Plane algebraic curves with many symmetries, and complete (k, n)-arcs

Gábor Korchmáros

University of Basilicata, Italy, and
Eötvös L. University of Budapest (Hungary) AC Research Group

joint work with H. Borghes, G.P. Nagy, P. Speziali, and T. Szőnyi

Finite Geometries, Sixth Irsee Conference

August 28 - September 3 2022, Irsee (Germany)

Automorphisms of algebraic curves

Automorphisms of algebraic curves

Curves with many automorphisms $\Rightarrow$ nice geometric and combinatorial properties,

Automorphisms of algebraic curves

Curves with many automorphisms $\Rightarrow$ nice geometric and combinatorial properties,
sometimes, with some new features impossible in zero characteristic.

Automorphisms of algebraic curves

Curves with many automorphisms $\Rightarrow$ nice geometric and combinatorial properties,
sometimes, with some new features impossible in zero characteristic.
Are some of these curves of interest in applications? (Finite geometry, Coding theory, etc.)

Automorphisms of algebraic curves

Curves with many automorphisms $\Rightarrow$ nice geometric and combinatorial properties,
sometimes, with some new features impossible in zero characteristic.
Are some of these curves of interest in applications? (Finite geometry, Coding theory, etc.)
Typical situation in applications:

Automorphisms of algebraic curves

Curves with many automorphisms $\Rightarrow$ nice geometric and combinatorial properties,
sometimes, with some new features impossible in zero characteristic.
Are some of these curves of interest in applications? (Finite geometry, Coding theory, etc.)
Typical situation in applications:
the curve is embedded in $PG(r, q)$ as an absolutely irreducible (not necessarily non-singular) curve, and

Automorphisms of algebraic curves

Curves with many automorphisms $\Rightarrow$ nice geometric and combinatorial properties,
sometimes, with some new features impossible in zero characteristic.
Are some of these curves of interest in applications? (Finite geometry, Coding theory, etc.)
Typical situation in applications:
the curve is embedded in $PG(r, q)$ as an absolutely irreducible (not necessarily non-singular) curve, and
application wants basic data of the geometry of the curve,

Automorphisms of algebraic curves

Curves with many automorphisms $\Rightarrow$ nice geometric and combinatorial properties,
sometimes, with some new features impossible in zero characteristic.
Are some of these curves of interest in applications? (Finite geometry, Coding theory, etc.)
Typical situation in applications:
the curve is embedded in $PG(r, q)$ as an absolutely irreducible (not necessarily non-singular) curve, and
application wants basic data of the geometry of the curve,
(degree, singular points, number of points over $\mathbb{F}_{q^m}$, combinatorial properties of the configuration of those points, intersection multiplicities with hyperplanes at a point of the curve).

Automorphisms of algebraic curves

Curves with many automorphisms $\Rightarrow$ nice geometric and combinatorial properties,
sometimes, with some new features impossible in zero characteristic.
Are some of these curves of interest in applications? (Finite geometry, Coding theory, etc.)
Typical situation in applications:
the curve is embedded in $PG(r, q)$ as an absolutely irreducible (not necessarily non-singular) curve, and
application wants basic data of the geometry of the curve, (degree, singular points, number of points over $\mathbb{F}_{q^m}$, combinatorial properties of the configuration of those points, intersection multiplicities with hyperplanes at a point of the curve).
The study of the geometry of a curve (like other objects) may benefit from its symmetries.

Automorphisms of algebraic curves

Curves with many automorphisms $\Rightarrow$ nice geometric and combinatorial properties,
sometimes, with some new features impossible in zero characteristic.
Are some of these curves of interest in applications? (Finite geometry, Coding theory, etc.)
Typical situation in applications:
the curve is embedded in $PG(r, q)$ as an absolutely irreducible (not necessarily non-singular) curve, and
application wants basic data of the geometry of the curve, (degree, singular points, number of points over $\mathbb{F}_{q^m}$, combinatorial properties of the configuration of those points, intersection multiplicities with hyperplanes at a point of the curve).
The study of the geometry of a curve (like other objects) may benefit from its symmetries.

Symmetries of a curve

Symmetries of a curve

The (classical) term *symmetry of a curve* = projective automorphism (or linear transformation, or projectivity) of $PG(r, q)$ which leaves the curve invariant.

Symmetries of a curve

The (classical) term *symmetry of a curve* = projective automorphism (or linear transformation, or projectivity) of $PG(r,q)$ which leaves the curve invariant.
Formally, $\mathcal{C}$ is a curve of $PG(r,q)$, σ is a projective automorphism of $\mathcal{C}$ $\Leftrightarrow$ $\sigma \in PGL(r+1,q)$ and $\sigma(\mathcal{C}) = \mathcal{C}$.

Symmetries of a curve

The (classical) term *symmetry of a curve* = projective automorphism (or linear transformation, or projectivity) of $PG(r,q)$ which leaves the curve invariant.

Formally, $\mathcal{C}$ is a curve of $PG(r,q)$, σ is a projective automorphism of $\mathcal{C} \Leftrightarrow \sigma \in PGL(r+1,q)$ and $\sigma(\mathcal{C}) = \mathcal{C}$.

In Algebraic geometry, an automorphism of an algebraic curve may also be birational (and not necessarily linear).

