in $\mathrm{PG}(3, q^n)$.*

In this article we prove that if an egg in $\mathrm{PG}(4n-1, q)$, q even, contains a pseudo pointed conic, then the egg is elementary and either $q = 4$ and the egg comes from an elliptic quadric, or $q = 8$ and the egg comes from a Tits ovoid.

2. Eggs and translation generalized quadrangles

A (finite) *generalized quadrangle* (GQ) (see [21] for a comprehensive introduction) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint (non-empty) sets of objects called *points* and *lines*, respectively, and for which $\mathrm{I} \subseteq (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$ is a symmetric point-line incidence relation satisfying the following axioms:

(i) Each point is incident with $1 + t$ lines ($t \geq 1$) and two distinct points are incident with at most one line;

(ii) Each line is incident with $1 + s$ points ($s \geq 1$) and two distinct lines are incident with at most one point;

(iii) If X is a point and ℓ is a line not incident with X, then there is a unique pair $(Y, m) \in \mathcal{P} \times \mathcal{B}$ for which $X \mathrm{~I~} m \mathrm{~I~} Y \mathrm{~I~} \ell$.

The integers s and t are the *parameters* of the GQ and $\mathcal{S}$ is said to have *order* (s, t). If $s = t$, then $\mathcal{S}$ is said to have order s. If $\mathcal{S}$ has order (s, t), then it follows that $|\mathcal{P}| = (s+1)(st+1)$ and $|\mathcal{B}| = (t+1)(st+1)$ ([21, 1.2.1]). A *subquadrangle* $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ of $\mathcal{S}$ is a GQ such that $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{B}' \subseteq \mathcal{B}$ and I' is the restriction of I to $(\mathcal{P}' \times \mathcal{B}') \cup (\mathcal{B}' \times \mathcal{P}')$. Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s, t), $s \neq 1$, $t \neq 1$. A collineation θ of $\mathcal{S}$ is an *elation* about the point P if $\theta = id$ or if θ fixes all lines incident with P and fixes no point of $\mathcal{P} \setminus P^{\perp}$. If there is a group G of elations about P acting regularly on $\mathcal{P} \setminus P^{\perp}$, then we say that $\mathcal{S}$ is an *elation generalized quadrangle* (EGQ) with *elation group* G and *base point* P. Briefly we say that $(\mathcal{S}^{(P)}, G)$ or $\mathcal{S}^{(P)}$ is an EGQ. If the group G is abelian, then we say that the EGQ $(\mathcal{S}^{(P)}, G)$ is a *translation generalized quadrangle* (TGQ) and G is the *translation group*.

In $\mathrm{PG}(2n+m-1, q)$ consider a set $\mathcal{E}(n, m, q)$ of $q^m + 1$ $(n-1)$-dimensional subspaces, every three of which generate a $\mathrm{PG}(3n-1, q)$ and such that each element E of $\mathcal{E}(n, m, q)$ is contained in an $(n+m-1)$-dimensional subspace T_E having no point in common with any element of $\mathcal{E}(n, m, q) \setminus \{E\}$. It is easy to check that T_E is uniquely determined for any element E of $\mathcal{E}(n, m, q)$. The space T_E is called the *tangent space of* $\mathcal{E}(n, m, q)$ *at* E. For $n = m = 1$ such a set $\mathcal{E}(1, 1, q)$ is an oval in $\mathrm{PG}(2, q)$ and more generally for $n = m$ such a set $\mathcal{E}(n, n, q)$ is a *pseudo-oval* of $\mathrm{PG}(3n-1, q)$. For $m = 2n = 2$ such a set

$\mathcal{E}(1,2,q)$ is an ovoid of $\mathrm{PG}(3,q)$ and more generally for $m=2n$ such a set $\mathcal{E}(n,2n,q)$ is a *pseudo-ovoid*. In general we call the sets $\mathcal{E}(n,m,q)$ *eggs*.

Now embed $\mathrm{PG}(2n+m-1,q)$ in a $\mathrm{PG}(2n+m,q)$, and construct a point-line geometry $T(n,m,q)$ as follows.

Points are of three types:

(i) the points of $\mathrm{PG}(2n+m,q)\setminus\mathrm{PG}(2n+m-1,q)$, called the *affine points*;

(ii) the $(n+m)$-dimensional subspaces of $\mathrm{PG}(2n+m,q)$ which intersect $\mathrm{PG}(2n+m-1,q)$ in a tangent space of $\mathcal{E}(n,m,q)$;

(iii) the symbol (∞).

Lines are of two types:

(a) the n-dimensional subspaces of $\mathrm{PG}(2n+m,q)$ which intersect $\mathrm{PG}(2n+m-1,q)$ in an element of $\mathcal{E}(n,m,q)$;

(b) the elements of $\mathcal{E}(n,m,q)$.

Incidence in $T(n,m,q)$ is defined as follows. A point of type (i) is incident only with lines of type (a); here the incidence is that of $\mathrm{PG}(2n+m,q)$. A point of type (ii) is incident with all lines of type (a) contained in it and with the unique element of $\mathcal{E}(n,m,q)$ contained in it. The point (∞) is incident with no line of type (a) and with all lines of type (b).

