Finite semifields and nonsingular tensors

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(S2) Left and right distributive laws hold

- $\forall x, y, z \in \mathbb{S} \ : \ x \circ (y + z) = x \circ y + x \circ z$
- $\forall x, y, z \in \mathbb{S} \ : \ (x + y) \circ z = x \circ z + y \circ z$

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- $\forall x, y \in \mathbb{S} \ : \ x \circ y = 0 \Rightarrow x = 0$ or $y = 0$

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- $\exists u \in \mathbb{S}, \forall x \in \mathbb{S} \ : \ x \circ u = u \circ x = x,$

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(without (S4) $\rightarrow$ pre-semifield)

From a pre-semifield to a semifield

Let $(\mathbb{S}, \circ)$ be a pre-semifield and $0 \neq u \in \mathbb{S}$.

Define a new multiplication:

$$(a \circ u) * (u \circ b) = a \circ b.$$

Then $(\mathbb{S}, *)$ is a semifield, with unit $u \circ u$.

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- Proper example of odd order q^{2k} (L. E. Dickson 1906)

$$\mathbb{S}_D \; : \; (\mathbb{F}_{q^k}^2, +, \circ) \begin{cases} (x,y)+(u,v) & = (x+u, y+v) \\ (x,y)\circ(u,v) & = (xu+\alpha y^q v^q, xv+yu) \end{cases}$$

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where α is a non-square in $\mathbb{F}_{q^k}$.

$\rightarrow$ Let's prove (S3): no zero divisors. Suppose $(x,y)\circ(u,v)=(0,0)$. If $u=0$ or $v=0$, then $(u,v)=(0,0)$. If $u\neq 0\neq v$, then

$$\begin{cases} xu+\alpha y^q v^q & = 0 \\ xv+yu & = 0 \end{cases} \Rightarrow \begin{cases} xuv+\alpha y^q v^{q+1} & = 0 \\ xvu+yu^2 & = 0 \end{cases}$$

If $y\neq 0$ then $\alpha y^{q-1}v^{q+1}=u^2$, a contradiction. Hence $y=0 \Rightarrow (x,y)=(0,0)$.

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  where α is a non-square in $\mathbb{F}_{q^k}$.
  Notice: $\mathbb{S}_D$ is commutative, but not associative.
- Generalized twisted fields (A. A. Albert 1961):

$$\mathbb{S}_{GT} \ : \ (\mathbb{F}_{q^n}, +, \circ) \text{ with } x \circ y = xy - \eta x^\alpha y^\beta,$$

  $\alpha, \beta \in Aut(\mathbb{F}_{q^n})$, $Fix(\alpha) = Fix(\beta) = \mathbb{F}_q$, where

$$\eta \in \mathbb{F}_{q^n} \setminus \{x^{\alpha-1} y^{\beta-1} : x, y \in \mathbb{F}_{q^n}\}$$

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- Hughes-Kleinfeld (1960): "Semi-nuclear extensions of Galois fields"
- Knuth (1965): "We are concerned with a certain type of algebraic system, called a semifield. Such a system has several names in the literature, where it is called, for example, a "nonassociative division ring" or a "distributive quasifield". Since these terms are rather lengthy, and since we make frequent reference to such systems in this paper, the more convenient name semifield will be used."

Since 1965, people have been using the name semifields.

Classification results*

* without assumptions on the nuclei

Classification results*

- A two-dimensional finite semifield is a finite field (Dickson 1906)
- A three-dimensional finite semifield is a twisted field or a field (Menichetti 1977) (Conjectured by Kaplansky)
- The smallest nonassociative semifield has size 16, and semifields have been classified by computer up to order 243 (Rúa-Combarro 2010)

* without assumptions on the nuclei

Translation planes from a semifield $\mathbb{S}$

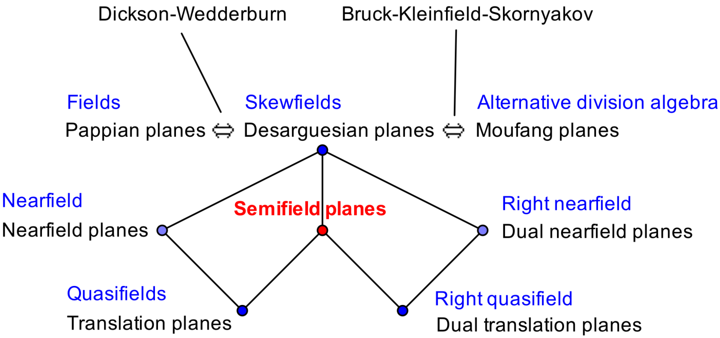
$(\mathbb{S}, \circ) \to$ projective plane $\pi(\mathbb{S}) := (\mathcal{P}, \mathcal{L}, \mathcal{I})$

- $\mathcal{P}$: points (a, b, c), i.e. $(0, 0, 1)$, $(0, 1, c)$, or $(1, b, c)$
- $\mathcal{L}$: lines $[x, y, z]$, i.e. $[0, 0, 1]$, $[0, 1, z]$, or $[1, y, z]$
- Incidence: $(a, b, c)\mathcal{I}[x, y, z] \Leftrightarrow az = b \circ y + cx$

Theorem

The incidence structure $\pi(\mathbb{S})$ is a projective plane. Moreover, it is a translation plane AND a its dual is also a translation plane.

