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EGGS IN $\mathrm{PG}(4n-1,q)$, q EVEN, CONTAINING A PSEUDO-CONIC

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Abstract

An ovoid of $\mathrm{PG}(3,q)$ can be defined as a set of q^2+1 points with the property that every three points span a plane and at every point there is a unique tangent plane. In 2000 M. R. Brown ([**7**]) proved that if an ovoid of $\mathrm{PG}(3,q)$, q even, contains a conic, then the ovoid is an elliptic quadric. Generalising the definition of an ovoid to a set of $(n-1)$-spaces of $\mathrm{PG}(4n-1,q)$ J. A. Thas [**21**] introduced the notion of pseudo-ovoids or eggs: a set of $q^{2n}+1$ $(n-1)$-spaces in $\mathrm{PG}(4n-1,q)$, with the property that any three egg elements span a $(3n-1)$-space and at every egg element there is a unique tangent $(3n-1)$-space. We prove that an egg in $\mathrm{PG}(4n-1,q)$, q even, contains a pseudo-conic, that is, a pseudo-oval arising from a conic of $\mathrm{PG}(2,q^n)$, if and only if the egg is classical, that is, arising from an elliptic quadric in $\mathrm{PG}(3,q^n)$.

1. *Introduction and preliminaries*

An *oval* of $\mathrm{PG}(2,q)$ is a set of $q+1$ points no three collinear. In 1954 it was shown by B. Segre [**20**] that if q is odd then an oval in $\mathrm{PG}(2,q)$ is a conic. For q even, many ovals are known which are not conics (see [**6**] for a recent survey). An *ovoid* of $\mathrm{PG}(3,q)$ is a set of q^2+1 points such that every three points span a plane. If we exclude $\mathrm{PG}(3,2)$, that is, assuming $q>2$, then q^2+1 is the maximal cardinality of a set of points satisfying this property. Moreover all the tangent lines to an ovoid at a certain point lie in a plane ([**2**], [**17**]); the *tangent plane* at that point. In 1955 A. Barlotti [**2**] and G. Panella [**17**] independently proved that an ovoid in $\mathrm{PG}(3,q)$, q odd, is an elliptic quadric. For q even, one other example of an ovoid is known; called the Tits ovoid, which exists for $q=2^{2e+1}$, $e\geq 1$. For results characterising the elliptic quadric and the Tits ovoid we refer to the survey [**6**]. A result fundamental to the proof of the main result of this paper is the following characterisation of the elliptic quadric ovoid.

Theorem 1 (M. R. Brown [**7**]). *Let $\mathcal{O}$ be an ovoid of $\mathrm{PG}(3,q)$, q even, and π a plane of $\mathrm{PG}(3,q)$ such that $\pi\cap\mathcal{O}$ is a conic. Then $\mathcal{O}$ is an elliptic quadric.*

An $(n-1)$-*spread* (*partial $(n-1)$-spread*) $\mathcal{S}$ of $\mathrm{PG}(rn-1,q)$ is a set of $(n-1)$-spaces such that any point of $\mathrm{PG}(rn-1,q)$ is contained in exactly (at most) one element of $\mathcal{S}$ (also called a *spread* if the dimension of the elements of $\mathcal{S}$ is understood). A spread $\mathcal{S}$ is called *Desarguesian* if the incidence geometry defined by taking the elements of $\mathcal{S}$ as points, the subspaces spanned by two different elements of $\mathcal{S}$ as

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lines, and the natural incidence relation (symmetric containment), is isomorphic to a Desarguesian projective space.

