Quasi-Polar spaces

Geertrui Van de Voorde

University of Canterbury
Aotearoa New Zealand

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NEW ZEALAND 2020-2021-2022

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Joint work inside the Hermit Kingdom

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Jeroen Schillewaert
Auckland University, NZ

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Let's start at the very beginning

...this talk is about certain point sets in $\mathrm{PG}(n, q)$

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- $\mathrm{PG}(n, q)$: n-dimensional projective space
- points have homogeneous coordinates over the finite field $\mathbb{F}_q$, $q = p^h$, p prime

LET'S START AT THE VERY BEGINNING

All projective planes in this talk are Desarguesian ($\mathrm{PG}(2, q)$).

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Theorem (Segre 1955)

A set of $q+1$ points in $\mathrm{PG}(2, q)$, q odd, no three of which are collinear, is a conic.

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A conic in a projective plane is a set of points whose coordinates satisfy a homogeneous quadratic equation (e.g. $Y^2 = XZ$).

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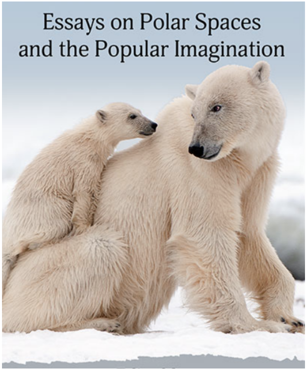
- And what if we impose the size?
  (no for even characteristic, yes for odd characteristic)

Polar spaces

Conics in $\mathrm{PG}(2, q) \rightarrow$ polar spaces in $\mathrm{PG}(n, q)$

Crash course in classical polar spaces



What Google thinks is a polar space

(Characteristic $\neq 2$ here)

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Conic in $\mathrm{PG}(2, q)$: points $X = (x, y, z)$ with $XAX^t = 0$, $A = A^t$:

$$[x, y, z] \begin{bmatrix} a & f & e \\ f & b & d \\ e & d & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\leftrightarrow ax^2 + by^2 + cz^2 + 2dyz + 2exz + 2fxy = 0.$$

Set of points $X = (x_0, x_1, \ldots, x_r)$ in $\mathrm{PG}(r, q)$ with $XAX^t = 0$, where A :

- Comes in orthogonal ($A = A^t$), symplectic $A = -A^t$ types

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CRASH COURSE IN CLASSICAL POLAR SPACES

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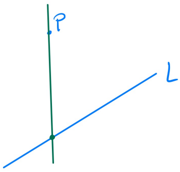
- polar space: Points of those sets + subspaces fully contained in them

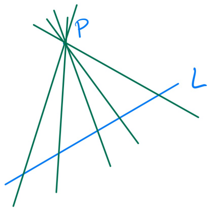
Crash course in classical polar spaces

Form	$n+1$	Name	Notation	Number of points	Collineation group
Alternating	$2r$	Symplectic	$W(2r-1,q)$	$(q^r+1)\theta_{r-1}(q)$	$\mathrm{P\Gamma Sp}(2r,q)$
Hermitian	$2r$	Hermitian	$H(2r-1,q)$	$(q^{r-1/2}+1)\theta_{r-1}(q)$	$\mathrm{P\Gamma U(2r,q)}$
Hermitian	$2r+1$	Hermitian	$H(2r,q)$	$(q^{r+1/2}+1)\theta_{r-1}(q)$	$\mathrm{P\Gamma U(2r+1,q)}$
Quadratic	$2r$	Hyperbolic	$Q^+(2r-1,q)$	$(q^{r-1}+1)\theta_{r-1}(q)$	$\mathrm{P\Gamma O}^+(2r,q)$
Quadratic	$2r+1$	Parabolic	$Q(2r,q)$	$(q^r+1)\theta_{r-1}(q)$	$\mathrm{P\Gamma O}(2r+1,q)$
Quadratic	$2r+2$	Elliptic	$Q^-(2r+1,q)$	$(q^{r+1}+1)\theta_{r-1}(q)$	$\mathrm{P\Gamma O}^-(2r+2,q)$

Crash course in classical polar spaces

Geometric point of view: point-line geometry$(\mathcal{P}, \mathcal{L})$ satisfying the 'one or all' axiom





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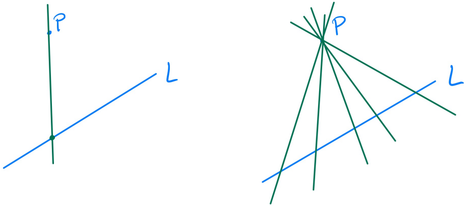

Theorem (Buekenhout-Shult-Veldkamp-Tits)

Finite polar spaces of rank at least 3 *(that is, there are planes lying in it) are classical.*

Characterising polar spaces

Segre-type problem

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First step

How does a hyperplane H intersect a non-singular polar space?