Types of finite translation planes



[Hughes - Piper, Projective Planes, Springer, 1973]

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- An isotopism from $(\mathbb{S}, \circ)$ to $(\mathbb{S}', \circ')$ is a triple (F, G, H) of bijections from $\mathbb{S}$ to $\mathbb{S}'$, linear over the characteristic field of $\mathbb{S}$, such that
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- If such an isotopism exists, then $\mathbb{S}$ and $\mathbb{S}'$ are called isotopic.
- Semifield $\mathbb{S} \longrightarrow$ isotopism class $[\mathbb{S}]$

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Let $\mathbb{S}, \circ$ be a pre-semifield and $0 \neq u \in \mathbb{S}$.

Define a new multiplication:

$$(a \circ u) * (u \circ b) = a \circ b.$$

Then $(\mathbb{S}, *)$ is a semifield isotopic to the pre-semifield $(\mathbb{S}, \circ)$:

$$a^{R_u} \circ b^{L_u} = a \circ b.$$

(Isotopism (R_u, L_u, id))

Nuclei

The left nucleus

$$\mathbb{N}_l(\mathbb{S}) := \{x \ : \ x \in \mathbb{S} \mid x \circ (y \circ z) = (x \circ y) \circ z, \ \forall y, z \in \mathbb{S}\},$$

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The center

$$Z(\mathbb{S}) := \{c \ : \ c \in \mathbb{N}_l(\mathbb{S}) \cap \mathbb{N}_m(\mathbb{S}) \cap \mathbb{N}_r(\mathbb{S}) \mid x \circ c = c \circ x, \forall x \in \mathbb{S}\}.$$

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$\Rightarrow$ left vector space over the left nucleus $\mathbb{N}_l(\mathbb{S}) =: V_l(\mathbb{S})$
$\Rightarrow$ right vector space over the right nucleus $\mathbb{N}_r(\mathbb{S}) =: V_r(\mathbb{S})$

$$y^{L_x} = x \circ y \Rightarrow L_x \in End(V_r(\mathbb{S}))$$

$$y^{R_x} = y \circ x \Rightarrow R_x \in End(V_l(\mathbb{S}))$$

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- If $\{e_1, \ldots, e_n\}$ is a basis for $\mathbb{S}$ over the center $Z(\mathbb{S})$, then the structure constants a_{ijk} are given by

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- Knuth orbit $\mathcal{K}(\mathbb{S})$:= $\{[\mathbb{S}_1], \ldots, [\mathbb{S}_6]\}$

The Knuthorbit of a semifield $\mathbb{S}$

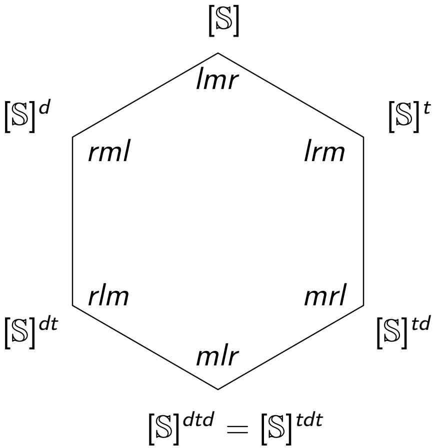

Figure: The nuclei are denoted by l, m, r

A GEOMETRIC APPROACH TO FINITE SEMIFIELDS

- Construction of examples
- Proving that these examples are "new"
- Extension of the Knuth orbit
- Classification results

1. General case
2. Two-dimensional case ($dim V_l(\mathbb{S}) = 2$)
3. Commutative semifields and symplectic semifields
4. Rank two commutative semifields (RTCS)

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- If T has dimension d, then $B_{\mathcal{D}}(T)$ is a linear set of rank $d+1$