An *egg* $\mathcal{E}$ in $\mathrm{PG}(4n-1,q)$ (or *pseudo-ovoid*) is a partial $(n-1)$-spread of size $q^{2n}+1$, such that every three egg elements span a $(3n-1)$-space and for every egg element E there exists a $(3n-1)$-space T_E (called the *tangent space of* $\mathcal{E}$ *at* E) which contains E and is skew from the other egg elements. A pseudo-oval (or an *egg in* $\mathrm{PG}(3n-1,q)$) is a partial $(n-1)$-spread of size q^n+1, such that every three elements of the pseudo-oval span $\mathrm{PG}(3n-1,q)$. The notion of eggs was introduced by J. A. Thas in 1971 ([**21**]). An egg $\mathcal{E}$ in $\mathrm{PG}(4n-1,q)$ is called *a good egg* if there is exists an egg element E such that every $(3n-1)$-space containing E and two other egg elements contains exactly q^n+1 egg elements. In that case E is called a *good element* of $\mathcal{E}$. If the elements of a pseudo-ovoid, respectively pseudo-oval, belong to a Desarguesian $(n-1)$-spread of $\mathrm{PG}(4n-1,q)$, respectively $\mathrm{PG}(3n-1,q)$, then the pseudo-ovoid, respectively pseudo-oval, is called *elementary.* It follows that an elementary pseudo-oval arises from an oval of $\mathrm{PG}(2,q^n)$ and an elementary pseudo-ovoid arises from an ovoid of $\mathrm{PG}(3,q^n)$. If the oval is a conic we say that the elementary pseudo-oval is a *pseudo-conic* or a *classical pseudo-oval*, if the ovoid is an elliptic quadric then we call the pseudo-ovoid a *classical pseudo-ovoid.* In 1974 J. A. Thas proved that if every four egg elements span $\mathrm{PG}(4n-1,q)$ or are contained in a $(3n-1)$-dimensional space, then the egg is elementary ([**22**]).

The only known examples of pseudo-ovals are elementary and pseudo-ovals have been classified by computer for $q^n \leq 16$ ([**19**]). More examples are known for pseudo-ovoids, all of them over a field of odd characteristic and they are connected to certain semifields (see Chapter 3 of [**12**] for a survey and [**13**] for recent results for the case when q is odd).

In this article we are concerned about pseudo-ovoids in the case when q is even. All known examples of eggs in $\mathrm{PG}(4n-1,q)$, q even, are elementary. Pseudo-ovoids have been classified by computer for $q^n \leq 4$ ([**14**]). In 2002 J. A. Thas published the following two theorems.

THEOREM 2 (J. A. Thas [**25**]). *An egg* $\mathcal{E}$ *of* $\mathrm{PG}(4n-1,q)$, *with* q *even, is classical if and only if* $\mathcal{E}$ *is good at some element and contains at least one pseudo-conic.*

THEOREM 3 (J. A. Thas [**25**]). *An egg* $\mathcal{E}$ *of* $\mathrm{PG}(4n-1,q)$, *with* q *even, is classical if and only if* $\mathcal{E}$ *contains at least two intersecting pseudo-conics.*

In this article we prove that the only assumption one needs to conclude that an egg in $\mathrm{PG}(4n-1,q)$, q even, is classical, is that it contains a pseudo-conic.

2. *Eggs and translation generalized quadrangles*

A (finite) *generalized quadrangle* (GQ) (see [**18**] for a comprehensive introduction) is an incidence structure $\mathcal{S}=(\mathcal{P},\mathcal{B},\mathrm{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint (non-empty) sets of objects called *points* and *lines*, respectively, and for which $\mathrm{I} \subseteq (\mathcal{P}\times\mathcal{B})\cup(\mathcal{B}\times\mathcal{P})$

is a symmetric point-line incidence relation satisfying the following axioms:

(i) Each point is incident with $1+t$ lines ($t \geq 1$) and two distinct points are incident with at most one line;
(ii) Each line is incident with $1+s$ points ($s \geq 1$) and two distinct lines are incident with at most one point;
(iii) If X is a point and ℓ is a line not incident with X, then there is a unique pair $(Y, m) \in \mathcal{P} \times \mathcal{B}$ for which $X \mathrm{I} m \mathrm{I} Y \mathrm{I} \ell$.

The integers s and t are the *parameters* of the GQ and $\mathcal{S}$ is said to have *order* (s,t). If $s = t$, then $\mathcal{S}$ is said to have order s. If $\mathcal{S}$ has order (s,t), then it follows that $|\mathcal{P}| = (s+1)(st+1)$ and $|\mathcal{B}| = (t+1)(st+1)$ ([**18**, 1.2.1]). A *subquadrangle* $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ of $\mathcal{S}$ is a GQ such that $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{B}' \subseteq \mathcal{B}$ and I' is the restriction of I to $(\mathcal{P}' \times \mathcal{B}') \cup (\mathcal{B}' \times \mathcal{P}')$. Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a GQ of order (s,t), $s \neq 1$, $t \neq 1$. A collineation θ of $\mathcal{S}$ is an *elation* about the point P if $\theta = id$ or if θ fixes all lines incident with P and fixes no point of $\mathcal{P} \setminus P^{\perp}$. If there is a group G of elations about P acting regularly on $\mathcal{P} \setminus P^{\perp}$, then we say that $\mathcal{S}$ is an *elation generalized quadrangle* (EGQ) with *elation group* G and *base point* P. Briefly we say that $(\mathcal{S}^{(P)}, G)$ or $\mathcal{S}^{(P)}$ is an EGQ. If the group G is abelian, then we say that the EGQ $(\mathcal{S}^{(P)}, G)$ is a *translation generalized quadrangle* (TGQ) and G is the *translation group*.

