A geometric approach to determine an optimal 2-dimensional flow on a graph

Joint work with Davide Mattiolo, Jozef Rajník and Gloria Tabarelli

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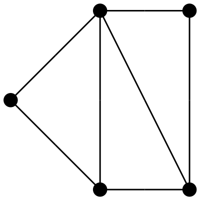
Flows

- G – Simple graph



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$(\mathbb{Z}_6, +)$

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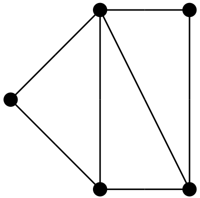

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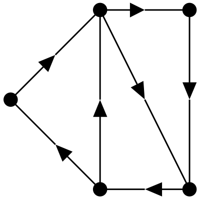

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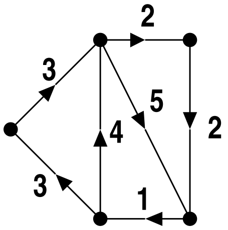

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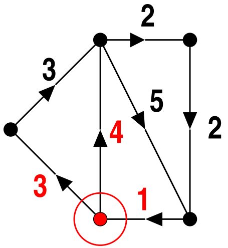

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$\mathbb{Z}_6$-flow

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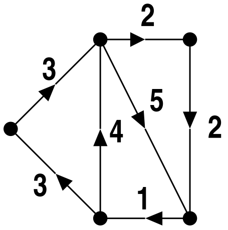
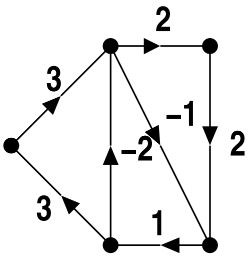
- Γ-flow (O, φ) on G



Nowhere-zero flows

- Nowhere-zero k-flow is a $\mathbb{Z}$-flow with values from $\{\pm 1, \pm 2, \ldots, \pm(k-1)\}$

Example of a nowhere-zero 4-flow



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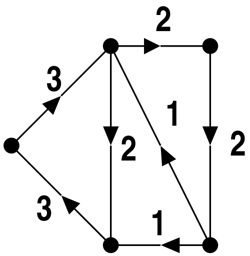

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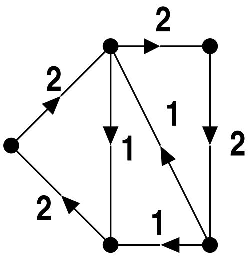

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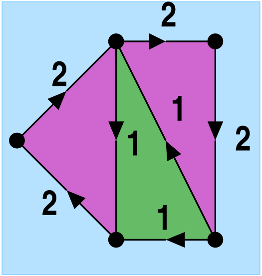

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Every bridgeless graph admits a nowhere-zero 5-flow.

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Remark

Constant 5 cannot be improved both for $\mathbb{Z}$-flows and for $\mathbb{R}$-flows.

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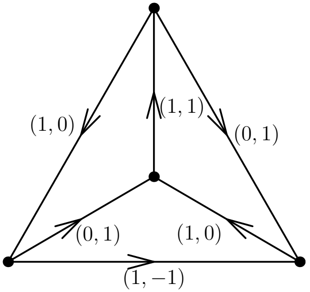
- $d = 1$: circular nowhere-zero r-flows
- $d = 2$: complex flows [Thomassen, 2014]
- $r = 2$: Unit vector flows [Wang et. al, Thomassen]

Multidimensional flows: example in $\mathbb{R}^2$

A $(1+\sqrt{2}, 2)$-flow on K_4



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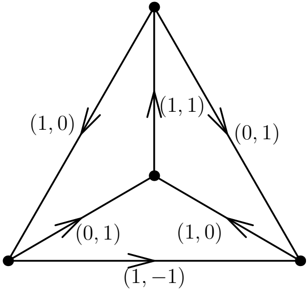

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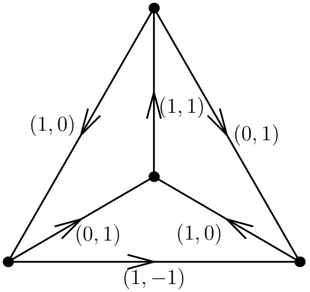

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YES, but we have not a trivial proof

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- continuous function $x \mapsto$ maximum norm

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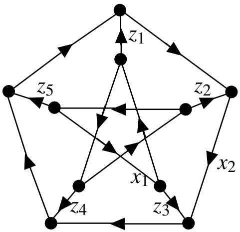
$(2,3)$-NZF of the Petersen graph P ($\phi_3(P) = 2$).