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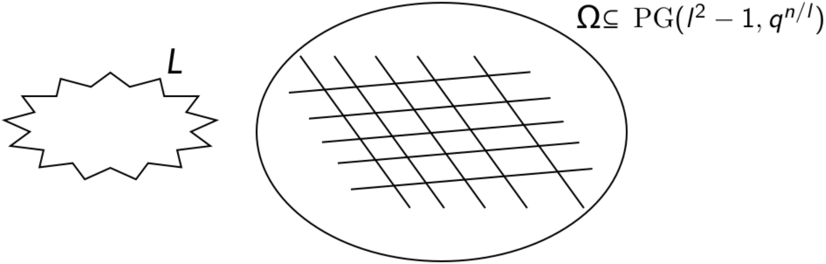
- Denote this secant variety by Ω



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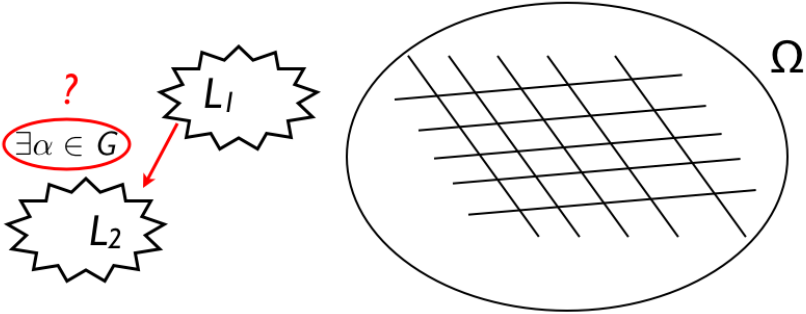
- Let X denote the set of linear sets of rank n disjoint from Ω.

Theorem (ML2011)

There is a one-to-one correspondence between the isotopism classes of semifields of order q^n, l-dimensional over their left nucleus and the orbits of G on the set X.



2. Rank two semifields (l=2) (2-dim over left nucleus)

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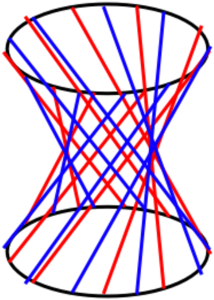

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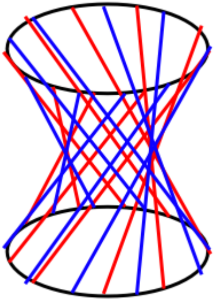

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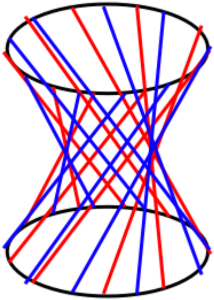

- Classification for $n = 4$ (Cardinali - Polverino - Trombetti, 2006)
- Towards a classification for $n = 6$ (Marino - Polverino - Trombetti)

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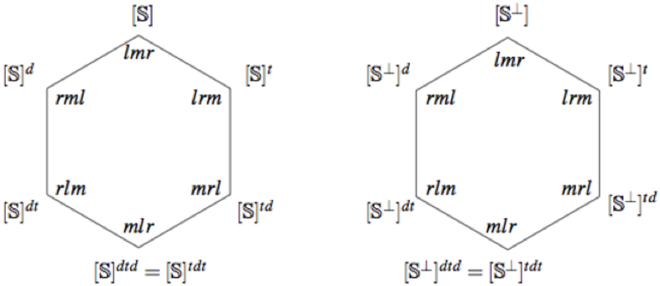
- Extension of the Knuth orbit $\rightarrow$ translation dual

$$\mathbb{S} \mapsto \mathbb{S}^{\perp}$$

[Lunardon - Marino - Polverino - Trombetti 2008], special case of "switching" from [Ball - Ebert - ML 2007]



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Theorem (Kantor 2003)

A pre–semifield $\mathbb{S}$ *is symplectic if and only if the pre–semifield* $\mathbb{S}^{dt}$ *is isotopic to a commutative semifield.*

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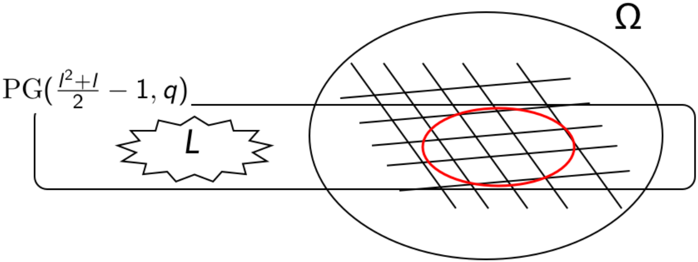
Theorem (Lunardon-Marino-Polverino-Trombetti 2011)

$\mathbb{S}$ is symplectic $\iff$ $L(\mathbb{S})$ is contained in an $(\frac{l^2+l}{2}-1)$-dimensional subspace intersecting $\mathcal{S}_{l,l}(q^{n/l})$ in a Veronese variety $\mathcal{V}_l(q^{n/l})$



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- Extension of the Knuth orbit if $l = 3 \rightarrow$ symplectic dual (using polarity in $\mathrm{PG}(5, q^{n/3})$ containing $\mathcal{V}_3(q^{n/3})$) [Lunardon, Marino, Polverino, Trombetti]

3. Symplectic semifields and commutative semifields

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- New examples
- Perfect nonlinear functions, equiangular lines, MUBS, ...