Symmetries of a curve

The (classical) term *symmetry of a curve* = projective automorphism (or linear transformation, or projectivity) of $PG(r, q)$ which leaves the curve invariant.
Formally, $\mathcal{C}$ is a curve of $PG(r, q)$, σ is a projective automorphism of $\mathcal{C} \Leftrightarrow \sigma \in PGL(r + 1, q)$ and $\sigma(\mathcal{C}) = \mathcal{C}$.
In Algebraic geometry, an automorphism of an algebraic curve may also be birational (and not necessarily linear).
Remark

Symmetries of a curve

The (classical) term *symmetry of a curve* = projective automorphism (or linear transformation, or projectivity) of $PG(r, q)$ which leaves the curve invariant.
Formally, $\mathcal{C}$ is a curve of $PG(r, q)$, σ is a projective automorphism of $\mathcal{C} \Leftrightarrow \sigma \in PGL(r+1, q)$ and $\sigma(\mathcal{C}) = \mathcal{C}$.
In Algebraic geometry, an automorphism of an algebraic curve may also be birational (and not necessarily linear).

Remark

Birational non-linear automorphism does not preserve the shape of our geometric object, i.e. its combinatorial properties,

Symmetries of a curve

The (classical) term *symmetry of a curve* = projective automorphism (or linear transformation, or projectivity) of $PG(r,q)$ which leaves the curve invariant.
Formally, $\mathcal{C}$ is a curve of $PG(r,q)$, σ is a projective automorphism of $\mathcal{C} \Leftrightarrow \sigma \in PGL(r+1,q)$ and $\sigma(\mathcal{C}) = \mathcal{C}$.
In Algebraic geometry, an automorphism of an algebraic curve may also be birational (and not necessarily linear).

Remark

Birational non-linear automorphism does not preserve the shape of our geometric object, i.e. its combinatorial properties, and therefore has minor or no interest in Finite geometry.

Symmetries of a curve

The (classical) term *symmetry of a curve* = projective automorphism (or linear transformation, or projectivity) of $PG(r, q)$ which leaves the curve invariant.

Formally, $\mathcal{C}$ is a curve of $PG(r, q)$, σ is a projective automorphism of $\mathcal{C} \Leftrightarrow \sigma \in PGL(r+1, q)$ and $\sigma(\mathcal{C}) = \mathcal{C}$.

In Algebraic geometry, an automorphism of an algebraic curve may also be birational (and not necessarily linear).

Remark

Birational non-linear automorphism does not preserve the shape of our geometric object, i.e. its combinatorial properties, and therefore has minor or no interest in Finite geometry.

$\Rightarrow$ motivation for the study of linear automorphism group of a curve.

Symmetries of a curve

The (classical) term *symmetry of a curve* = projective automorphism (or linear transformation, or projectivity) of $PG(r, q)$ which leaves the curve invariant.

Formally, $\mathcal{C}$ is a curve of $PG(r, q)$, σ is a projective automorphism of $\mathcal{C} \Leftrightarrow \sigma \in PGL(r+1, q)$ and $\sigma(\mathcal{C}) = \mathcal{C}$.

In Algebraic geometry, an automorphism of an algebraic curve may also be birational (and not necessarily linear).

Remark

Birational non-linear automorphism does not preserve the shape of our geometric object, i.e. its combinatorial properties, and therefore has minor or no interest in Finite geometry.

$\Rightarrow$ motivation for the study of linear automorphism group of a curve.

For a given curve, the problem of finding its linear automorphisms is frequently challenging,

Symmetries of a curve

The (classical) term *symmetry of a curve* = projective automorphism (or linear transformation, or projectivity) of $PG(r, q)$ which leaves the curve invariant.

Formally, $\mathcal{C}$ is a curve of $PG(r, q)$, σ is a projective automorphism of $\mathcal{C} \Leftrightarrow \sigma \in PGL(r+1, q)$ and $\sigma(\mathcal{C}) = \mathcal{C}$.

In Algebraic geometry, an automorphism of an algebraic curve may also be birational (and not necessarily linear).

Remark

Birational non-linear automorphism does not preserve the shape of our geometric object, i.e. its combinatorial properties, and therefore has minor or no interest in Finite geometry.

$\Rightarrow$ motivation for the study of linear automorphism group of a curve.

For a given curve, the problem of finding its linear automorphisms is frequently challenging, although the action on points (and/or on lines, blocks etc.) is bounded by the specific geometric (and combinatorial) properties of the curve.

Set up for the study of the linear automorphism group of a plane algebraic curve.

Set up for the study of the linear automorphism group of a plane algebraic curve.

$p \geq 2 :=$ prime, q a power of p

Set up for the study of the linear automorphism group of a plane algebraic curve.

$p \geq 2$:= prime, q a power of p
$PGL(3, q)$:= 3-dimensional projective linear group defined over a finite field $\mathbb{F}_q$

Set up for the study of the linear automorphism group of a plane algebraic curve.

$p \geq 2$:= prime, q a power of p
$PGL(3,q)$:= 3-dimensional projective linear group defined over a finite field $\mathbb{F}_q$
G:= subgroup of $PGL(3,q)$

Set up for the study of the linear automorphism group of a plane algebraic curve.

$p \geq 2$:= prime, q a power of p
$PGL(3,q)$:= 3-dimensional projective linear group defined over a finite field $\mathbb{F}_q$
G:= subgroup of $PGL(3,q)$
Remark $PGL(3,q)$ is a subgroup of $PGL(3,q^m)$ for $m \geq 1 \Rightarrow$,
$G \leq PGL(3,q^m)$

Set up for the study of the linear automorphism group of a plane algebraic curve.

$p \geq 2$:= prime, q a power of p
$PGL(3,q)$:= 3-dimensional projective linear group defined over a finite field $\mathbb{F}_q$
G:= subgroup of $PGL(3,q)$
Remark $PGL(3,q)$ is a subgroup of $PGL(3,q^m)$ for $m \geq 1 \Rightarrow$,
$G \leq PGL(3,q^m)$
It makes sense to investigate G-invariant plane curves $\mathcal{C}$ of $PG(2,q^m)$ where $m \geq 1$.