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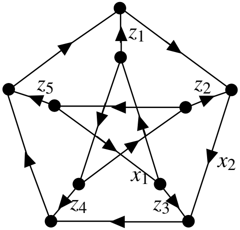

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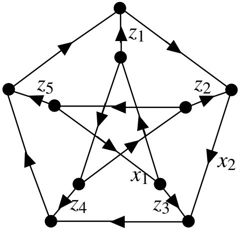

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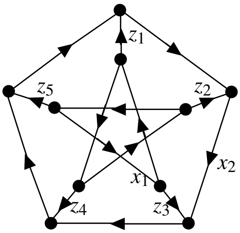

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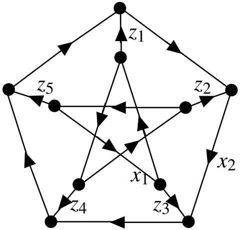

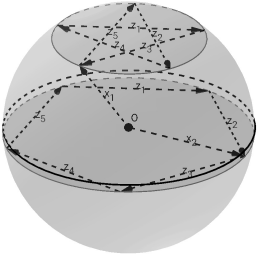

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- $\phi_7(G) = 2$ (observed by Thomassen [2014])

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Conjecture (Berge-Fulkerson 1971)

Each bridgeless cubic graph G has 6 perfect matchings such that every edge of G is contained in exactly two of them. Or, equivalently, G has a 6-cycle 4-cover.

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Theorem [Thomassen, 2014]

For a graph G, the following statements are equivalent

(i) G has a $(3, 1)$-NZF,

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(iii) G has a $(2, 2)$-NZF with values from $\{z \in \mathbb{C} \mid z^3 = 1\}$.

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For this kind of questions an answer for cubic graphs implies an answer for the general case!

Looking for an upper-bound for ϕ_2

Proposition (D.Mattiolo, G.M., J.Rajnik, G.Tabarelli, 2021)

For each 3-edge-colourable cubic graph G,

$$\phi_2(G) \leq 1 + \sqrt{2}.$$

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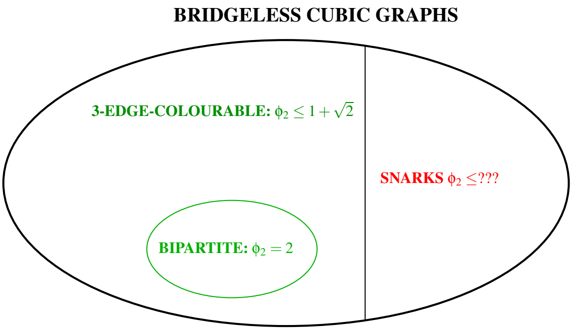

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- $1 \leq ||\varphi(e)|| \leq \sqrt{1^2 + 2^2} = \sqrt{5}$

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Theorem (Mattiolo, M., Rajnik, Tabarelli)

If G has an oriented k-cycle double cover, then

- $\phi_2(G) = 2$, if $k = 3$;
- $\phi_2(G) \leq 1 + \sqrt{2}$, if $k = 4$;
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- The proof is based on finding a set of k points in the complex plane such that the ratio of maximum distance to minimum distance is the smallest possible.
- For $k = 3, 4, 5$ the best possible configurations are proved to be the vertices of a regular k-gon. [Bateman, Erdös (1951)]

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Oriented 5-CDC Conjecture (Jaeger, 1988)

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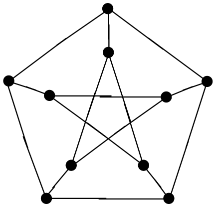
Corollary (Mattiolo, M., Rajnik, Tabarelli, 2021)

If the oriented 5-CDC conjecture holds true, then $\phi_2(G) \leq \Phi^2 \approx 2.618$ for every bridgeless graph G.

Looking for a graph with large 2-dimensional flow number. 1^{st} attempt: Petersen graph



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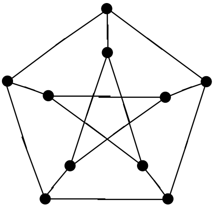

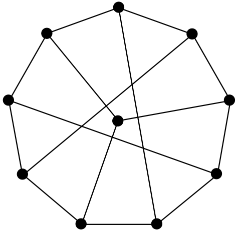

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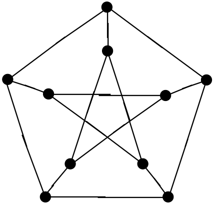
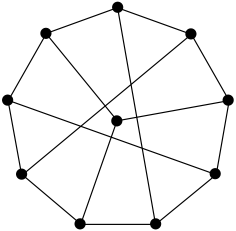
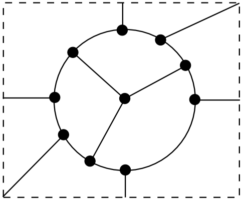
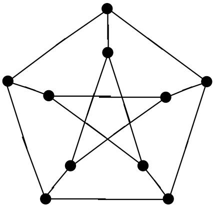
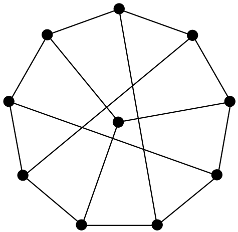