Theorem 2.1 (8.7.1 of Payne and Thas [21]). *The incidence geometry $T(n,m,q)$ is a TGQ of order (q^n,q^m) with base point (∞). Conversely, every TGQ is isomorphic to a $T(n,m,q)$. It follows that the theory of TGQ is equivalent to the theory of the sets $\mathcal{E}(n,m,q)$.*

In the case where $n=m=1$ and $\mathcal{E}(1,1,q)$ is the oval $\mathcal{O}$ the GQ $T(1,1,q)$ is the Tits GQ $T_2(\mathcal{O})$. When $m=2n=2$ and $\mathcal{E}(1,2,q)$ is the ovoid Ω, the GQ $T(1,2,q)$ is the Tits GQ $T_3(\Omega)$. Note that $T_2(\mathcal{O})\cong Q(4,q)$, if and only if $\mathcal{O}$ is a conic and non-classical otherwise, while $T_3(\Omega)\cong Q(5,q)$ if and only if Ω is an elliptic quadric (see [21, Chapter 3]).

The *kernel* of $\mathcal{S}=T(n,m,q)$ is the field $\mathrm{GF}(q')$ for which there exists an $O(n',m',q')$ representing $\mathcal{S}$ and $\mathcal{S}$ may be represented by an $\mathcal{E}(n'',m'',q'')$ if and only if $\mathrm{GF}(q'')\subseteq\mathrm{GF}(q')$ (see [21, Chapter 8]).

Let $\mathcal{E}$ be an egg in $\mathrm{PG}(4n-1,q)$ and $T(\mathcal{E})$ the corresponding TGQ. If $\mathcal{O}$ is a pseudo-oval of $\mathcal{E}$ contained in $\mathrm{PG}(3n-1,q)$ and $\mathrm{PG}(3n,q)$ any subspace containing $\mathrm{PG}(3n-1,q)$ not contained in $\mathrm{PG}(4n-1,q)$, then $\mathrm{PG}(3n,q)$ induces a subquadrangle $\mathcal{T}(\mathcal{O})$ of $T(\mathcal{E})$.

3. Ovoids of GQs and subquadrangles

Let $\mathcal{S}=(\mathcal{P},\mathcal{B},\mathrm{I})$ be a GQ of order (s,t). An *ovoid* Ω of $\mathcal{S}$ is a set of points of $\mathcal{S}$ such that each line of $\mathcal{S}$ is incident with precisely one point of Ω. It follows that $|\Omega|=st+1$.

Now suppose that $\mathcal{S}=(\mathcal{P},\mathcal{B},\mathrm{I})$ is a GQ of order (s,t) and $\mathcal{S}'=(\mathcal{P}',\mathcal{B}',\mathrm{I}')$ a subquadrangle of $\mathcal{S}$ of order (s,t'). If $P\in\mathcal{P}\setminus\mathcal{P}'$ is incident with no line of $\mathcal{S}'$, then it is called an *external* point of $\mathcal{S}'$. Each external point of $\mathcal{S}'$ is collinear with the $st'+1$ points of an ovoid of $\mathcal{S}'$ (see [21, 2.2.1]). Such an ovoid is said to be *subtended* by the external point.

In the particular case where $\mathcal{S}$ has order (s,s^2) and $\mathcal{S}'$ is a subquadrangle of order s, each point of $\mathcal{P}\setminus\mathcal{P}'$ is an external point of $\mathcal{S}'$. By a result of Bose and Shrikhande ([3]) any three points of $\mathcal{S}$ pairwise non-collinear (a *triad*) have exactly $s+1$ common collinear points. Hence an ovoid of $\mathcal{S}'$ may be subtended by at most two points of $\mathcal{P}\setminus\mathcal{P}'$. In this case the ovoid is said to be *doubly subtended*.

Now suppose that Ω is an ovoid of $\mathrm{PG}(3,q)$ containing an oval $\mathcal{O}$. If we construct the Tits GQ $T_2(\mathcal{O})$, then the set of points

$$\overline{\Omega} = (\Omega\setminus\mathcal{O})\cup\{\pi_P : \pi_P \text{ is the tangent plane to } \Omega \text{ at the point } P\in\mathcal{O}\},$$

is an ovoid of $T_2(\mathcal{O})$ (see [7]). Such an ovoid of $T_2(\mathcal{O})$ is called a *projective ovoid*. If π_∞ is the plane of $\mathcal{O}$ and π is another plane of $\mathrm{PG}(3,q)$ skew from $\mathcal{O}$, then the set of points

$$\{(\infty)\}\cup\pi\setminus(\pi\cap\pi_\infty),$$

is also an ovoid of $T_2(\mathcal{O})$ called a *planar ovoid*.

4. Eggs in $\mathrm{PG}(4n-1,q)$, q even, containing a pseudo pointed conic

In this section we study the structure of TGQs whose corresponding egg contains a pseudo pointed conic.

We begin with a statement and sketch proof of an important lemma. The proof is a combination of results of [11], [26], [14], [27] and [18].

Lemma 4.1. *Every $(2n-1)$-dimensional space in $\mathrm{PG}(3n-1,q)$, q even, skew from a pseudo pointed conic is the span of two elements of the Desarguesian spread induced by the pseudo pointed conic.*