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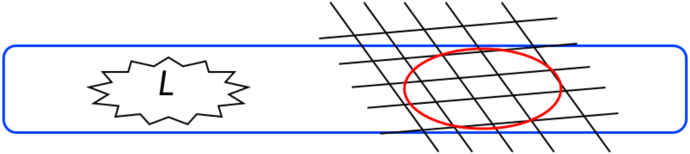

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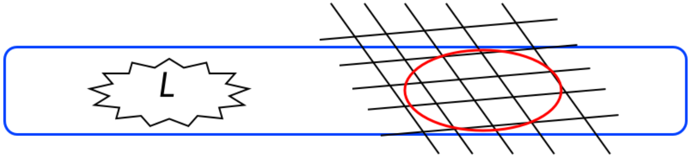

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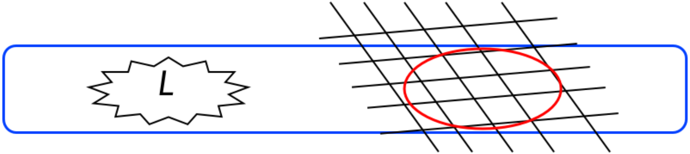
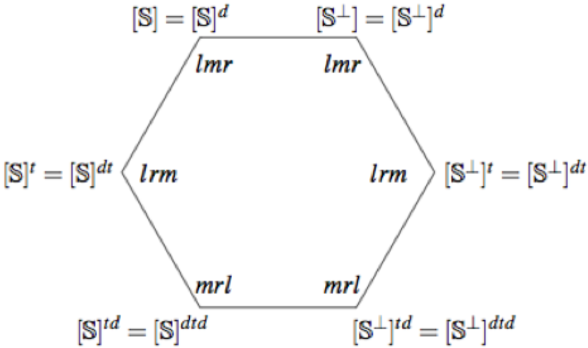


- many links with other structures in finite geometry, e.g. flocks of a quadratic cone and translation ovoids of GQ, pseudo-ovoids ("eggs") and TGQ
- few examples known: field, Dickson (1906), Cohen-Ganley (1982), Penttila-Williams (1999)

4. Rank two commutative semifields (RTCS)

- Knuth-orbit [Ball - Brown 2004]



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Theorem (Ball - Blokhuis - ML 2003, ML 2006)

Let $\mathbb{S}$ be an RTCS of order q^{2n}, q an odd prime power (p^{2n}, p an odd prime), with center $\mathbb{F}_q$. If $q \geq 4n^2 - 8n + 2$ ($p > 2n^2 - (4 - 2\sqrt{3})n + (3 - 2\sqrt{3})$), then $\mathbb{S}$ is either a field or a RTCS of Dickson type.

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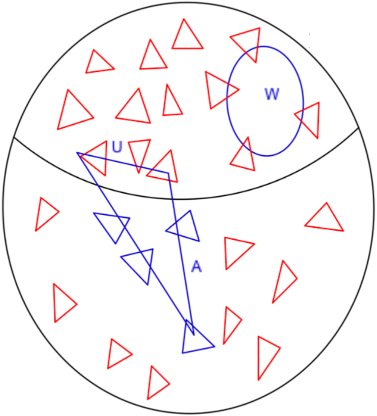

- If $r = 2$, U and W have the same dimension: $(\mathcal{D}, U, W) \to (\mathcal{D}, W, U)$: switching

TENSOR PRODUCT APPROACH

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Idea from a paper by R. Liebler (1981):

"*We have used [...] in hopes of making it clear that coordinates play no essential role in the study of class V planes. In fact, it is my view that coordinatization has played a role in the study of projective planes that is analogous to the role Artin [...] argues matrices have played in linear algebra.*"

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$T_{\mathbb{S}} = \varphi^{-1}(h_{\mathbb{S}})$, where φ is defined by

$$\varphi : V^{\vee} \otimes V^{\vee} \otimes V \to \mathrm{Hom}_{\mathbb{F}}(V \otimes V, V)$$

$$(v_1 \otimes v_2)(u^{\vee} \otimes v^{\vee} \otimes w)^{\varphi} := u^{\vee}(v_1)v^{\vee}(v_2)w.$$

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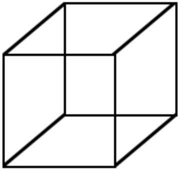
(Knuth 1965) *With bases of V_i $\rightarrow$ cube of structure constants a_{ijk}*



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- Thank you for your attention!