Set up for the study of the linear automorphism group of a plane algebraic curve.

$p \geq 2$:= prime, q a power of p
$PGL(3, q)$:= 3-dimensional projective linear group defined over a finite field $\mathbb{F}_q$
G:= subgroup of $PGL(3, q)$
Remark $PGL(3, q)$ is a subgroup of $PGL(3, q^m)$ for $m \geq 1 \Rightarrow$, $G \leq PGL(3, q^m)$
It makes sense to investigate G-invariant plane curves $\mathcal{C}$ of $PG(2, q^m)$ where $m \geq 1$.
Intuitively, if G is large (with respect to q) then the degree of $\mathcal{C}$ must also be large.

Set up for the study of the linear automorphism group of a plane algebraic curve.

$p \geq 2 :=$ prime, q a power of p
$PGL(3, q) :=$ 3-dimensional projective linear group defined over a finite field $\mathbb{F}_q$
$G :=$ subgroup of $PGL(3, q)$
Remark $PGL(3, q)$ is a subgroup of $PGL(3, q^m)$ for $m \geq 1 \Rightarrow$, $G \leq PGL(3, q^m)$
It makes sense to investigate G-invariant plane curves $\mathcal{C}$ of $PG(2, q^m)$ where $m \geq 1$.
Intuitively, if G is large (with respect to q) then the degree of $\mathcal{C}$ must also be large.

How large $\deg(\mathcal{C})$ must be at least for a G-invariant plane curve $\mathcal{C}$?

Set up for the study of the linear automorphism group of a plane algebraic curve.

$p \geq 2$:= prime, q a power of p
$PGL(3, q)$:= 3-dimensional projective linear group defined over a finite field $\mathbb{F}_q$
G:= subgroup of $PGL(3, q)$
Remark $PGL(3, q)$ is a subgroup of $PGL(3, q^m)$ for $m \geq 1 \Rightarrow$, $G \leq PGL(3, q^m)$
It makes sense to investigate G-invariant plane curves $\mathcal{C}$ of $PG(2, q^m)$ where $m \geq 1$.
Intuitively, if G is large (with respect to q) then the degree of $\mathcal{C}$ must also be large.

How large $\deg(\mathcal{C})$ must be at least for a G-invariant plane curve $\mathcal{C}$?
What about the plane curves $\mathcal{C}$ hitting the minimum?

Projective automorphisms of a plane algebraic curve.

Some more notation

Projective automorphisms of a plane algebraic curve.

Some more notation

$d(G)$:= the smallest integer that is the degree of a G-invariant irreducible plane curve $\mathcal{C}$, other than a line.

Projective automorphisms of a plane algebraic curve.

Some more notation

$d(G)$:= the smallest integer that is the degree of a G-invariant irreducible plane curve $\mathcal{C}$, other than a line.

Remark

$d(G)$ only depends on the conjugacy class of G in $PGL(3, q)$.

Some more notation

$d(G)$:= the smallest integer that is the degree of a G-invariant irreducible plane curve $\mathcal{C}$, other than a line.

Remark

$d(G)$ only depends on the conjugacy class of G in $PGL(3, q)$.

Σ:=Spectrum of the degrees of G-invariant curves.

Some more notation

$d(G)$:= the smallest integer that is the degree of a G-invariant irreducible plane curve $\mathcal{C}$, other than a line.

Remark

$d(G)$ only depends on the conjugacy class of G in $PGL(3, q)$.

Σ:=Spectrum of the degrees of G-invariant curves.

Main problems

Projective automorphisms of a plane algebraic curve.

Some more notation

$d(G)$:= the smallest integer that is the degree of a G-invariant irreducible plane curve $\mathcal{C}$, other than a line.

Remark

$d(G)$ only depends on the conjugacy class of G in $PGL(3, q)$.

Σ:=Spectrum of the degrees of G-invariant curves.

Main problems

(i) find $d(G)$ for a given subgroup G of $PGL(3, q)$; (i.e. $d(G)$ is the smallest value in Σ)

Projective automorphisms of a plane algebraic curve.

Some more notation

$d(G)$:= the smallest integer that is the degree of a G-invariant irreducible plane curve $\mathcal{C}$, other than a line.

Remark

$d(G)$ only depends on the conjugacy class of G in $PGL(3, q)$.

Σ:=Spectrum of the degrees of G-invariant curves.

Main problems

(i) find $d(G)$ for a given subgroup G of $PGL(3, q)$; (i.e. $d(G)$ is the smallest value in Σ)

(ii) find the largest positive integer $\varepsilon(G)$ depending on q s.t. there is no G-invariant irreducible plane curve of degree $< d(G) + \varepsilon(G)$. (i.e. $d(G) + \varepsilon(G)$ is the second smallest value in Σ, and $\varepsilon(G) - 1$ is the first gap).

Some more notation

$d(G)$:= the smallest integer that is the degree of a G-invariant irreducible plane curve $\mathcal{C}$, other than a line.

Remark

$d(G)$ only depends on the conjugacy class of G in $PGL(3, q)$.

Σ:=Spectrum of the degrees of G-invariant curves.

Main problems

(i) find $d(G)$ for a given subgroup G of $PGL(3, q)$; (i.e. $d(G)$ is the smallest value in Σ)

(ii) find the largest positive integer $\varepsilon(G)$ depending on q s.t. there is no G-invariant irreducible plane curve of degree $< d(G) + \varepsilon(G)$. (i.e. $d(G) + \varepsilon(G)$ is the second smallest value in Σ, and $\varepsilon(G) - 1$ is the first gap).