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How does a hyperplane H intersect a non-singular polar space?

- non-singular polar space in H or
- cone with vertex a point and base a non-singular polar space of the same type

Example

$\mathcal{Q}(6, q)$:

Hyperplanes meet either in

- non-singular quadric: $\mathcal{Q}^+(5, q)$ or $\mathcal{Q}^-(5, q)$
- cone with vertex a point, base a $\mathcal{Q}(4, q)$

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Three different intersection sizes with respect to hyperplanes.

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Characterising polar spaces

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Two different intersection sizes with respect to hyperplanes.

Characterising polar spaces

Example

$\mathcal{Q}^-(5, q)$:

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Characterising polar spaces

Example

$\mathcal{H}(r, q^2)$:

Hyperplanes meet either in

- non-singular Hermitian polar space: $\mathcal{H}(r-1, q^2)$
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Characterising polar spaces

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Back to the Segre-type problem

If a point set has the same intersection sizes with respect to hyperplanes as a polar space, is it a polar space?

Definition

A *quasi-polar space* is a point set that has the same intersection numbers with respect to hyperplanes as a classical polar space.

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Do we need to ask for the size?

THE SIZE OF A QUASI-POLAR SPACE

PROPOSITION (SCHILLEWAERT-VDV)

A quasi-polar space in $\mathrm{PG}(n, q)$ *of elliptic, hyperbolic, or Hermitian type has the same size as the polar space of that type.*

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  size is $|\mathcal{Q}^-(3, q)|$ (ovoid) or $q+1$ (line)
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Note that in both cases, the polar space had rank 1.

BACK TO THE SEGRE-TYPE PROBLEM

Apart from those exceptions, is every quasi-polar space a polar space?

Characterising polar spaces

Back to the Segre-type problem

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Theorem (De Winter- Schillewaert 2010)

A quasi-polar space that has the same intersection numbers with co-dimension two spaces as a polar space, is a polar space.

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Do we need the co-dimension two spaces?

De Clerck, Hamilton, O'Keefe, Penttila 2000

Quasi-quadrics and related structures

Frank De Clerck

Vakgroep Zuivere Wiskunde en Computeralgebra, Universiteit Gent
Galglaan 2, B-9000 Gent, Belgium
e-mail: fdc@cage.rug.ac.be

Nicholas Hamilton

Department of Mathematics, The University of Queensland
Brisbane 4072, Australia
e-mail: nick@maths.uq.edu.au

Christine M. O'Keefe

Department of Pure Mathematics, The University of Adelaide
Adelaide 5005, Australia
e-mail: cokeefe@maths.adelaide.edu.au

Tim Penttila

Department of Mathematics, University of Western Australia
Nedlands, Western Australia 6009, Australia
e-mail: penttila@maths.uwa.edu.au

Abstract

In a projective space $\mathrm{PG}(n, q)$ a *quasi-quadric* is a set of points that has the same intersection numbers with respect to hyperplanes as a non-degenerate quadric in that space. Of course, non-degenerate quadrics themselves are examples of quasi-quadrics, but many other examples exist. In the case that n is odd, quasi-quadrics have two sizes of intersections with hyperplanes and so are *two-character sets*. These sets are

Quasi-polar spaces

We will focus on the pivoting technique of De Clerck, Hamilton, O'Keefe, Penttila;

Quasi-polar spaces

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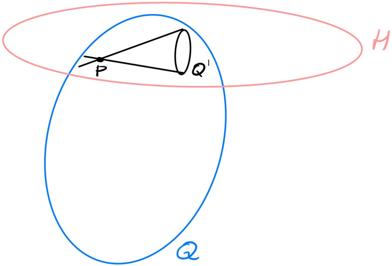
- there are other constructions: *See a.o. Aguglia, Cossidente, Korchmáros.*
- particular type of a set with few intersection numbers. *See a.o. Abatangelo, Barlotti, Bruen, Donati, Durante, Ferri, Napolitano, Olanda, Panella, Tallini, Tallini Scafati, Thas, Zanetti, Zuanni*