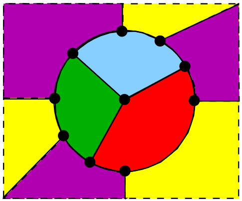




Looking for a graph with large 2-dimensional flow number. 1^{st} attempt: Petersen graph







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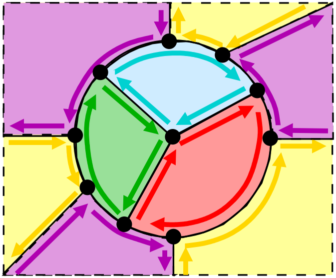
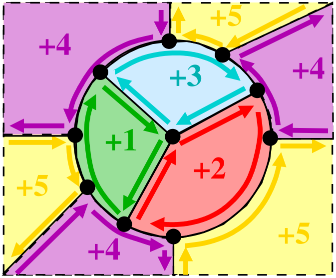


An oriented 5-CDC of the Petersen graph

Looking for a graph with large 2-dimensional flow number. 1^{st} attempt: Petersen graph



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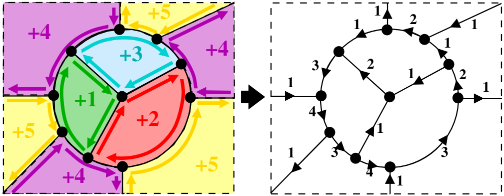

A 5-NZF of the Petersen graph

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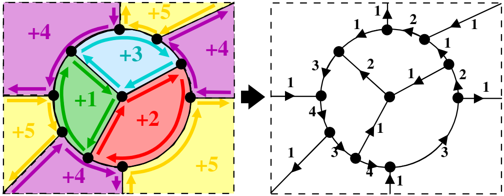
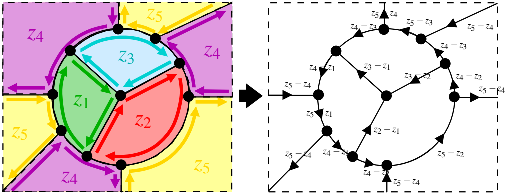


A 5-NZF of the Petersen graph

It is well-known that $\phi_1(P) = 5$

Looking for a graph with large 2-dimensional flow number. 1^{st} attempt: Petersen graph



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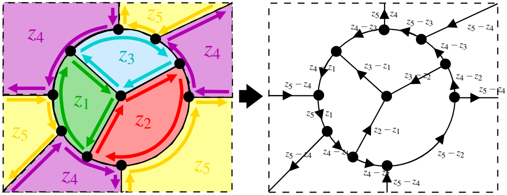
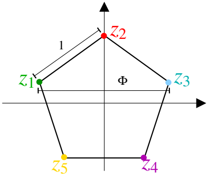




A $(\Phi^2, 2)$-NZF of the Petersen graph (Recall $1 + \Phi = \Phi^2$)

Looking for a graph with large 2-dimensional flow number: a better 2-dimensional flow on the Petersen graph

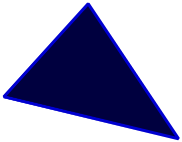
We use a different geometric construction to improve previous result.

Best upper bound known so far

$\phi_2(P) \leq 1 + \sqrt{7/3} \approx 2.5275$ $(\Phi^2 \approx 2.6180)$

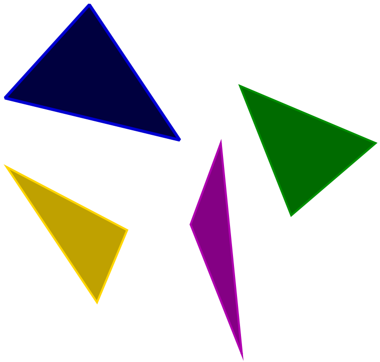
Flow triangulations

- Triangle: set of points (sides and interior)



Flow triangulations

- Triangle: set of points (sides and interior)
- Attachable sides:



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- Triangle: set of points (sides and interior)
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  - same length
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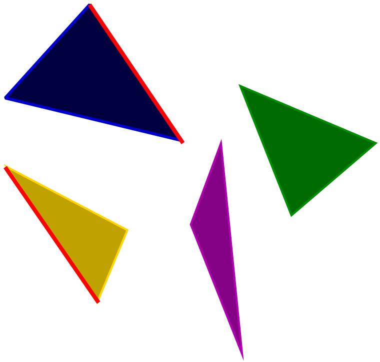

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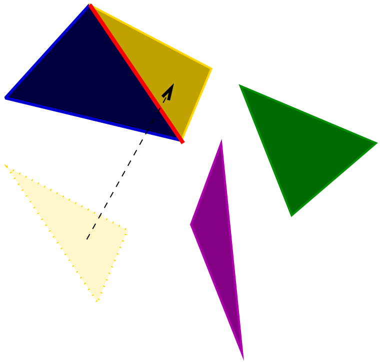

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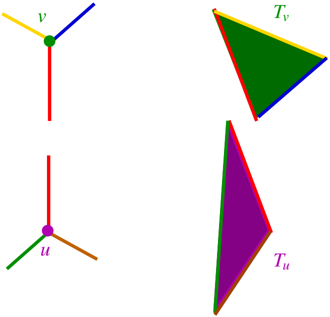

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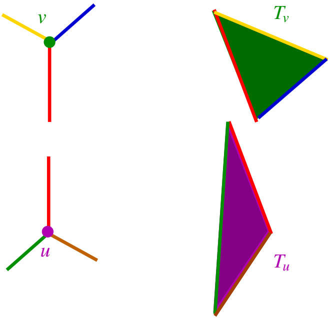

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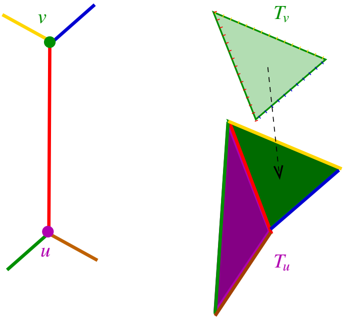

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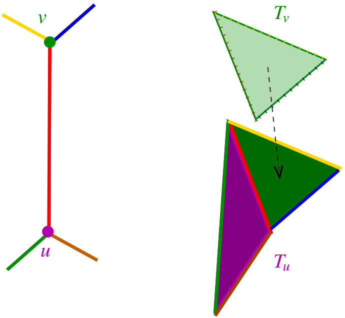
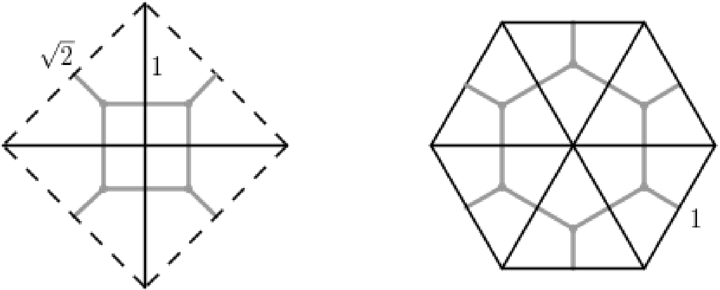
- side lengths from $[1, r-1]$



Examples of flow triangulations



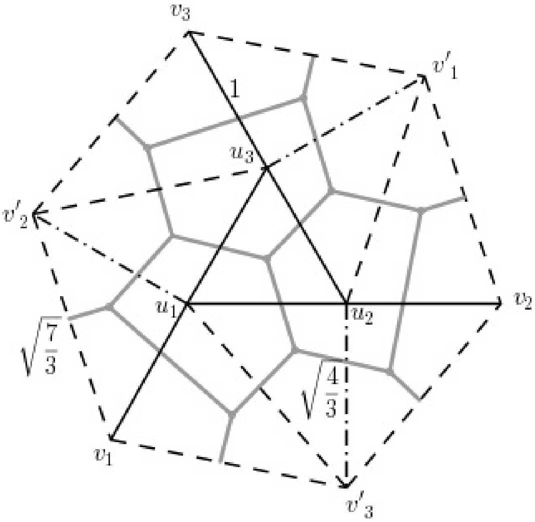
Flow triangulations of K_4 and $K_{3,3}$

Flow triangulations

Proposition (Mattiolo, M., Rajnik, Tabarelli)

A bridgeless cubic graph G has an r-flow triangulation if and only if G has an $(r, 2)$-flow.

$\phi_2(P) \leq 1 + \sqrt{7/3} \approx 2.5275$



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What triangulation is *nice*

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Problem

Is a nice r-flow triangulation possible for every $(r, 2)$-flow of G?