Proof : Let U be a $(2n-1)$-space skew from a pseudo pointed conic $\mathcal{O}$ in $\mathrm{PG}(3n-1,q)$. The q^n+1 tangent spaces of the pseudo pointed conic meet pairwise in a fixed $(n-1)$-dimensional space N in $\mathrm{PG}(3n-1,q)$, which is also an element of the Desareguesian spread induced by the pseudo pointed conic (since it corresponds to the nucleus of the pointed conic). An elementary count shows that U and N are skew. Since $\mathcal{O}\cup\{N\}$ contains a pseudo-conic, it follows that U is also skew from a pseudo-conic. Note that this pseudo-conic induces the same Desareguesian spread as the pseudo pointed conic. Dualising in $\mathrm{PG}(3n-1,q)$ we obtain an $(n-1)$-space U' disjoint from a dual pseudo-conic, i.e. the set of q^n+1 $(2n-1)$-spaces corresponding to the q^n+1 lines of a dual conic in $\mathrm{PG}(2,q^n)$. By embedding $\mathrm{PG}(2,q^n)$ in $\mathrm{PG}(3,q^n)$ and dualising in $\mathrm{PG}(3,q^n)$ one sees that the set of affine points of any n-space intersecting $\mathrm{PG}(2,q^n)$ in U' becomes a set of planes forming a semifield flock of a quadratic cone in $\mathrm{PG}(3,q^n)$ and since q is even the corresponding semifield is a field, which implies that U corresponds to a line in $\mathrm{PG}(2,q^n)$. □

Lemma 4.2. *Let $\mathcal{S}$ be a TGQ of order (s,s^2) with a translation point (∞) and a subquadrangle $\mathcal{S}'=(\mathcal{P}',\mathcal{B}',\mathrm{I}')$ of order s containing the point (∞). Then the egg corresponding to $\mathcal{S}$ contains a pseudo-oval $\mathcal{O}$ and $\mathcal{S}'$ is a TGQ isomorphic to $T(\mathcal{O})$.*

Proof : Suppose that the kernel of $\mathcal{S}$ contains GF(q) and $s = q^n$. Then let $\mathcal{E}$ be the corresponding egg in PG$(4n-1, q)$ and represent $\mathcal{S}$ as $T(\mathcal{E})$. The $q^n + 1$ lines of $\mathcal{S}'$ incident with the point (∞) determine a set $\mathcal{O}$ of $q^n + 1$ egg elements $\{E_0, E_1, \dots, E_{q^n}\}$. Let $\mathcal{A}$ denote the set of affine points of $\mathcal{S}'$. Let $Q \in \mathcal{A}$ and consider the line $\langle E_0, Q\rangle$ in $\mathcal{S}'$. It follows that every affine point of $\langle E_0, Q\rangle$ is contained in $\mathcal{A}$. Let P be an affine point in $\langle E_0, Q, E_1\rangle \setminus \langle E_0, Q\rangle$. Then $\langle E_1, P\rangle$ intersects $\langle E_0, Q\rangle$ in an affine point $R \in \mathcal{A}$, and hence $P \in \mathcal{A}$. Hence all affine points of $\langle E_0, Q, E_1\rangle$ are contained in $\mathcal{A}$. Now consider any affine point P in $\langle E_0, E_1, E_2, Q\rangle \setminus \langle E_0, Q, E_1\rangle$. Then $\langle E_2, P\rangle$ intersects $\langle E_0, E_1, Q\rangle$ in a point $R \in \mathcal{A}$. It follows that $\mathcal{A}$ is the set of affine points of $\langle E_0, E_1, E_2, Q\rangle$ and $\mathcal{O}$ is contained in $\langle E_0, E_1, E_2\rangle$. This implies that $\mathcal{O}$ is a pseudo-oval contained in $\mathcal{E}$ and $\mathcal{S}'$ is a TGQ isomorphic to $T(\mathcal{O})$. □

Theorem 4.3. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a TGQ of order (s, s^2), s even, with a translation point (∞) and a subquadrangle $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ isomorphic to $T_2(\mathcal{O})$ where $\mathcal{O}$ is a pointed conic of* PG$(2, s)$*. Further suppose that $\mathcal{S}'$ contains (∞) and that (∞) is a translation point of $\mathcal{S}'$ (and so may be considered as the point (∞) of $T_2(\mathcal{O})$). Then either $s = 4$ and $\mathcal{S} \cong Q(5, 4)$; or $s = 8$ and each ovoid of $\mathcal{S}'$ subtended by a point of $\mathcal{P} \setminus (\mathcal{P}' \cup (\infty)^{\perp})$ is a projective ovoid of $T_2(\mathcal{O})$ arising from a Tits ovoid of* PG$(3, 8)$.

Proof : Suppose that the kernel of $\mathcal{S}$ contains GF(q) and $s = q^n$. Then let $\mathcal{E}$ be the corresponding egg in PG$(4n-1, q)$ and represent $\mathcal{S}$ as $T(\mathcal{E})$. Now $\mathcal{S}'$ is a subquadrangle of order q^n containing (∞). By Lemma 4.2 $\mathcal{E}$ contains a pseudo pointed conic in PG$(3n-1, q)$ and $\mathcal{S}'$ is constructed from a PG$(3n, q)$ containing PG$(3n-1, q)$.

Suppose that X is a point of type (ii) of $\mathcal{S} \setminus \mathcal{S}'$, that is a subspace of dimension $3n$ meeting PG$(4n-1, q)$ in the tangent space at an egg element, and $\mathcal{O}_X$ the ovoid of $\mathcal{S}'$ subtended by X. Then $\mathcal{O}_X$ consists of the point (∞) plus the q^{2n} points $(X \cap \mathrm{PG}(3n, q)) \setminus \mathrm{PG}(3n-1, q)$. The subspace $X \cap \mathrm{PG}(3n-1, q)$ is a $(2n-1)$-dimensional subspace skew from the pseudo pointed conic $\mathcal{O}$. From Lemma 4.1 we have that this is the span of two elements of the Desarguesian spread induced by the pseudo pointed conic. Representing $\mathcal{S}'$ over GF(q^n), that is, as $T_2(O)$ where O is a pointed conic in PG$(2, q^n)$, we see that $\mathcal{O}_X$ consists of (∞) and the affine points of a plane of PG$(3, q^n)$ skew from O. Counting reveals that there are $q^{2n}(q^n-1)/2$ ovoids of $T_2(O)$ of this form and $q^{2n}(q^n-1)$ points of $\mathcal{P} \setminus \mathcal{P}'$ collinear with (∞) and hence subtending an ovoid of $\mathcal{S}'$ containing (∞). Since each such point may subtend at most two ovoids of $T_2(O)$, it follows that each planar ovoid of $T_2(O)$ is doubly subtended.