Pivoting in quadrics

Pivoting

- Take a quadric $\mathcal{Q}$
- Take a singular hyperplane H: $H \cap \mathcal{Q}$ is a cone $P\mathcal{Q}'$
- Replace $P\mathcal{Q}'$ with $P\mathcal{Q}''$ (cone of same type)



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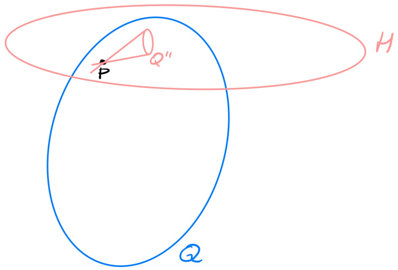
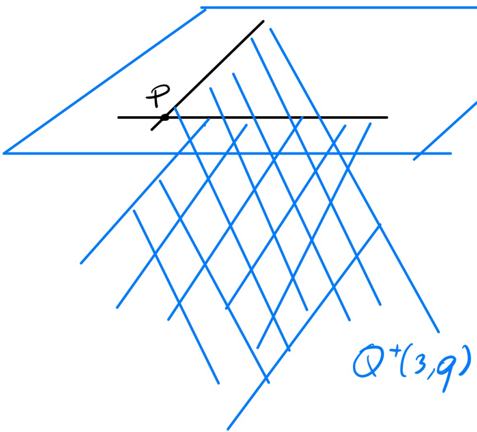


WHY DOES PIVOTING WORK?

EXAMPLE

- Take $\mathcal{Q}^+(3, q)$: every plane meets in $q + 1$ or $2q + 1$ points
- Take a singular plane π: meeting $\mathcal{Q}^+(3, q)$ in two lines L_1, L_2 intersecting in P
- Replace those lines by two other lines L'_1, L'_2 through $P \to \widetilde{Q}$.

Why does pivoting work?



We need to show that every plane meets $\widetilde{Q}$ in $q+1$ or $2q+1$ points.

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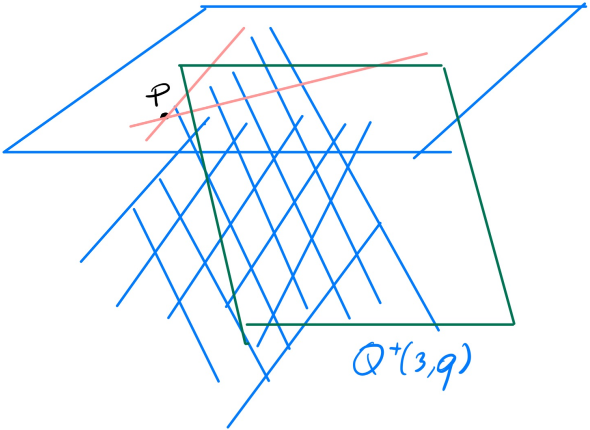

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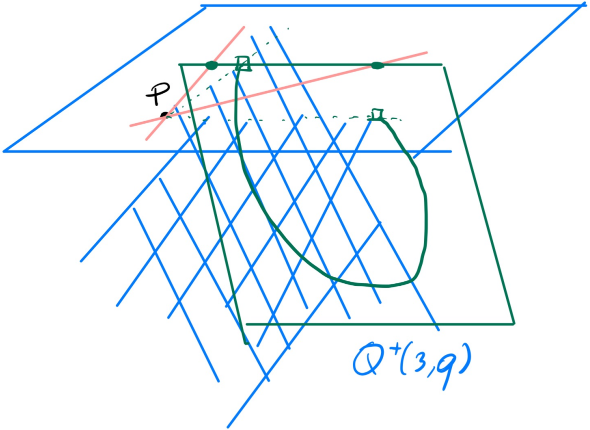