(iii) find all G-invariant irreducible plane curves of degree $d(G)$.

Projective automorphisms of a plane algebraic curve.

Some more notation

$d(G)$:= the smallest integer that is the degree of a G-invariant irreducible plane curve $\mathcal{C}$, other than a line.

Remark

$d(G)$ only depends on the conjugacy class of G in $PGL(3, q)$.

Σ:=Spectrum of the degrees of G-invariant curves.

Main problems

(i) find $d(G)$ for a given subgroup G of $PGL(3, q)$; (i.e. $d(G)$ is the smallest value in Σ)

(ii) find the largest positive integer $\varepsilon(G)$ depending on q s.t. there is no G-invariant irreducible plane curve of degree $< d(G) + \varepsilon(G)$. (i.e. $d(G) + \varepsilon(G)$ is the second smallest value in Σ, and $\varepsilon(G) - 1$ is the first gap).

(iii) find all G-invariant irreducible plane curves of degree $d(G)$.

Essential tool in investigating the above problems: Pencil of plane algebraic curves

Projective automorphisms of a plane algebraic curve.

Some more notation

$d(G)$:= the smallest integer that is the degree of a G-invariant irreducible plane curve $\mathcal{C}$, other than a line.

Remark

$d(G)$ only depends on the conjugacy class of G in $PGL(3,q)$.

Σ:=Spectrum of the degrees of G-invariant curves.

Main problems

(i) find $d(G)$ for a given subgroup G of $PGL(3,q)$; (i.e. $d(G)$ is the smallest value in Σ)

(ii) find the largest positive integer $\varepsilon(G)$ depending on q s.t. there is no G-invariant irreducible plane curve of degree $< d(G)+\varepsilon(G)$. (i.e. $d(G)+\varepsilon(G)$ is the second smallest value in Σ, and $\varepsilon(G)-1$ is the first gap).

(iii) find all G-invariant irreducible plane curves of degree $d(G)$.

Essential tool in investigating the above problems: Pencil of plane algebraic curves

G-fixed pencil of plane algebraic curves

G-fixed pencil of plane algebraic curves

$F_1, F_2 \in \mathbb{F}_{q^m}[x_1, x_2, x_3]$, homogenous polynomials of degree d

G-fixed pencil of plane algebraic curves

$F_1, F_2 \in \mathbb{F}_{q^m}[x_1, x_2, x_3]$, homogenous polynomials of degree d
C_λ:= (degree d) plane curve of equation $F_1 + \lambda F_2 = 0$

G-fixed pencil of plane algebraic curves

$F_1, F_2 \in \mathbb{F}_{q^m}[x_1, x_2, x_3]$, homogenous polynomials of degree d
$C_\lambda :=$ (degree d) plane curve of equation $F_1 + \lambda F_2 = 0$
$C_\infty :=$the plane curve of equation $F_2 = 0$

G-fixed pencil of plane algebraic curves

$F_1, F_2 \in \mathbb{F}_{q^m}[x_1, x_2, x_3]$, homogenous polynomials of degree d
$C_\lambda :=$ (degree d) plane curve of equation $F_1 + \lambda F_2 = 0$
$C_\infty :=$ the plane curve of equation $F_2 = 0$
pencil $\Lambda := \{C_\lambda | \lambda \in \mathbb{F}_{q^{rm}}, r \geq 1\} \cup \{C_\infty\}$

G-fixed pencil of plane algebraic curves

$F_1, F_2 \in \mathbb{F}_{q^m}[x_1, x_2, x_3]$, homogenous polynomials of degree d
$C_\lambda :=$ (degree d) plane curve of equation $F_1 + \lambda F_2 = 0$
C_∞ :=the plane curve of equation $F_2 = 0$
pencil $\Lambda := \{C_\lambda | \lambda \in \mathbb{F}_{q^{rm}}, r \geq 1\} \cup \{C_\infty\} = < C_0, C_\infty >$.

G-fixed pencil of plane algebraic curves

$F_1, F_2 \in \mathbb{F}_{q^m}[x_1, x_2, x_3]$, homogenous polynomials of degree d
$C_\lambda :=$ (degree d) plane curve of equation $F_1 + \lambda F_2 = 0$
$C_\infty :=$the plane curve of equation $F_2 = 0$
pencil $\Lambda := \{C_\lambda | \lambda \in \mathbb{F}_{q^{rm}}, r \geq 1\} \cup \{C_\infty\} = < C_0, C_\infty >$.
Λ is *G-fixed pencil* if G preserves each curve in Λ.

G-fixed pencil of plane algebraic curves

$F_1, F_2 \in \mathbb{F}_{q^m}[x_1, x_2, x_3]$, homogenous polynomials of degree d
$C_\lambda :=$ (degree d) plane curve of equation $F_1 + \lambda F_2 = 0$
C_∞ :=the plane curve of equation $F_2 = 0$
pencil $\Lambda := \{C_\lambda | \lambda \in \mathbb{F}_{q^{rm}}, r \geq 1\} \cup \{C_\infty\} = < C_0, C_\infty >$.
Λ is *G-fixed pencil* if G preserves each curve in Λ.

Theorem *Let Λ be a G-fixed pencil of curves of degree d without common component. Let $\mathcal{U}$ be any further G-invariant curve. Then* $\deg(\mathcal{U}) \geq |G|/d$.