In $\mathrm{PG}(2n+m-1, q)$ consider a set $\mathcal{E}(n,m,q)$ of $q^m + 1$ $(n-1)$-dimensional subspaces, every three of which generate a $\mathrm{PG}(3n-1, q)$ and such that each element E of $\mathcal{E}(n,m,q)$ is contained in an $(n+m-1)$-dimensional subspace T_E having no point in common with any element of $\mathcal{E}(n,m,q) \setminus \{E\}$. It is easy to check that T_E is uniquely determined for any element E of $\mathcal{E}(n,m,q)$. The space T_E is called the *tangent space of* $\mathcal{E}(n,m,q)$ *at* E. For $n = m = 1$ such a set $\mathcal{E}(1,1,q)$ is an oval in $\mathrm{PG}(2,q)$ and more generally for $n = m$ such a set $\mathcal{E}(n,n,q)$ is a *pseudo-oval* of $\mathrm{PG}(3n-1,q)$. For $m = 2n = 2$ such a set $\mathcal{E}(1,2,q)$ is an ovoid of $\mathrm{PG}(3,q)$ and more generally for $m = 2n$ such a set $\mathcal{E}(n,2n,q)$ is a *pseudo-ovoid*. In general we call the sets $\mathcal{E}(n,m,q)$ *eggs*.

Now embed $\mathrm{PG}(2n+m-1,q)$ in a $\mathrm{PG}(2n+m,q)$, and construct a point-line geometry $T(n,m,q)$ as follows. Points are of three types:

(i) the points of $\mathrm{PG}(2n+m,q) \setminus \mathrm{PG}(2n+m-1,q)$, called the *affine points*;
(ii) the $(n+m)$-dimensional subspaces of $\mathrm{PG}(2n+m,q)$ which intersect $\mathrm{PG}(2n+m-1,q)$ in a tangent space of $\mathcal{E}(n,m,q)$;
(iii) the symbol (∞).

Lines are of two types:

(a) the n-dimensional subspaces of $\mathrm{PG}(2n+m,q)$ which intersect $\mathrm{PG}(2n+m-1,q)$ in an element of $\mathcal{E}(n,m,q)$;
(b) the elements of $\mathcal{E}(n,m,q)$.

Incidence in $T(n,m,q)$ is defined as follows. A point of type (i) is incident only with lines of type (a); here the incidence is that of $\mathrm{PG}(2n+m,q)$. A point of type (ii) is incident with all lines of type (a) contained in it and with the unique element of $\mathcal{E}(n,m,q)$ contained in it. The point (∞) is incident with no line of type (a) and with all lines of type (b).

THEOREM 4 (8.7.1 of Payne and Thas [**18**]). *The incidence geometry $T(n,m,q)$ is a TGQ of order (q^n, q^m) with base point (∞). Conversely, every TGQ is isomorphic to a $T(n,m,q)$. It follows that the theory of TGQ is equivalent to the theory of the sets $\mathcal{E}(n,m,q)$.*

In the case where $n = m = 1$ and $\mathcal{E}(1,1,q)$ is the oval $\mathcal{O}$ the GQ $T(1,1,q)$ is the Tits GQ $T_2(\mathcal{O})$. When $m = 2n = 2$ and $\mathcal{E}(1,2,q)$ is the ovoid Ω, the GQ $T(1,2,q)$ is the Tits GQ $T_3(\Omega)$. Note that $T_2(\mathcal{O}) \cong Q(4,q)$, if and only if $\mathcal{O}$ is a conic and non-classical otherwise, while $T_3(\Omega) \cong Q(5,q)$ if and only if Ω is an elliptic quadric (see [**18**, Chapter 3]). The *kernel* of $\mathcal{S} = T(n,m,q)$ is the maximum cardinality field $\mathrm{GF}(q')$ for which there exists an $O(n',m',q')$ representing $\mathcal{S}$ and $\mathcal{S}$ may be represented by an $\mathcal{E}(n'',m'',q'')$ if and only if $\mathrm{GF}(q'') \subseteq \mathrm{GF}(q)$ (see [**18**, Chapter 8]). Let $\mathcal{E}$ be an egg in $\mathrm{PG}(4n-1,q)$ and $T(\mathcal{E})$ the corresponding TGQ. If $\mathcal{O}$ is a pseudo-oval of $\mathcal{E}$ contained in $\mathrm{PG}(3n-1,q)$ and $\mathrm{PG}(3n,q)$ any subspace containing $\mathrm{PG}(3n-1,q)$ not contained in $\mathrm{PG}(4n-1,q)$, then $\mathrm{PG}(3n,q)$ induces a subquadrangle of $T(\mathcal{E})$ isomorphic to $T(\mathcal{O})$.