Plane not through P: $q+1-2+2=q+1$

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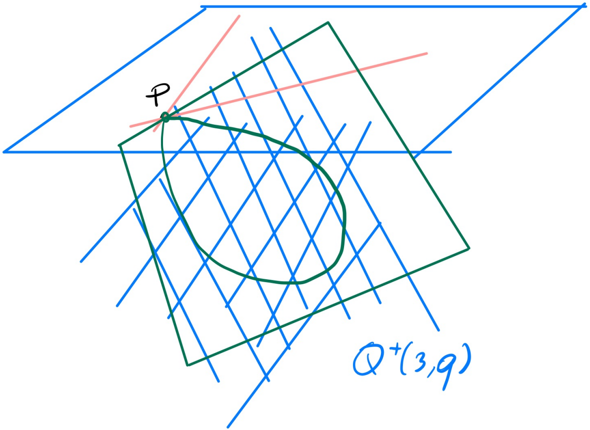

Plane through P, not through L_1, L_2, L_1', L_2': $q + 1$

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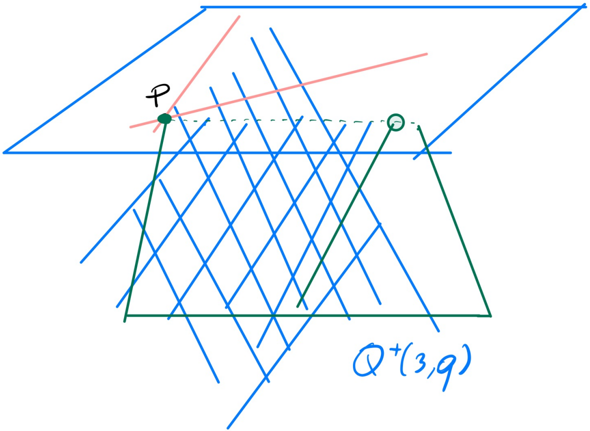

Plane through L_i: $q + 1 = 2q + 1 - (q + 1) + 1$

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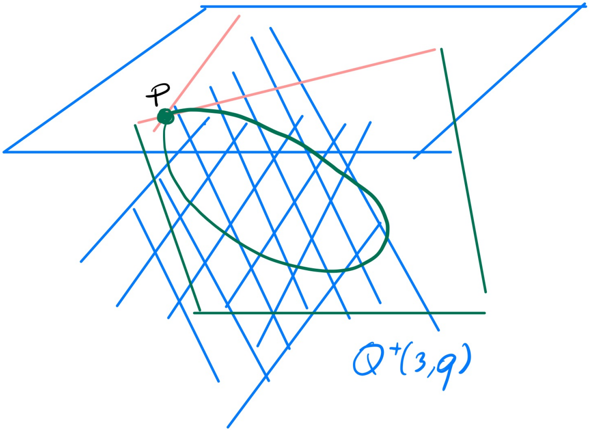

Plane through L_i': $2q + 1 = q + 1 - (1) + (q + 1)$

De Clerck, Hamilton, O'Keefe, Penttila

General question

Can we construct other quasi-quadrics from (quasi)-quadrics by changing the points in a hyperplane?

De Clerck, Hamilton, O'Keefe, Penttila

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We call this switching.

Switching in quasi-polar spaces

Theorem (Schillewaert-VdV)

Switching preserves the type of a quasi-polar space

SWITCHING IN QUASI-POLAR SPACES

THEOREM (SCHILLEWAERT-VDV)

Switching preserves the type of a quasi-polar space unless

- *elliptic and hyperbolic quadric in* $\mathrm{PG}(2n+1, 2)$
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Corollary

In the elliptic, hyperbolic, Hermitian case: switching preserves size.

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Switching in a quadric or Hermitian variety in $\mathrm{PG}(m, q)$*,* $m \geq 3$*, is only possible in singular hyperplanes unless*

- $q = 2$ *for elliptic and hyperbolic quadrics*
- $q = 2, 3, 4$ *for parabolic quadrics*

Switching in polar spaces

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Switching in a singular hyperplane of a polar space is pivoting unless for $\mathcal{Q}(2n, q)$, q even.

REPEATED SWITCHING IN QUASI-POLAR SPACES

- De Clerck–Durante:

Constructions and characterizations of classical sets in $\mathrm{PG}(n,q)$ 19

Note that the pivoting construction is breaking up the lines and subspaces on the quadric, moreover one can repeat pivoting as much as one wants, implying that the family of quasi-quadrics is quite wild. However, the following theorems are worthwhile to mention in this context.