G-fixed pencil of plane algebraic curves

$F_1, F_2 \in \mathbb{F}_{q^m}[x_1, x_2, x_3]$, homogenous polynomials of degree d
$C_\lambda :=$ (degree d) plane curve of equation $F_1 + \lambda F_2 = 0$
C_∞ :=the plane curve of equation $F_2 = 0$
pencil $\Lambda := \{C_\lambda | \lambda \in \mathbb{F}_{q^{rm}}, r \geq 1\} \cup \{C_\infty\} = < C_0, C_\infty >$.
Λ is *G-fixed pencil* if G preserves each curve in Λ.

Theorem *Let Λ be a G-fixed pencil of curves of degree d without common component. Let $\mathcal{U}$ be any further G-invariant curve. Then* $\deg(\mathcal{U}) \geq |G|/d$.
Proof $|G|$ is a lower bound on the number of common points of $\mathcal{U}$ with a generically chosen (absolutely irreducible) curve $\mathcal{C}$ from Λ. Comparison of this lower bound with the upper bound derived from the Bézout theorem yields $d \cdot \deg(\mathcal{U}) \geq |G|$.

G-fixed pencil of plane algebraic curves

$F_1, F_2 \in \mathbb{F}_{q^m}[x_1, x_2, x_3]$, homogenous polynomials of degree d
$C_\lambda :=$ (degree d) plane curve of equation $F_1 + \lambda F_2 = 0$
C_∞ :=the plane curve of equation $F_2 = 0$
pencil $\Lambda := \{C_\lambda | \lambda \in \mathbb{F}_{q^{rm}}, r \geq 1\} \cup \{C_\infty\} = < C_0, C_\infty >$.
Λ is *G-fixed pencil* if G preserves each curve in Λ.

Theorem *Let Λ be a G-fixed pencil of curves of degree d without common component. Let $\mathcal{U}$ be any further G-invariant curve. Then* $\deg(\mathcal{U}) \geq |G|/d$.
Proof $|G|$ is a lower bound on the number of common points of $\mathcal{U}$ with a generically chosen (absolutely irreducible) curve $\mathcal{C}$ from Λ. Comparison of this lower bound with the upper bound derived from the Bézout theorem yields $d \cdot \deg(\mathcal{U}) \geq |G|$.
Corollary *Let $|G| > d^2$. Then $d(G) \leq d$. If* $\deg(C) = d(G)$ *then C is either in Λ, or C is a nonlinear component of a curve in Λ.*

G-fixed pencil of plane algebraic curves

$F_1, F_2 \in \mathbb{F}_{q^m}[x_1, x_2, x_3]$, homogenous polynomials of degree d
$C_\lambda :=$ (degree d) plane curve of equation $F_1 + \lambda F_2 = 0$
C_∞ :=the plane curve of equation $F_2 = 0$
pencil $\Lambda := \{C_\lambda | \lambda \in \mathbb{F}_{q^{rm}}, r \geq 1\} \cup \{C_\infty\} = < C_0, C_\infty >$.
Λ is *G-fixed pencil* if G preserves each curve in Λ.

Theorem *Let Λ be a G-fixed pencil of curves of degree d without common component. Let $\mathcal{U}$ be any further G-invariant curve. Then* $\deg(\mathcal{U}) \geq |G|/d$.
Proof $|G|$ is a lower bound on the number of common points of $\mathcal{U}$ with a generically chosen (absolutely irreducible) curve $\mathcal{C}$ from Λ. Comparison of this lower bound with the upper bound derived from the Bézout theorem yields $d \cdot \deg(\mathcal{U}) \geq |G|$.
Corollary *Let $|G| > d^2$. Then $d(G) \leq d$. If* $\deg(C) = d(G)$ *then C is either in Λ, or C is a nonlinear component of a curve in Λ.*
Problem Find G-invariant pencils!

G-fixed pencil of plane algebraic curves

$F_1, F_2 \in \mathbb{F}_{q^m}[x_1, x_2, x_3]$, homogenous polynomials of degree d
$C_\lambda :=$ (degree d) plane curve of equation $F_1 + \lambda F_2 = 0$
C_∞ :=the plane curve of equation $F_2 = 0$
pencil $\Lambda := \{C_\lambda | \lambda \in \mathbb{F}_{q^{rm}}, r \geq 1\} \cup \{C_\infty\} = < C_0, C_\infty >$.
Λ is *G-fixed pencil* if G preserves each curve in Λ.

Theorem *Let Λ be a G-fixed pencil of curves of degree d without common component. Let $\mathcal{U}$ be any further G-invariant curve. Then* $\deg(\mathcal{U}) \geq |G|/d$.
Proof $|G|$ is a lower bound on the number of common points of $\mathcal{U}$ with a generically chosen (absolutely irreducible) curve $\mathcal{C}$ from Λ. Comparison of this lower bound with the upper bound derived from the Bézout theorem yields $d \cdot \deg(\mathcal{U}) \geq |G|$.
Corollary *Let $|G| > d^2$. Then $d(G) \leq d$. If* $\deg(C) = d(G)$ *then C is either in Λ, or C is a nonlinear component of a curve in Λ.*
Problem Find G-invariant pencils! (in general difficult, no general method from classical Algebraic geometry)

G-fixed pencil of plane algebraic curves

$F_1, F_2 \in \mathbb{F}_{q^m}[x_1, x_2, x_3]$, homogenous polynomials of degree d
$C_\lambda :=$ (degree d) plane curve of equation $F_1 + \lambda F_2 = 0$
C_∞ :=the plane curve of equation $F_2 = 0$
pencil $\Lambda := \{C_\lambda | \lambda \in \mathbb{F}_{q^{rm}}, r \geq 1\} \cup \{C_\infty\} = < C_0, C_\infty >$.
Λ is *G-fixed pencil* if G preserves each curve in Λ.