3. *Eggs in* $\mathrm{PG}(4n-1,q)$*, q even, containing a pseudo-conic*

In this section we characterise the classical GQ $Q(5,q)$ as a TGQ with a single classical subquadrangle on the translation point. As a corollary we have the analogue of Theorem 1 for eggs.
We begin with a statement and sketch proof of an important lemma. The proof is a combination of results of [**10**], [**23**], [**11**], [**24**] and [**15**], and already noted in [**25**].

LEMMA 5. *Every $(2n-1)$-dimensional space in $\mathrm{PG}(3n-1,q)$, q even, skew from a pseudo-conic is the span of two elements of the Desarguesian spread induced by the pseudo-conic.*

Proof. Let U be a $(2n-1)$-space skew from a pseudo-conic in $\mathrm{PG}(3n-1,q)$. Dualising in $\mathrm{PG}(3n-1,q)$ we obtain an $(n-1)$-space U' disjoint from a dual pseudo-conic, i.e. the set of q^n+1 $(2n-1)$-spaces corresponding to the q^n+1 lines of a dual conic in $\mathrm{PG}(2,q^n)$. By embedding $\mathrm{PG}(2,q^n)$ in $\mathrm{PG}(3,q^n)$ and dualising in $\mathrm{PG}(3,q^n)$ one sees that the set of affine points of any n-space intersecting $\mathrm{PG}(2,q^n)$ in U' becomes a set of planes forming a semifield flock of a quadratic cone in $\mathrm{PG}(3,q^n)$ and since q is even the corresponding semifield is a field, which implies that U corresponds to a line in $\mathrm{PG}(2,q^n)$. □

LEMMA 6. *Let $\mathcal{S}$ be a TGQ of order (s,s^2) with a translation point (∞) and a subquadrangle $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ of order s containing the point (∞). Then the egg corresponding to $\mathcal{S}$ contains a pseudo-oval $\mathcal{O}$ and $\mathcal{S}'$ is a TGQ isomorphic to $T(\mathcal{O})$.*

Proof. Suppose that the kernel of $\mathcal{S}$ contains $\mathrm{GF}(q)$ and $s = q^n$. Then let $\mathcal{E}$ be the corresponding egg in $\mathrm{PG}(4n-1,q)$ and represent $\mathcal{S}$ as $T(\mathcal{E})$. The q^n+1 lines of $\mathcal{S}'$ incident with the point (∞) determine a set $\mathcal{O}$ of q^n+1 egg elements

$\{E_0, E_1, \ldots, E_{q^n}\}$. Let $\mathcal{A}$ denote the set of affine points of $\mathcal{S}'$. Let $Q \in \mathcal{A}$ and consider the line $\langle E_0, Q\rangle$ in $\mathcal{S}'$. It follows that every affine point of $\langle E_0, Q\rangle$ is contained in $\mathcal{A}$. Let P be an affine point in $\langle E_0, Q, E_1\rangle \setminus \langle E_0, Q\rangle$. Then $\langle E_1, P\rangle$ intersects $\langle E_0, Q\rangle$ in an affine point $R \in \mathcal{A}$, and hence $P \in \mathcal{A}$. Hence all affine points of $\langle E_0, Q, E_1\rangle$ are contained in $\mathcal{A}$. Now consider any affine point P in $\langle E_0, E_1, E_2, Q\rangle \setminus \langle E_0, Q, E_1\rangle$. Then $\langle E_2, P\rangle$ intersects $\langle E_0, E_1, Q\rangle$ in a point $R \in \mathcal{A}$. It follows that $\mathcal{A}$ is the set of affine points of $\langle E_0, E_1, E_2, Q\rangle$ and $\mathcal{O}$ is contained in $\langle E_0, E_1, E_2\rangle$. This implies that $\mathcal{O}$ is a pseudo-oval contained in $\mathcal{E}$ and $\mathcal{S}'$ is a TGQ isomorphic to $T(\mathcal{O})$. □