Now let Y be a point of $\mathcal{P} \setminus \mathcal{P}'$ not collinear with (∞) and $\mathcal{O}_Y$ the ovoid it subtends in $\mathcal{S}'$. We will consider this ovoid in the $T_2(O)$ model of $\mathcal{S}'$. Since $Y \not\sim (\infty)$ it follows that $\mathcal{O}_Y = \mathcal{A} \cup \{\pi_P : P \in O\}$, where $\mathcal{A}$ is a set of $q^{2n} - q^n$ affine points of $T_2(O)$ and π_P is a point of type (ii) of $T_2(\mathcal{O})$ which is a plane containing $P \in O$. We now investigate the intersections of a plane π of PG$(3, q^n)$ with $\mathcal{A}$. If π contains no point of O, then $\pi \cup (\infty)$ is a planar ovoid subtended by two points, X and X' of $\mathcal{S} \setminus \mathcal{S}'$. If Y is collinear with X or X', then $\pi \cap \mathcal{A}$ is a single point. If Y is not collinear with X nor with X', then $\{X, X', Y\}$ is a triad of $\mathcal{S}$ and hence has $q^n + 1$ centres. Hence $|\pi \cap \mathcal{A}| = q^n + 1$. Next suppose that π contains a unique point P of O. If $\pi = \pi_P \subset \mathcal{O}_Y$, then π contains no point of $\mathcal{A}$. If $\pi \neq \pi_P$, then the q^n lines of π incident with P and not in the plane of O are lines of $T_2(O)$ and so contain precisely one point of $\mathcal{A}$. Hence $|\pi \cap \mathcal{A}| = q^n$. Next suppose that π contains two points, P and Q, of O. Of the $q^n + 1$ projective lines in π incident with P one is contained in PG$(2, q^n)$, one is contained in π_P and $q^n - 1$ are lines of $T_2(O)$ containing a unique point of $\mathcal{A}$. Hence $|\pi \cap \mathcal{A}| = q^n - 1$. Finally, if $\pi = \mathrm{PG}(2, q^n)$, then π contains no point of $\mathcal{A}$.

Consider the set of points of PG$(3, q^n)$ defined by $\overline{\mathcal{O}_Y} = \mathcal{A} \cup O$. By the above the plane intersections with $\overline{\mathcal{O}_Y}$ have size 1 or $q^n + 1$ and a straightforward count shows that $\overline{\mathcal{O}_Y}$ is an ovoid of PG$(3, q^n)$.

Further, since $\overline{\mathcal{O}_Y}$ contains the pointed conic O then by Theorem 1.1 either $q^n = 4$ and $\overline{\mathcal{O}_Y}$ is an elliptic quadric or $q^n = 8$ and $\overline{\mathcal{O}_Y}$ is a Tits ovoid.

In the case $q^n = 4$ by the isomorphism from $Q(4,4)$ to $T_2(O)$ ([21]) it is clear that the planar ovoids of $T_2(O)$ are elliptic quadric ovoids (in fact, this is true in general for q even and O a conic), as are the ovoids $\overline{\mathcal{O}_Y}$. Thus we have that every ovoid of $\mathcal{S}' \cong Q(4,4)$ subtended by a point of $\mathcal{P} \setminus \mathcal{P}'$ is an elliptic quadric ovoid. By a theorem due independently to Brown ([8]) and Brouns, Thas and Van Maldeghem ([4]) it now follows that $\mathcal{S}$ is the classical GQ $Q(5,4)$. □

Following on from this theorem we need to prove that if $\mathcal{O}$ is a pointed conic of PG(2, 8) and $T_2(\mathcal{O})$ is contained as a subquadrangle in a GQ $\mathcal{S}$ of order (8, 64) such that each subtended ovoid of $T_2(\mathcal{O})$ is either a planar or a projective ovoid (arising from a Tits ovoid), then $\mathcal{S} \cong T_3(\Omega)$, where Ω is a Tits ovoid.