Theorem 3.3 ([57]). *Let $\mathcal{K}$ be a set of points in $\mathrm{PG}(n,q)$, where $n \geqslant 4$ and $|\mathcal{K}| \geq q^3 + q^2 + q + 1$, such that $\mathcal{K}$ intersects all planes in 1, a, or b points, $b \geq 2q+1$, $\mathcal{K}$ intersects all solids in c, $c+q$, or $c+2q$ points, $c \leq q^2+1$, and there exist solids intersecting $\mathcal{K}$ in c points and in $c+q$ points; then $\mathcal{K}$ is a non-singular quadric of $\mathrm{PG}(4,q)$.*

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- Schillewaert-VdV: construction for repeated switching $q+1$ times

Parabolic quasi-quadrics

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PARABOLIC QUASI-QUADRICS

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Parabolic quasi-quadrics

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- For q even: there is a nucleus (all lines through N intersect the parabolic quadric in exactly one point)
- Definition of parabolic quasi-quadric?

Parabolic quasi-quadrics for q even

De Clerck-Hamilton-O'Keefe-Penttila define a parabolic quasi-quadric with nucleus N in $\mathrm{PG}(2n, q)$, q even, to be a set $\mathcal{S}$ of points such that

(A) $|\mathcal{S}| = \frac{q^{2n}-1}{q-1}$;

(B) Every hyperplane not through N intersects $\mathcal{S}$ in $|\mathcal{Q}^-(2n-1, q)|$ or $|\mathcal{Q}^+(2n-1, q)|$ points;

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OBSERVATION

We see that $N \notin \mathcal{S}$ and (a) follows immediately from (c).

Definition of quasi-quadric

Recall our definition:

(B') every hyperplane intersects $\mathcal{S}$ in $|\mathcal{Q}^-(2n-1,q)|$, $|Q^+(2n-1,q)|$ or $|P\mathcal{Q}(2n-2,q)|$ points.

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Observation

A quasi-quadric with nucleus as defined by DC-H-OK-P is a parabolic quasi-quadric as defined by us.

Definition of quasi-quadric

The definition of the nucleus

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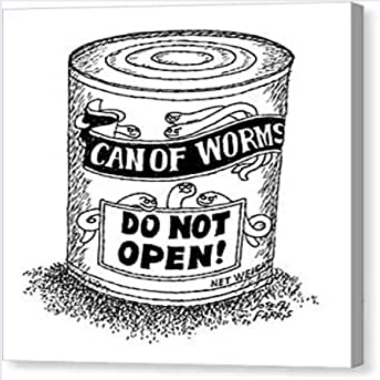

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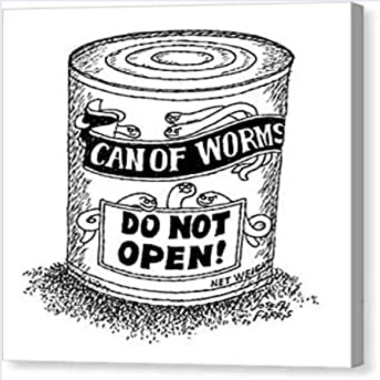

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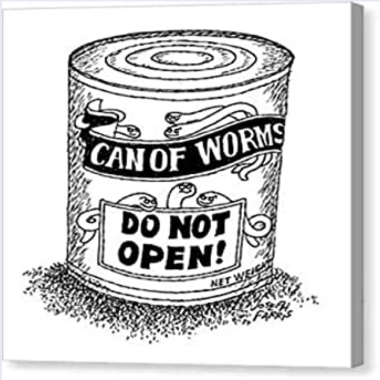

- Hyperplanes meeting $\mathcal{S}$ in $|P\mathcal{Q}(2n-2,q)|$ points contain a common point N
- + all hyperplanes through N contain $|P\mathcal{Q}(2n-2,q)|$ points?

THE EXISTENCE OF A NUCLEUS

$n = 1$

Every oval in $\mathrm{PG}(2, q)$, q even, admits a nucleus.

The existence of a nucleus

$n = 1$

Every oval in $\mathrm{PG}(2, q)$, q even, admits a nucleus.

No longer true for $n > 1$:

Theorem

A parabolic quasi-quadric in $\mathrm{PG}(2n, q)$, q even, does not necessarily have a nucleus.

The existence of a nucleus

But:

Theorem

If every codimension 2-space is contained in at least one singular hyperplane, then a parabolic quasi-quadric of size $|\mathcal{Q}(2n, q)|$, q even, has a nucleus.