THEOREM 7. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a TGQ of order (s, s^2), s even, with a translation point (∞) and a subquadrangle $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ isomorphic to $Q(4, s)$ containing (∞). Then $\mathcal{S} \cong Q(5, s)$.*

Proof. Suppose that the kernel of $\mathcal{S}$ contains $\mathrm{GF}(q)$ and $s = q^n$. Then let $\mathcal{E}$ be the corresponding egg in $\mathrm{PG}(4n-1, q)$ and represent $\mathcal{S}$ as $T(\mathcal{E})$. Now $\mathcal{S}'$ is a (classical) subquadrangle of order q^n containing (∞). By Lemma 6 $\mathcal{E}$ contains a pseudo-conic in $\mathrm{PG}(3n-1, q)$ and $\mathcal{S}'$ is constructed from a $\mathrm{PG}(3n, q)$ containing $\mathrm{PG}(3n-1, q)$.
If X is a point of $\mathcal{P} \setminus \mathcal{P}'$, then the lines incident with X intersect $\mathcal{S}'$ in a set $\mathcal{O}_X$ of $q^{2n}+1$ points of $\mathcal{S}'$, no two collinear, called an *ovoid* of $\mathcal{S}'$ ([**18**, 2.2.1]). The ovoid $\mathcal{O}_X$ is said to be *subtended* by X. Suppose that X is a point of type (ii) of $\mathcal{S}$, that is a subspace of dimension $3n$ meeting $\mathrm{PG}(4n-1, q)$ in the tangent space at an egg element. Then $\mathcal{O}_X$ consists of the point (∞) plus the q^{2n} points $(X \cap \mathrm{PG}(3n, q)) \setminus \mathrm{PG}(3n-1, q)$. The subspace $X \cap \mathrm{PG}(3n-1, q)$ is a $(2n-1)$-dimensional subspace skew from the pseudo-conic $\mathcal{C}$. From Lemma 5 we have that this is the span of two elements of the Desarguesian spread induced by the pseudo-conic. Representing $\mathcal{S}'$ over $\mathrm{GF}(q^n)$, that is, as $T_2(C)$ where C is a conic in $\mathrm{PG}(2, q^n)$, we see that $\mathcal{O}_X$ consists of (∞) and the affine points of a plane of $\mathrm{PG}(3, q^n)$ skew from C. By the isomorphism from $Q(4, q^n)$ to $T_2(\mathcal{C})$ ([**18**]) it is clear that the ovoids of $T_2(\mathcal{C})$ consisting of (∞) and the affine points of a plane skew to $\mathcal{C}$ correspond to the elliptic quadric ovoids of $Q(4, q^n)$ containing a fixed point. By a result of Bose and Shrikhande ([**4**]) any triad of $\mathcal{S}$ has q^n+1 centres and so a subtended ovoid of $\mathcal{S}'$ may be subtended by at most two points of $\mathcal{S} \setminus \mathcal{S}'$, in which case the ovoid is said to be *doubly subtended.* Counting reveals that there are $q^{2n}(q^n-1)/2$ elliptic quadric ovoids of $\mathcal{S}'$ containing (∞) and $q^{2n}(q^n-1)$ points of $\mathcal{P} \setminus \mathcal{P}'$ collinear with (∞) and hence subtending an ovoid of $\mathcal{S}'$ containing (∞). Thus each such ovoid is doubly subtended.
Now let Y be a point of $\mathcal{P} \setminus \mathcal{P}'$ not collinear with (∞) and $\mathcal{O}_Y$ the ovoid it subtends in $\mathcal{S}'$. We will consider this ovoid in the $T_2(C)$ model of $\mathcal{S}'$. Since $Y \not\sim (\infty)$ it follows that $\mathcal{O}_Y = \mathcal{A} \cup \{\pi_P : P \in C\}$, where $\mathcal{A}$ is a set of $q^{2n} - q^n$ affine points of $T_2(C)$ and π_P is a point of type (ii) of $T_2(\mathcal{C})$ which is a plane containing $P \in C$. We now investigatethe intersections of a plane π of $\mathrm{PG}(3, q^n)$ with $\mathcal{A}$. If π contains no point of C, then $\pi \cup (\infty)$ is an elliptic quadric subtended by two points, X and X' of $\mathcal{S} \setminus \mathcal{S}'$. If Y is collinear with X or X', then $\pi \cap \mathcal{A}$ is a single point. If Y is not collinear with X nor with X', then $\{X, X', Y\}$ is a triad of $\mathcal{S}$ and hence has q^n+1 centres. Hence $|\pi \cap \mathcal{A}| = q^n + 1$. Next suppose that π contains a unique point P of C. If $\pi = \pi_P \subset \mathcal{O}_Y$, then π contains no point of $\mathcal{A}$. If $\pi \neq \pi_P$, then the q^n lines of π incident with P and not in the plane of C are lines of the $T_2(C)$ and so contain precisely one point of $\mathcal{A}$. Hence $|\pi \cap \mathcal{A}| = q^n$. Next suppose that π contains