5. Configurations of Tits ovoids and the proof of the main theorem

Let $q = 2^{2e+1}$, $e \geq 1$, and $\mathcal{O}$ be an oval of $\pi_\infty \cong \mathrm{PG}(2,q)$ equivalent to $\{(1,t,t^\sigma) : t \in \mathrm{GF}(q)\} \cup \{(0,0,1)\}$ where σ is the automorphism of GF(q) such that $\sigma : x \mapsto x^{2^{e+1}}$. Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (q,q^2) containing $T_2(\mathcal{O}) = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ as a subquadrangle such that each point of $\mathcal{P} \setminus \mathcal{P}'$ collinear with (∞) subtends a planar ovoid of $T_2(\mathcal{O})$ and each point of $\mathcal{P} \setminus \mathcal{P}'$ not collinear with (∞) subtends a projective ovoid of $T_2(\mathcal{O})$ arising from a Tits ovoid. Suppose that ℓ is a line of $\mathcal{B} \setminus \mathcal{B}'$ incident with the affine point P of $T_2(\mathcal{O})$. Of the other q points of $\mathcal{S}$ incident with ℓ, one is collinear with (∞) and so subtends a planar ovoid of $T_2(\mathcal{O})$ and the other $q-1$ subtend projective ovoids. If the planar ovoid arises from the plane π of PG(3, q) and the projective ovoids arise from the Tits ovoids $\Omega_1, \Omega_2, \ldots, \Omega_{q-1}$, then it follows that $\Omega_i \cap \Omega_j = \mathcal{O} \cup \{P\}$, $i,j \in \{1,2,\ldots,q-1\}$, $i \neq j$ and π is the tangent plane to Ω_i at P for $i = 1,2,\ldots,q-1$.

In general we will call such a set of ovoids a *stack* of ovoids. The point P is called the *base point* of the stack and π_∞ the *base plane*. The "classical" way to generate a stack of ovoids is to take a single ovoid Ω_1 and act on it by the group of homologies of PG(3, q) with centre P and axis π_∞. Such a stack is called a *homologous stack*.

Suppose that Ω is contained in a stack S with base point P and base plane π_∞ and $\perp$ denotes the symplectic polarity defined by Ω. If Ω' is a second ovoid in S, then the polarity defined by Ω' shares with the polarity of Ω (at least) the singular lines that are the lines incident with P in $P^\perp$ and the lines incident with $\pi_\infty^\perp$ in π_∞. There are exactly $q-1$ symplectic polarities of PG(3, q) which have these singular lines (this is straightforward to see in the Klein quadric $Q^+(5,q)$). In the case of a homologous stack each of the $q-1$ ovoids defines a distinct symplectic polarity, however it is not clear prima facie that this should be true for a general stack.

Theorem 5.1. *Let $\mathcal{O}$ be an oval of* PG(2, q) *and $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ a GQ of order (q,q^2) containing $T_2(\mathcal{O}) = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ as a subquadrangle. Suppose that each point of $\mathcal{P} \setminus \mathcal{P}'$ collinear with (∞) subtends a planar ovoid of $T_2(\mathcal{O})$ and each point of $\mathcal{P} \setminus \mathcal{P}'$ not collinear with (∞) subtends a projective ovoid of $T_2(\mathcal{O})$. If for each line of $\mathcal{B} \setminus \mathcal{B}'$ incident with an affine point of $\mathcal{S}$ the stack of ovoids induced by this line is homologous, then $\mathcal{S} \cong T_3(\Omega)$ for some ovoid Ω of* PG(3, q) *with $\mathcal{O} \subset \Omega$.*

Proof : Let X be a point of $\mathcal{P} \setminus \mathcal{P}'$ not collinear with (∞), $\mathcal{O}_X$ the ovoid of $T_2(\mathcal{O})$ subtended by X and Ω_X the corresonding ovoid of PG(3, q). For each point $P \in \Omega_X \setminus \mathcal{O}$ the line PX of $\mathcal{S}$ gives rise to a stack of ovoids with base point P, base plane π_∞ and containing $\mathcal{O}_X$. This stack is homologous and so is uniquely determined by P, π_∞ and Ω_X. Next we note that the subgraph of the point graph of $\mathcal{S}$ defined on $\mathcal{P} \setminus (\mathcal{P}' \cup \{(\infty)\}^\perp)$, where $\perp$ indicates collinearity in $\mathcal{S}$, is connected. Hence if $\mathcal{O}_X$ and $\mathcal{O}_Y$ are any two subtended projective ovoids of $T_2(\mathcal{O})$, X and Y are connected in this graph and it follows that $\mathcal{O}_Y$ is determined by $\mathcal{O}_X$. In other words the set of subtended projective ovoids is uniquely determined by one of the ovoids.

So suppose Ω is the ovoid of $\mathrm{PG}(3,q)$ giving rise to one of the subtended projective ovoids of $T_2(\mathcal{O})$. Construct the Tits GQ $T_3(\Omega)$ in $\mathrm{PG}(4,q)$. Then $T_2(\mathcal{O})$ is a subquadrangle of $T_3(\Omega)$ constructed from a 3-dimensional subspace Σ of $\mathrm{PG}(4,q)$ meeting $\mathrm{PG}(3,q)$ in the plane π_∞ containing $\mathcal{O}$. Let ℓ be an arbitrary line of $T_3(\Omega) \setminus T_2(\mathcal{O})$ that is not incident with (∞) and not concurrent with an element of $\mathcal{O}$. Suppose ℓ is incident with affine points $\{X_1, X_2, \ldots, X_{q-1}, P\}$ where P is a point of $T_2(\mathcal{O})$. Then the group of homologies of $\mathrm{PG}(4,q)$ with centre P and axis $\mathrm{PG}(3,q)$ induce a group of collineations of $T_3(\Omega)$ that acts transitively on $\{X_1, X_2, \ldots, X_{q-1}\}$. Since $T_2(\mathcal{O})$ is also fixed by these collineations it follows that the group acts transitively on the subtended ovoids corresponding to $X_1, \ldots, X_{q-1}$. Since this group is a homology group with centre P and axis π_∞ in Σ, it follows that the corresponding stack of ovoids is homologous.