- What if G is bipartite?

Once again: General upper bound for $\phi_2(G)$?

- New possible conjecture: $\phi_2(G) \leq 1 + \sqrt{7/3} \approx 2.5275$ for all G?

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- Computer assisted: $\phi_2(P_\Delta) \leq 2.590296429$
- We conjecture: $\phi_2(G) \leq \Phi^2 \approx 2.6180$

1-dimensional flows and 3-edge-colourings

1-dimensional flow number of cubic graphs

If G is a cubic graph, then on of the following holds:

- $\phi_1(G) = 3$ and G is bipartite
- $\phi_1(G) = 4$ and G is 3-edge-colourable (non-bipartite)
- $\phi_1(G) > 4$ and G is not 3-edge-colourable

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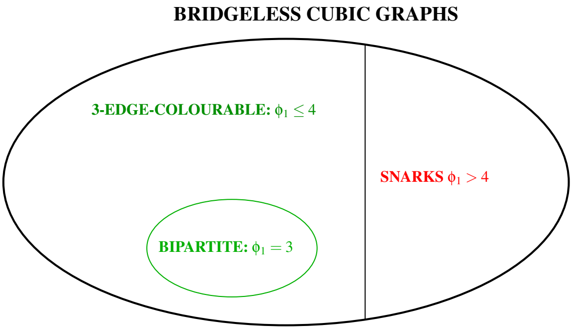
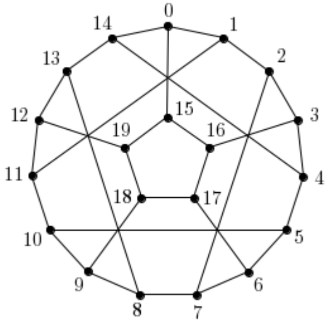
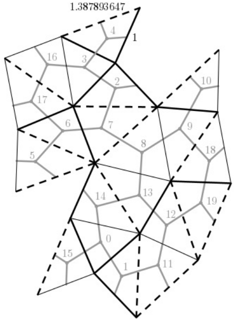


No such classification for ϕ_2:

- $\phi_2(J_5) \leq 2.387893647 < 1 + \sqrt{2} = \phi_2(K_4)$





A general lower bound for $\phi_2(G)$?

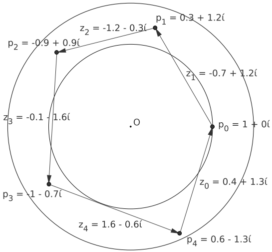
Theorem - Mattiolo, M., Rajnik, Tabarelli 2022

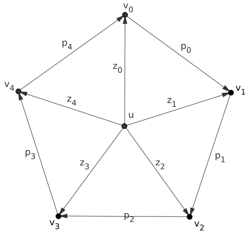
Let G be a cubic graph and let g be its odd-girth. Then,

$$\phi_2(G) \geq \begin{cases} 1 + 2\sin(\frac{\pi}{6} \cdot \frac{g}{g-1}) & \text{if } g \equiv 1, 3 \mod 6, \\ 1 + 2\sin(\frac{\pi}{6} \cdot \frac{g+1}{g}) & \text{if } g \equiv 5 \mod 6. \end{cases}$$

A general lower bound for $\phi_2(G)$?

$$\phi_2(G) \geq \phi_2(W_g)$$



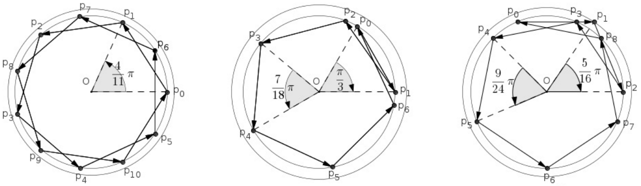


Theorem - Mattiolo, M., Rajnik, Tabarelli 2022

Let W_g be a wheel of order $g+1$. Then,

$$\phi_2(W_g) = \begin{cases} 2 & \text{if } g \text{ even,} \\ 1 + 2\sin(\frac{\pi}{6} \cdot \frac{g}{g-1}) & \text{if } g \equiv 1, 3 \mod 6, \\ 1 + 2\sin(\frac{\pi}{6} \cdot \frac{g+1}{g}) & \text{if } g \equiv 5 \mod 6. \end{cases}$$

Proof (more than 15 pages): the following three types of configurations are optimal



$n = 11$ $n = 7$ $n = 9$

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- $\phi_2(G) \leq \Phi^2$ for every bridgeless graph G?
- Does a graph G such that $\phi_2(G) = \Phi^2$ exist?

Thanks for your attention