Theorem *Let Λ be a G-fixed pencil of curves of degree d without common component. Let $\mathcal{U}$ be any further G-invariant curve. Then* $\deg(\mathcal{U}) \geq |G|/d$.
Proof $|G|$ is a lower bound on the number of common points of $\mathcal{U}$ with a generically chosen (absolutely irreducible) curve $\mathcal{C}$ from Λ. Comparison of this lower bound with the upper bound derived from the Bézout theorem yields $d \cdot \deg(\mathcal{U}) \geq |G|$.
Corollary *Let $|G| > d^2$. Then $d(G) \leq d$. If* $\deg(C) = d(G)$ *then C is either in Λ, or C is a nonlinear component of a curve in Λ.*
Problem Find G-invariant pencils! (in general difficult, no general method from classical Algebraic geometry)

Sufficient condition for a pencil to be G-fixed

Sufficient condition for a pencil to be G-fixed

Lemma If Λ has at least three G-invariant curves, then Λ is a G-fixed pencil.

Sufficient condition for a pencil to be G-fixed

Lemma If Λ has at least three G-invariant curves, then Λ is a G-fixed pencil.

Remark Two G-invariant curves in Λ are not enough in general, unless in particular cases.

Sufficient condition for a pencil to be G-fixed

Lemma If Λ has at least three G-invariant curves, then Λ is a G-fixed pencil.

Remark Two G-invariant curves in Λ are not enough in general, unless in particular cases.

One of these cases:

Sufficient condition for a pencil to be G-fixed

Lemma If Λ has at least three G-invariant curves, then Λ is a G-fixed pencil.

Remark Two G-invariant curves in Λ are not enough in general, unless in particular cases.

One of these cases: Take $\Gamma \leq \mathrm{GL}(3,q)$ which is the pullback of G in the natural homomorphism $\mathrm{GL}(3,q) \rightarrow PGL(3,q)$.

Sufficient condition for a pencil to be G-fixed

Lemma If Λ has at least three G-invariant curves, then Λ is a G-fixed pencil.

Remark Two G-invariant curves in Λ are not enough in general, unless in particular cases.

One of these cases: Take $\Gamma \leq \mathrm{GL}(3,q)$ which is the pullback of G in the natural homomorphism $\mathrm{GL}(3,q) \to PGL(3,q)$.

If $F_1, F_2 \in \mathbb{F}_{q^m}[x_1, x_2, x_3]$ are both Γ-invariant homogeneous polynomials of the same degree d, then any linear combination $F = F_1 + \lambda F_2$ is also a Γ-invariant form.

Sufficient condition for a pencil to be G-fixed

Lemma If Λ has at least three G-invariant curves, then Λ is a G-fixed pencil.

Remark Two G-invariant curves in Λ are not enough in general, unless in particular cases.

One of these cases: Take $\Gamma \leq \mathrm{GL}(3, q)$ which is the pullback of G in the natural homomorphism $\mathrm{GL}(3, q) \to PGL(3, q)$.

If $F_1, F_2 \in \mathbb{F}_{q^m}[x_1, x_2, x_3]$ are both Γ-invariant homogeneous polynomials of the same degree d, then any linear combination $F = F_1 + \lambda F_2$ is also a Γ-invariant form.

By *projectivization*, the pencil $< F_1, F_2 >$ is G-fixed.

Sufficient condition for a pencil to be G-fixed

Lemma If Λ has at least three G-invariant curves, then Λ is a G-fixed pencil.
Remark Two G-invariant curves in Λ are not enough in general, unless in particular cases.
One of these cases: Take $\Gamma \leq \mathrm{GL}(3, q)$ which is the pullback of G in the natural homomorphism $\mathrm{GL}(3, q) \to PGL(3, q)$.
If $F_1, F_2 \in \mathbb{F}_{q^m}[x_1, x_2, x_3]$ are both Γ-invariant homogeneous polynomials of the same degree d, then any linear combination $F = F_1 + \lambda F_2$ is also a Γ-invariant form.
By *projectivization*, the pencil $< F_1, F_2 >$ is G-fixed.
Remark It is enough that the rational function F_1/F_2 is Γ-invariant.

Sufficient condition for a pencil to be G-fixed

Lemma If Λ has at least three G-invariant curves, then Λ is a G-fixed pencil.

Remark Two G-invariant curves in Λ are not enough in general, unless in particular cases.

One of these cases: Take $\Gamma \leq \mathrm{GL}(3, q)$ which is the pullback of G in the natural homomorphism $\mathrm{GL}(3, q) \to PGL(3, q)$.

If $F_1, F_2 \in \mathbb{F}_{q^m}[x_1, x_2, x_3]$ are both Γ-invariant homogeneous polynomials of the same degree d, then any linear combination $F = F_1 + \lambda F_2$ is also a Γ-invariant form.

By *projectivization*, the pencil $< F_1, F_2 >$ is G-fixed.

Remark It is enough that the rational function F_1/F_2 is Γ-invariant.

Focus on the following problem:

Sufficient condition for a pencil to be G-fixed

Lemma If Λ has at least three G-invariant curves, then Λ is a G-fixed pencil.

Remark Two G-invariant curves in Λ are not enough in general, unless in particular cases.

One of these cases: Take $\Gamma \leq \mathrm{GL}(3, q)$ which is the pullback of G in the natural homomorphism $\mathrm{GL}(3, q) \to PGL(3, q)$.

If $F_1, F_2 \in \mathbb{F}_{q^m}[x_1, x_2, x_3]$ are both Γ-invariant homogeneous polynomials of the same degree d, then any linear combination $F = F_1 + \lambda F_2$ is also a Γ-invariant form.