From the above it follows that the set of subtended projective ovoids of $T_2(\mathcal{O})$ in $\mathcal{S}$ is exactly the same as that of $T_2(\mathcal{O})$ in $T_3(\Omega)$. It remains to show that this implies that $\mathcal{S} \cong T_3(\Omega)$. First suppose that Ω is an elliptic quadric. Then $\mathcal{O}$ is a conic and $T_2(\mathcal{O})$ is a subquadrangle of $\mathcal{S}$ isomorphic to $Q(4,q)$ each subtended ovoid of which is an elliptic quadric, hence by [4, 8] $\mathcal{S} \cong Q(5,q) \cong T_3(\Omega)$. So now we suppose that Ω is not an elliptic quadric. Suppose further that there is a subtended projective ovoid $\mathcal{O}_X$ of $T_2(\mathcal{O})$ in $T_3(\Omega)$ that is subtended by two distinct points X and Y. The line XY of $\mathrm{PG}(4,q)$ meets the subspace Σ in some point P. The collineation of $\mathrm{PG}(4,q)$ with centre P and axis $\mathrm{PG}(3,q)$ mapping X to Y induces a collineation of $T_3(\Omega)$ mapping X to Y and fixing $T_2(\mathcal{O})$ and so also fixes $\mathcal{O}_X$. If Ω_X is the ovoid of Σ corresonding to $\mathcal{O}_X$, then Ω_X is stabilised by a central collineation of Σ. By [9] this implies that Ω_X is in fact an elliptic quadric. Hence it follows that each subtended projective ovoid of $T_2(\mathcal{O})$ is subtended by exactly one point. It also follows that each subtended projective ovoid of $T_2(\mathcal{O})$ in $\mathcal{S}$ is subtended by exactly one point.

So now we have the same representation of the geometries $T_3(\Omega) \setminus (\infty)^\perp$ and $\mathcal{S} \setminus (\infty)^\perp$ in $T_2(\mathcal{O})$ (where we abuse notation and let $\perp$ denote collinearity in both $T_3(\Omega)$ and $\mathcal{S}$):

Points	:	(i)	Points of $T_2(\mathcal{O}) \setminus (\infty)^\perp$;
		(ii)	ovoids arising from subtended projective ovoids of $T_2(\mathcal{O})$.
Lines	:	(a)	Lines of $T_2(\mathcal{O})$ not incident with (∞);
		(b)	pairs consisting of a subtended stack of ovoids and the base point of the stack.
Incidence	:		natural.

Since any GQ may be reconstructed uniquely from the incidence structure created by removing a point, all the lines incident with that point and the points incident with those lines, it follows that the above incidence structure uniquely reconstructs both $T_3(\Omega)$ and $\mathcal{S}$. Hence $T_3(\Omega) \cong \mathcal{S}$. □

Returning to our original discussion in this section we consider Tits ovoids and possible configurations leading to stacks of Tits ovoids. First we give properties of the Tits ovoid that will be useful for our calculations.

Lemma 5.2 (see [19]). *Let Ω be a Tits ovoid of* $\mathrm{PG}(3,q)$ *and G the homography stabiliser of Ω, which is isomorphic to the Suzuki group $Sz(q)$.*

1. *G acts 2-transitively on the Ω.*

2. *For each $P \in \Omega$ there is a unique line ℓ of $\mathrm{PG}(3,q)$ such that $P \in \ell$ and for each plane π with $\ell \subset \pi$ and π not the tangent plane of Ω at P, $\pi \cap \Omega$ is a translation oval with axis ℓ.*

   For convenience we call ℓ the axis of Ω at P.

3. *The stabiliser in G of two points $P, Q \in \Omega$ is cyclic of order $q-1$ and acts regularly on the $q-1$ oval sections, not containing P, with the axis of Ω at Q as a tangent.*

Lemma 5.3. *Let Ω be a Tits ovoid of $\mathrm{PG}(3,q)$. Then the homography stabiliser group of Ω is transitive on pairs $(\mathcal{O}, P)$ where $\mathcal{O}$ is an oval section of Ω and $P \in \Omega \setminus \mathcal{O}$.*

Proof : Let $(\mathcal{O}, P)$ and $(\mathcal{O}', P')$ be two oval section, point pairs of Ω with $P \notin \mathcal{O}$ and $P' \notin \mathcal{O}'$. Let Q and Q' be the points of Ω on the axis of $\mathcal{O}$ and $\mathcal{O}'$, respectively. By the 2-transitivity of Ω we can map $(P', Q') \mapsto (P, Q)$. Now the homography group fixing Ω and the pair (P, Q) has order $q-1$ and acts regularly on oval sections of Ω not containing P and with the axis of Ω at Q as a tangent. Hence we can find a map taking $(\mathcal{O}, P)$ to $(\mathcal{O}', P')$. □

Lemma 5.4. *Let Ω and Ω' be two Tits ovoids of $\mathrm{PG}(3,8)$ such that $\Omega \cap \Omega' = \mathcal{O} \cup \{P\}$, where $\mathcal{O}$ is an oval in the plane π_∞ and P is a point not contained in $\mathcal{O}$. Further suppose that Ω' is the image of Ω under a non-trivial homology of $\mathrm{PG}(3,8)$ with centre P and axis π_∞. If Ω'' is a Tits ovoid of $\mathrm{PG}(3,8)$ defining the same symplectic polarity as Ω and such that $\Omega' \cap \Omega'' = \mathcal{O} \cup \{P\}$, then $\Omega'' = \Omega$.*