An open problem

Question

Does a parabolic quasi-quadric necessarily have size $|\mathcal{Q}(2n, q)|$?

For $n = 1$, no, any arc will do.

AN OPEN PROBLEM

QUESTION

Does a parabolic quasi-quadric necessarily have size $|\mathcal{Q}(2n, q)|$?

For $n = 1$, no, any arc will do. But for $n > 1$?

$n = 2$

Let $\mathcal{S}$ be a set of points in $\mathrm{PG}(4, q)$, such that every 3-space meets $\mathcal{S}$ in $q^2 + 1, q^2 + q + 1$, or $q^2 + 2q + 1$ points. Is $|\mathcal{S}| = q^3 + q^2 + q + 1$?

Unitals

Hermitian quasi-polar spaces in $\mathrm{PG}(2, q^2)$: set of points such that every line meets in 1 or $q + 1$ points.

Exercise

A Hermitian quasi-polar space in $\mathrm{PG}(2, q^2)$:

UNITALS

Hermitian quasi-polar spaces in $\mathrm{PG}(2, q^2)$: set of points such that every line meets in 1 or $q + 1$ points.

EXERCISE

A Hermitian quasi-polar space in $\mathrm{PG}(2, q^2)$:

- either $q^2 + q + 1$ points, and then is a Baer subplane
- or $q^3 + 1$ points, and is called a unital

UNITALS

Unitals, embedded in $\mathrm{PG}(2, q^2)$, are well-studied, interesting point sets, and many characterisations of the Hermitian curve are known.

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Unitals, embedded in $\mathrm{PG}(2, q^2)$, are well-studied, interesting point sets, and many characterisations of the Hermitian curve are known.

See for example:

Abatangelo, Aguglia, Baker, Ball, Barwick, Billioti, Blokhuis, Casse, Cossidente, De Bruyn, De Resmini, Donati, Durante, Ebert, Feng, Giuzzi, Hui, Korchmáros, Larato, Li, Lunardon, Metsch, Nagy, Siciliano, Sonnino, Stroppel, Szőnyi, Rottey, Thas, Van Maldeghem, Wantz, Wong ...

UNITALS

A characterisation for the coding theorists:

UNITALS

A characterisation for the coding theorists:

THEOREM (BLOKHUIS, BROUWER, WILBRINK (1991))

A unital in $\mathrm{PG}(2, q^2)$ *is a Hermitian curve if and only if its incidence vector is contained in the code of points and lines of* $\mathrm{PG}(2, q^2)$.

(Recently generelised by Aguglia, Bartoli, Storme, Weiner).

Unitals in $\mathrm{PG}(2, q^2)$

- All known unitals of $\mathrm{PG}(2, q^2)$ can be obtained using Buekenhout's construction using André/Bruck-Bose.

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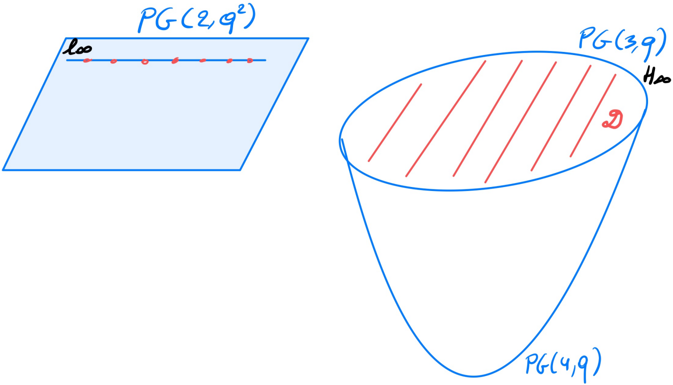

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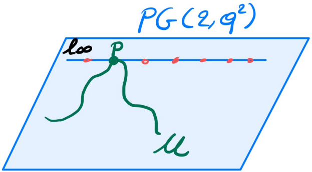

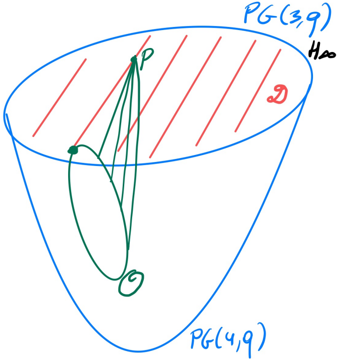

UNITALS IN PG$(2, q^2)$

TERMINOLOGY

- If $\mathcal{O}$ is an elliptic quadric: (ovoidal) Buekenhout-Metz unital
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UNITALS IN $\mathrm{PG}(2, q^2)$