By *projectivization*, the pencil $< F_1, F_2 >$ is G-fixed.

Remark It is enough that the rational function F_1/F_2 is Γ-invariant.

Focus on the following problem:

Problem *How to find G-invariant pencils for large subgroups G of $PGL(3, q)$, in particular for maximal subgroups G of $PGL(3, q)$?*

Sufficient condition for a pencil to be G-fixed

Lemma If Λ has at least three G-invariant curves, then Λ is a G-fixed pencil.
Remark Two G-invariant curves in Λ are not enough in general, unless in particular cases.
One of these cases: Take $\Gamma \leq \mathrm{GL}(3, q)$ which is the pullback of G in the natural homomorphism $\mathrm{GL}(3, q) \to PGL(3, q)$.
If $F_1, F_2 \in \mathbb{F}_{q^m}[x_1, x_2, x_3]$ are both Γ-invariant homogeneous polynomials of the same degree d, then any linear combination $F = F_1 + \lambda F_2$ is also a Γ-invariant form.
By *projectivization*, the pencil $< F_1, F_2 >$ is G-fixed.
Remark It is enough that the rational function F_1/F_2 is Γ-invariant.
Focus on the following problem:
Problem *How to find G-invariant pencils for large subgroups G of $PGL(3, q)$, in particular for maximal subgroups G of $PGL(3, q)$?*
These G-invariant pencils change depending on which maximal subgroup is taken for G;

Sufficient condition for a pencil to be G-fixed

Lemma If Λ has at least three G-invariant curves, then Λ is a G-fixed pencil.

Remark Two G-invariant curves in Λ are not enough in general, unless in particular cases.

One of these cases: Take $\Gamma \leq \mathrm{GL}(3,q)$ which is the pullback of G in the natural homomorphism $\mathrm{GL}(3,q) \to PGL(3,q)$.

If $F_1, F_2 \in \mathbb{F}_{q^m}[x_1, x_2, x_3]$ are both Γ-invariant homogeneous polynomials of the same degree d, then any linear combination $F = F_1 + \lambda F_2$ is also a Γ-invariant form.

By *projectivization*, the pencil $< F_1, F_2 >$ is G-fixed.

Remark It is enough that the rational function F_1/F_2 is Γ-invariant.

Focus on the following problem:

Problem *How to find G-invariant pencils for large subgroups G of $PGL(3,q)$, in particular for maximal subgroups G of $PGL(3,q)$?*

These G-invariant pencils change depending on which maximal subgroup is taken for G; a case-by-case analysis is needed.

Maximal subgroups of $PGL(3, q)$

Maximal subgroups of $PGL(3, q)$

(i) $PSL(3, q)$ for $q \equiv 1 \pmod 3$, having order $\frac{1}{3}(q^2 + q + 1)q^3(q + 1)(q - 1)^2$

(ii) the stabilizer of a point of $PG(2, q)$, having order $q^3(q + 1)(q - 1)^2$

(iii) the stabilizer of a line of $PG(2, q)$, having order $q^3(q + 1)(q - 1)^2$

(iv) the stabilizer of an Hermitian curve of $PG(2, q)$ for $q = n^2$, having order $n^3(n^3 + 1)(n - 1)^2$

(v) the stabilizer of a triangle of $PG(2, q)$, having order $6(q - 1)^2$

(vi) the stabilizer of an imaginary triangle (i.e., a triangle in $PG(2, q^3) \setminus PG(2, q)$), having order $3(q^2 + q + 1)$

(vii) for q odd, the stabilizer of an irreducible conic, having order $q(q + 1)(q - 1)$

(viii) sporadic subgroups (of order ≤ 2520)

Case $G = PGL(3,q)$, $|G| = (q^2+q+1)q^3(q+1)(q-1)^2$

Example (Borges 2009)

Case $G = PGL(3,q)$, $|G| = (q^2+q+1)q^3(q+1)(q-1)^2$

Example (Borges 2009) $m, n :=$ positive integers, $\gcd(m,n) = 1$

Case $G = PGL(3,q)$, $|G| = (q^2+q+1)q^3(q+1)(q-1)^2$

Example (Borges 2009) $m, n :=$ positive integers, $\gcd(m,n) = 1$
$\mathcal{F}_{n,m} :=$ plane curve in $PG(2,q)$ with affine equation:

$$\frac{(X^{q^n} - X)(Y^{q^m} - Y) - (Y^{q^n} - Y)(X^{q^m} - X)}{(X^{q^2} - X)(Y^{q} - Y) - (Y^{q^2} - Y)(X^{q} - X)} = 0.$$

Case $G = PGL(3,q)$, $|G| = (q^2+q+1)q^3(q+1)(q-1)^2$

Example (Borges 2009) $m, n :=$ positive integers, $\gcd(m,n) = 1$
$\mathcal{F}_{n,m} :=$ plane curve in $PG(2,q)$ with affine equation:

$$\frac{(X^{q^n}-X)(Y^{q^m}-Y)-(Y^{q^n}-Y)(X^{q^m}-X)}{(X^{q^2}-X)(Y^{q}-Y)-(Y^{q^2}-Y)(X^{q}-X)} = 0.$$

$\mathcal{F}_{3,1}$, named DGZ (Dickson-Guralnick-Zieve) curve,

Case $G = PGL(3,q)$, $|G| = (q^2+q+1)q^3(q+1)(q-1)^2$

Example (Borges 2009) $m, n :=$ positive integers, $\gcd(m,n) = 1$
$\mathcal{F}_{n,m} :=$ plane curve in $PG(2,q)$ with affine equation:

$$\frac{(X^{q^n}-X)(Y^{q^m}-Y)-(Y^{q^n}-Y)(X^{q^m}-X)}{(X^{q^2}-X)(Y^{q}-Y)-(Y^{q^2}-Y)(X^{q}-X)} = 0.$$

$\mathcal{F}_{3,1}$, named DGZ (Dickson-Guralnick-Zieve) curve,
$\mathcal{F}_{3,2}$, the dual DGZ curve.