Proof : We start with calculations over general $q = 2^{2e+1}$, $e \geq 1$, and then specialise to $q = 8$ later in the proof. Let $\Omega = \{(1, s, t, st + s^\sigma + t^{\sigma+2}) : s, t \in \mathrm{GF}(q)\} \cup \{(0,0,0,1)\}$ which has symplectic polarity with form $x_0y_3 + x_3y_0 + x_1y_2 + x_2y_1 = 0$. By Lemma 5.3 we may suppose that π_∞ is the plane $x_2 = 0$ with $\mathcal{O} = \pi_\infty \cap \Omega = \{(1, s, 0, s^\sigma) : s \in \mathrm{GF}(q)\} \cup \{(0,0,0,1)\}$ and that $P = (1,0,1,1)$. Any non-trivial homology with centre P and axis π_∞ has the form

$$(x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, x_2, x_3) + \lambda x_2(1,0,1,1) = (x_0 + \lambda x_2, x_1, (\lambda+1)x_2, x_3 + \lambda x_2)$$

for $\lambda \in \mathrm{GF}(q) \setminus \{0,1\}$. Hence

$$\Omega' = \{(1 + \lambda t, s, (\lambda+1)t, st + s^\sigma + t^{\sigma+2} + \lambda t) : s, t \in \mathrm{GF}(q)\} \cup \{(0,0,0,1)\},$$

for some $\lambda \in \mathrm{GF}(q) \setminus \{0,1\}$.

Now suppose that Ω'' is a Tits ovoid distinct from Ω, containing P and $\mathcal{O}$ and also defining the same symplectic polarity as Ω. Since any automorphic collineation of $\mathrm{PG}(3,q)$ fixes Ω we may assume that Ω is mapped to Ω'' by a homography T of $\mathrm{PG}(3,q)$. Further, by Lemma 5.3 we may suppose that T fixes $\mathcal{O}$, P and commutes with the symplectic polarity defined by Ω. In particular, T must fix the point $(0,0,0,1)$ of $\mathcal{O}$ on the axis of $\mathcal{O}$, the tangent plane $x_0 = 0$ at this point, the nucleus $(0,1,0,0)$ of $\mathcal{O}$ and the plane $\pi_\infty : x_2 = 0$. From this we have that

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ b & c & d & 0 \\ 0 & 0 & e & 0 \\ f & 0 & g & h \end{pmatrix} \quad \text{with } c, e, h \neq 0.$$

Next, T also fixes $P = (1, 0, 1, 1)$, and so it follows that $b = d$ and $1 = e = f + g + h \neq 0$.

$T(1, s, 0, s^\sigma) = (1, b + cs, 0, f + hs^\sigma) \in \mathcal{O}$ for $s \in \mathrm{GF}(q)$. Hence

$$b^\sigma + c^\sigma s^\sigma = f + hs^\sigma, \text{ for all } s \in \mathrm{GF}(q).$$

Thus $f = b^\sigma$ and $h = c^\sigma$. Since T commutes with the symplectic polarity defined by Ω, we obtain that $g + b = 0$, and so $g = b$, while $c^\sigma = c$ from which it follows that $c = 1$.

Now $b = g = 1 + b^\sigma + 1$. Hence $b = 0$ or $b = 1$. The case $b = 0$ implies that T is the identity, while $b = 1$ yields

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

Hence $\Omega'' = \{(1, 1 + \bar{s} + \bar{t}, \bar{t}, 1 + \bar{t} + \bar{s}\bar{t} + \bar{s}^\sigma + \bar{t}^{\sigma+2}) : \bar{s}, \bar{t} \in \mathrm{GF}(q)\} \cup \{(0, 0, 0, 1)\}$.

Now we look for Ω' and Ω'' to have a point of intersection outside of $\mathcal{O} \cup \{P\}$. That is,

$$(1 + \lambda t, s, (\lambda + 1)t, st + s^\sigma + t^{\sigma+2} + \lambda t) \equiv (1, 1 + \bar{s} + \bar{t}, \bar{t}, 1 + \bar{t} + \bar{s}\bar{t} + \bar{s}^\sigma + \bar{t}^{\sigma+2}),$$

where $t \neq 0$, $\bar{t} \neq 0$, $(s, t) \neq (0, 1)$ and $(\bar{s}, \bar{t}) \neq (0, 1)$. The first coordinates imply that $1 + \lambda t \neq 0$. We have a solution if and only if the following equations are satisfied.

$$\bar{t} = \frac{(\lambda + 1)t}{1 + \lambda t}, \quad \bar{s} = \frac{s + t + 1}{1 + \lambda t}, \quad \frac{st + s^\sigma + t^{\sigma+2} + \lambda t}{1 + \lambda t} = 1 + \bar{t} + \bar{s}\bar{t} + \bar{s}^\sigma + \bar{t}^{\sigma+2}.$$

This is the case if and only if

$$\begin{aligned} & s^\sigma \left[(1 + \lambda t)^{\sigma+1} + (1 + \lambda t)^2\right] + s\left[t(1 + \lambda t)^{\sigma+1} + (\lambda + 1)t(1 + \lambda t)^\sigma\right] \\ & + \left[(t^{\sigma+2} + \lambda t)(1 + \lambda t)^{\sigma+1} + (1 + \lambda t)^{\sigma+2} + (\lambda + 1)t(1 + \lambda t)^{\sigma+1}\right. \\ & \left. + t(t + 1)(\lambda + 1)(1 + \lambda t)^\sigma + (t + 1)^\sigma(1 + \lambda t)^2 + (\lambda + 1)^{\sigma+2}t^{\sigma+2}\right] = 0. \end{aligned}$$