two points, P and Q, of C. Of the q^n+1 projective lines in π incident with P one is contained in $\mathrm{PG}(2,q^n)$, one is contained in π_P and q^n-1 are lines of $T_2(C)$ containing a unique point of $\mathcal{A}$. Hence $|\pi \cap \mathcal{A}| = q^n - 1$. Finally, if $\pi = \mathrm{PG}(2,q^n)$, then π contains no point of $\mathcal{A}$.

Consider the set of points of $\mathrm{PG}(3,q^n)$ defined by $\overline{\mathcal{O}_Y} = \mathcal{A} \cup C$. By the above the plane intersections with $\overline{\mathcal{O}_Y}$ have size 1 or q^n+1 and a straightforward count shows that $\overline{\mathcal{O}_Y}$ is an ovoid of $\mathrm{PG}(3,q^n)$. Further, since $\overline{\mathcal{O}_Y}$ contains the conic C it is an elliptic quadric by Theorem 1. Hence the ovoid $\mathcal{O}_Y$ is an elliptic quadric ovoid of $\mathcal{S}'$ in the $Q(4,q^n)$ model. Thus we have that every ovoid of $\mathcal{S}' \cong Q(4,q^n)$ subtended by a point of $\mathcal{P} \setminus \mathcal{P}'$ is an elliptic quadric ovoid. By a theorem due independently to Brown ([**8**]) and Brouns, Thas and Van Maldeghem ([**5**]) it now follows that $\mathcal{S}$ is the classical GQ $Q(5,q^n)$. □

Remark 1. In general, suppose that $\mathcal{S}$ is a TGQ of order (s,s^2), s even, represented by an egg $\mathcal{E}$ in $\mathrm{PG}(4n-1,q)$. Suppose that $\mathcal{S}'$ is a subquadrangle of $\mathcal{S}$ of order s, containing the base point (∞) of $\mathcal{S}$. Then the argument at the start of the proof of Theorem 7 proves that $\mathcal{S}'$ is isomorphic to $T(\mathcal{O})$ for $\mathcal{O}$ a pseudo-oval contained in $\mathcal{E}$. This solves an open case in [**9**].

As a corollary we now have the main result of the paper.

Theorem 8. *An egg $\mathcal{E}$ in $\mathrm{PG}(4n-1,q)$, q even, contains a pseudo-conic if and only if the egg is classical, that is arising from an elliptic quadric in $\mathrm{PG}(3,q^n)$.*

Proof. Since an elliptic quadric contains conics, any egg arising from an elliptic quadric contains pseudo-conics. Now suppose $\mathcal{E}$ is an egg of $\mathrm{PG}(4n-1,q)$ containing a pseudo-conic. Then $T(\mathcal{E})$ is a TGQ of order (q^n,q^{2n}) containing a classical subquadrangle of order q^n containing (∞). By Theorem 7 $T(\mathcal{E})$ is the classical GQ $Q(5,q^n)$ and so by [**1**, Lemma 1] $\mathcal{E}$ arises from an elliptic quadric in $\mathrm{PG}(3,q^n)$. □