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Their stabiliser in $\mathrm{PGL}(3, q^2)$, $q = 2^{2e+1}$, *is* $G \cong (C_4)^{2e+1}$ *and in* $\mathrm{P\Gamma L}$ *is* $G \ltimes K$, *where* K *is cyclic of order* $16e + 8$ *(and explicitely described).*

UNITALS

FEET IN ORTHOGONAL-BUEKENHOUT-METZ UNITALS

N. ABARZÚA, R. POMAREDA, AND O. VEGA

ABSTRACT. Given an Orthogonal-Buekenhout-Metz unital $U_{\alpha,\beta}$, embedded in $PG(2,q^2)$, and a point $P \notin U_{\alpha,\beta}$, we study the set of feet, $\tau_P(U_{\alpha,\beta})$, of P in $U_{\alpha,\beta}$. We characterize geometrically each of these sets as either $q+1$ collinear points or as $q+1$ points partitioned into two arcs. Other results about the geometry of these sets are also given.

UNITALS

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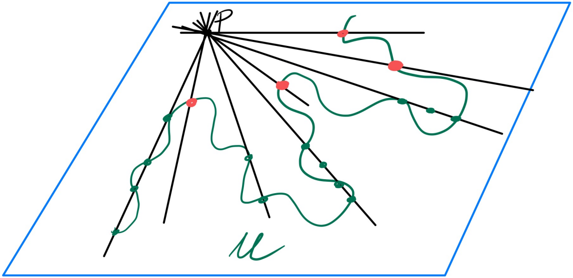

Unitals

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UNITALS

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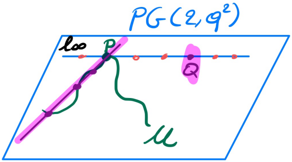
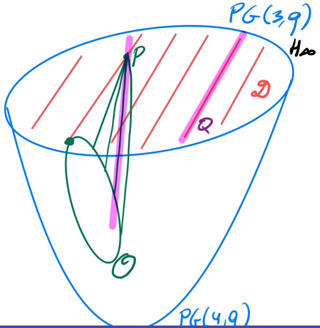
CLASSICAL RESULTS

- For a Hermitian curve: feet are always collinear
- If the feet are collinear for every point, then the unital is classical (J.A. Thas)
- For a non-classical Buekenhout-Metz unital: feet are collinear if and only if $P \in \ell_\infty$ (Ebert)





Unitals

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If the feet of a point are not collinear, which configurations are possible?

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Theorem (Abarzua, Pomareda, Vega 2018)

A line meets the feet set of a Buekenhout-Metz unital in $0, 1, 2, 4$ *or* $q+1$ *points; if not collinear, the feet set of a point form two arcs.*

Unitals

Theorem (Faulkner-VdV)

A line meets the feet set of a Buekenhout-Tits unital in $0, 1, 2, 3, 4$ *or* $q + 1$ *points.*

- *the possibility* 3 *occurs*
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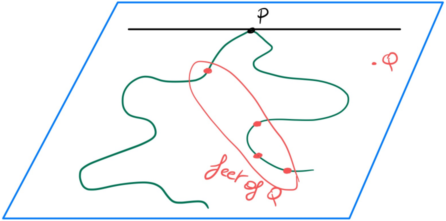

UNITALS

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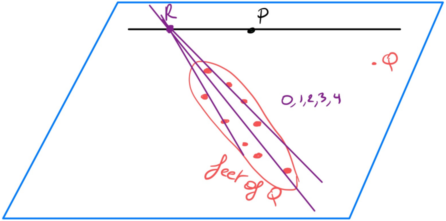

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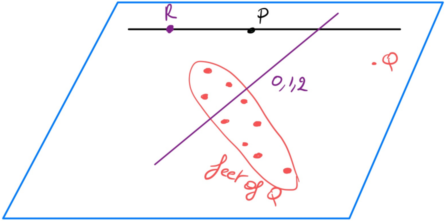

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Proof: non-trivial computations involving linearised polynomials + Lemma of Ceria, Cossidente, Marino, Pavese (2021)

CONCLUSION

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The horn of plenty of good old-fashioned finite geometry keeps on giving

Thank you!