Now we set $s = 0$ in the above equation to obtain the equation

$$t^{2\sigma+3}[\lambda^{\sigma+1}] + t^{2\sigma+2}[\lambda^\sigma] + t^{\sigma+3}[\lambda] + t^{\sigma+1}[\lambda^{\sigma+1}] + t^\sigma[\lambda^\sigma + 1] + t^2 + t[\lambda] = 0$$

in t and look for solutions with $t \neq 0, 1, 1/\lambda$. At this point we specialise to the case $q = 8$ and obtain the equation

$$t(\lambda t^6 + \lambda^5 t^4 + (\lambda^5 + \lambda^4 + 1)t^3 + \lambda^4 t^2 + t + \lambda) = 0.$$

If $\mathrm{Tr}(\lambda) = 1$, then set $t = \lambda^2 \neq 0, 1, 1/\lambda$ which gives a solution to the equation. If $\mathrm{Tr}(\lambda) = 0$, then set $t = \lambda \neq 0, 1, 1/\lambda$ which also gives a solution to the above equation. Thus we have established that in the case $q = 8$ the Tits ovoids Ω' and Ω'' intersect in more than $\mathcal{O} \cup \{P\}$. Hence Ω' and Ω'' cannot be in a common stack. Thus the only ovoid defining the symplectic polarity with form $x_0y_3 + x_3y_0 + x_1y_2 + x_2y_1 = 0$ and contained in a stack with Ω' having base point P and base plane π_∞ is Ω. □

Lemma 5.5. *In* $\mathrm{PG}(3, 8)$ *every stack of ovoids is homologous.*

Proof : By [12] every ovoid of $\mathrm{PG}(3,8)$ is a Tits ovoids, so we are considering stacks of Tits ovoids. Let Ω be a Tits ovoid of $\mathrm{PG}(3,8)$ with oval section $\mathcal{O} = \pi_\infty \cap \Omega$ and $P \in \Omega \setminus \mathcal{O}$. Consider the 7 symplectic polarities of $\mathrm{PG}(3,8)$ that have as singular lines the lines of π_∞ tangent to Ω and the lines tangent to Ω at P. One of these polarities is that defined by Ω and by Lemma 5.4 for each of the other polarities, there is a unique ovoid defining that polarity and intersecting Ω in exactly $\mathcal{O} \cup \{P\}$. In particular this ovoid is the image of Ω under a homology with centre P and axis π_∞. Hence every stack of ovoids of $\mathrm{PG}(3,8)$ must be homologous. □

Theorem 5.6. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a TGQ of order (s, s^2), s even, with a translation point (∞) and a subquadrangle $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ isomorphic to $T_2(\mathcal{O})$ where $\mathcal{O}$ is a pointed conic of* $\mathrm{PG}(2,s)$. *Further suppose that $\mathcal{S}'$ contains (∞) and that (∞) is a translation point of $\mathcal{S}'$ (and so may be considered as the point (∞) of $T_2(\mathcal{O})$). Then either $s = 4$ and $\mathcal{S} \cong Q(5,4)$; or $s = 8$ and $\mathcal{S} \cong T_3(\Omega)$ where Ω is a Tits ovoid of* $\mathrm{PG}(3,8)$.

Proof : By Theorem 4.3 we have only to consider the case where $s = 8$ and each ovoid of $\mathcal{S}'$ subtended by a point of $\mathcal{P} \setminus (\mathcal{P}' \cup (\infty)^\perp)$ is a projective ovoid of $T_2(\mathcal{O})$ arising from a Tits ovoid of $\mathrm{PG}(3,8)$. By Lemma 5.5 every stack of ovoids of $\mathrm{PG}(3,8)$ is homologous and so by Theorem 5.1 it follows that $\mathcal{S} \cong T_3(\Omega)$ where Ω is a Tits ovoid of $\mathrm{PG}(3,8)$. □

As a corollary we now have the main result of the paper.

Corollary 5.7. *An egg $\mathcal{E}$ in* $\mathrm{PG}(4n-1,q)$, *q even, contains a pseudo pointed conic if and only if the egg is elementary and either the ovoid is an elliptic quadric in* $\mathrm{PG}(3,4)$, *or the ovoid is a Tits ovoid in* $\mathrm{PG}(3,8)$.

Proof : Since an elliptic quadric in $\mathrm{PG}(3,4)$ and a Tits ovoid in $\mathrm{PG}(3,8)$ contain pointed conics any egg arising from these ovoids contains pseudo pointed conics.

Now suppose $\mathcal{E}$ is an egg of $\mathrm{PG}(4n-1,q)$ containing a pseudo pointed conic. Then $T(\mathcal{E})$ is a TGQ of order (q^n, q^{2n}) containing a subquadrangle of order q^n containing (∞) that is isomorphic to $T_2(\mathcal{O})$, where $\mathcal{O}$ is a pointed conic of $\mathrm{PG}(2,q^n)$. Consequently the point (∞) of $T(\mathcal{E})$ may be considered to be the point (∞) of $T_2(\mathcal{O})$. By Theorem 5.6 either $q^n = 4$ and $\mathcal{S} \cong Q(5,4)$; or $q^n = 8$ and $\mathcal{S} \cong T_3(\Omega)$ where Ω is a Tits ovoid of $\mathrm{PG}(3,8)$. Hence by [1, Lemma 1] $\mathcal{E}$ arises from an elliptic quadric in $\mathrm{PG}(3,4)$ or a Tits ovoid in $\mathrm{PG}(3,8)$. □

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