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References

1. L. Bader, G. Lunardon, I. Pinneri; A new semifield flock. *J. Combin. Theory Ser. A* **86** (1999), no.1, 49–62.
2. A. Barlotti; Un'estensione del teorema di Segre-Kustaanheimo. *Boll. Un. Mat. Ital.* **10** (1955) 96–98.
3. A. Barlotti; *Some topics in finite geometrical structures.* Institute of Statistics Mimeo Series, no. 439, University of North Carolina, 1965.
4. R. C. Bose and S. S. Shrikhande; Geometric and pseudo-geometric graphs $(q^2+1, q+1, q)$. *J. Geom.* **2/1** (1973) 75–94.
5. L. Brouns, J. A. Thas, H. Van Maldeghem; A characterisation of $Q(5,q)$ using one subquadrangle $Q(4,q)$. *European J. Combin.* **23** (2002), no. 2, 163–177.
6. Matthew Brown; (Hyper)ovals and ovoids in projective spaces. *Socrates Intensive Course Finite Geometry and its Applications Ghent*, 3-14 April 2000, lecture notes. Available from http://cage.rug.ac.be/~fdc/intensivecourse2/brown_2.pdf

7. Matthew R. Brown; Ovoids of $PG(3,q), q$ even, with a conic section. *J. London Math. Soc.* (2) **62** (2000), no. 2, 569–582.

8. Matthew R. Brown; A characterisation of the generalized quadrangle $Q(5,q)$ using cohomology. *J. Algebraic Combin.* **15** (2002) no. 2, 107–125.

9. Matthew R. Brown, J.A. Thas; Subquadrangles of order s of generalized quadrangles of order (s,s^2), Part II. Submitted *J. Combin. Theory Ser. A.*

10. Stephen D. Cohen, Michael J. Ganley; Commutative semifields, two-dimensional over their middle nuclei. *J. Algebra* **75** (1982), no. 2, 373–385.

11. N. L. Johnson; Semifield flocks of quadratic cones. *Simon Stevin* **61** (1987), no. 3-4, 313–326.

12. Michel Lavrauw; Scattered spaces with respect to spreads, and eggs in finite projective spaces. Dissertation, Technische Universiteit Eindhoven, Eindhoven, 2001. Eindhoven University of Technology, Eindhoven, 2001. viii+115 pp.

13. Michel Lavrauw; Characterisations and properties of good eggs in $PG(4n-1,q)$, q odd. Submitted.

14. Michel Lavrauw, Tim Penttila; On eggs and translation generalised quadrangles. *J. Combin. Theory Ser. A.* **96** (2001), no. 2, 303–315.

15. Guglielmo Lunardon; Flocks, ovoids of $Q(4,q)$ and designs. *Geom. Dedicata* **66** (1997), no. 2, 163–173.

16. G. Lunardon, J. A. Thas; Finite Generalized Quadrangles. Lecture notes, *Intensive Course on Galois Geometry and Generalized Polygons*, (1998), http://cage.rug.ac.be/~fdc/intensivecourse/intensivecourse_final.html.

17. G. Panella; Caratterizzazione delle quadriche di uno spazio (tridimenensionale) lineare sopra un corpo finito. *Boll. Un. Mat. Ital.* **10** (1955), 507–513.

18. S. E. Payne, J. A. Thas; *Finite generalized quadrangles.* Research Notes in Mathematics, 110. Pitman (Advanced Publishing Program), Boston, MA, 1984. vi+312 pp. ISBN 0-273-08655-3

19. Tim Penttila; Translation generalised quadrangles and elation Laguerre planes of order 16. European J. Combin., to appear.

20. Beniamino Segre; Sulle ovali nei piani lineari finiti. (Italian) *Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat.* (8) **17**, (1954). 141–142.

21. J. A. Thas; The m-dimensional projective space $S_m(M_n(GF(q)))$ over the total matrix algebra $M_n(GF(q))$ of the $n \times n$-matrices with elements in the Galois field $GF(q)$. *Rend. Mat.* (6) **4** (1971), 459–532.

22. J. A. Thas; Geometric characterization of the $[n-1]$-ovaloids of the projective space $PG(4n-1,q)$. *Simon Stevin* **47** (1973/74), 97–106.

23. J. A. Thas; Generalized quadrangles and flocks of cones. *European J. Combin.* **8** (1987), no. 4, 441–452.

24. J. A. Thas; Symplectic spreads in $PG(3,q)$, inversive planes and projective planes. Combinatorics (Rome and Montesilvano, 1994). *Discrete Math.* **174** (1997), no. 1-3, 329–336.

25. J. A. Thas; Translation generalized quadrangles of order (s,s^2), s even, and eggs. *J. Combin. Theory Ser. A* **99** (2002), no. 1, 40–50.

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