Example (Borges 2009) $m, n :=$ positive integers, $\gcd(m, n) = 1$
$\mathcal{F}_{n,m} :=$ plane curve in $PG(2, q)$ with affine equation:

$$\frac{(X^{q^n} - X)(Y^{q^m} - Y) - (Y^{q^n} - Y)(X^{q^m} - X)}{(X^{q^2} - X)(Y^q - Y) - (Y^{q^2} - Y)(X^q - X)} = 0.$$

$\mathcal{F}_{3,1}$, named DGZ (Dickson-Guralnick-Zieve) curve,
$\mathcal{F}_{3,2}$, the dual DGZ curve.
$< \mathcal{F}_{3,1}^{q+1}, \mathcal{F}_{3,2}^{q} >$ is a $PGL(3, q)$-fixed pencil.

Case $G = PGL(3,q)$, $|G| = (q^2+q+1)q^3(q+1)(q-1)^2$

Example (Borges 2009) $m, n :=$ positive integers, $\gcd(m,n) = 1$
$\mathcal{F}_{n,m} :=$ plane curve in $PG(2,q)$ with affine equation:

$$\frac{(X^{q^n}-X)(Y^{q^m}-Y)-(Y^{q^n}-Y)(X^{q^m}-X)}{(X^{q^2}-X)(Y^{q}-Y)-(Y^{q^2}-Y)(X^{q}-X)} = 0.$$

$\mathcal{F}_{3,1}$, named DGZ (Dickson-Guralnick-Zieve) curve,
$\mathcal{F}_{3,2}$, the dual DGZ curve.
$< \mathcal{F}_{3,1}^{q+1}, \mathcal{F}_{3,2}^{q} >$ is a $PGL(3,q)$-fixed pencil.

$$d(PGL(3,q)) = q^3 - q^2,\ d(PGL(3,q)) + \varepsilon(PGL(3,q)) = q^3 - q,$$

Example (Borges 2009) $m, n :=$ positive integers, $\gcd(m,n) = 1$
$\mathcal{F}_{n,m} :=$ plane curve in $PG(2,q)$ with affine equation:

$$\frac{(X^{q^n} - X)(Y^{q^m} - Y) - (Y^{q^n} - Y)(X^{q^m} - X)}{(X^{q^2} - X)(Y^{q} - Y) - (Y^{q^2} - Y)(X^{q} - X)} = 0.$$

$\mathcal{F}_{3,1}$, named DGZ (Dickson-Guralnick-Zieve) curve,
$\mathcal{F}_{3,2}$, the dual DGZ curve.
$< \mathcal{F}_{3,1}^{q+1}, \mathcal{F}_{3,2}^{q} >$ is a $PGL(3,q)$-fixed pencil.

$$d(PGL(3,q)) = q^3 - q^2,\ d(PGL(3,q)) + \varepsilon(PGL(3,q)) = q^3 - q,$$

the DGZ curve is the unique $PGL(3,q)$-invariant irreducible plane curve of degree $q^3 - q^2$

Case $G = PGL(3,q)$, $|G| = (q^2+q+1)q^3(q+1)(q-1)^2$

Example (Borges 2009) $m, n :=$ positive integers, $\gcd(m,n) = 1$
$\mathcal{F}_{n,m} :=$ plane curve in $PG(2,q)$ with affine equation:

$$\frac{(X^{q^n}-X)(Y^{q^m}-Y)-(Y^{q^n}-Y)(X^{q^m}-X)}{(X^{q^2}-X)(Y^{q}-Y)-(Y^{q^2}-Y)(X^{q}-X)} = 0.$$

$\mathcal{F}_{3,1}$, named DGZ (Dickson-Guralnick-Zieve) curve,
$\mathcal{F}_{3,2}$, the dual DGZ curve.
$< \mathcal{F}_{3,1}^{q+1}, \mathcal{F}_{3,2}^{q} >$ is a $PGL(3,q)$-fixed pencil.

$$d(PGL(3,q)) = q^3 - q^2,\ d(PGL(3,q)) + \varepsilon(PGL(3,q)) = q^3 - q,$$

the DGZ curve is the unique $PGL(3,q)$-invariant irreducible plane curve of degree $q^3 - q^2$
the dual DGZ curve is an example for
$d(PGL(3,q)) + \varepsilon(PGL(3,q)) = q^3 - q$.

Case $G = PGL(3,q)$, $|G| = (q^2+q+1)q^3(q+1)(q-1)^2$

Example (Borges 2009) $m, n :=$ positive integers, $\gcd(m,n) = 1$
$\mathcal{F}_{n,m} :=$ plane curve in $PG(2,q)$ with affine equation:

$$\frac{(X^{q^n}-X)(Y^{q^m}-Y)-(Y^{q^n}-Y)(X^{q^m}-X)}{(X^{q^2}-X)(Y^{q}-Y)-(Y^{q^2}-Y)(X^{q}-X)} = 0.$$

$\mathcal{F}_{3,1}$, named DGZ (Dickson-Guralnick-Zieve) curve,
$\mathcal{F}_{3,2}$, the dual DGZ curve.
$< \mathcal{F}_{3,1}^{q+1}, \mathcal{F}_{3,2}^{q} >$ is a $PGL(3,q)$-fixed pencil.

$$d(PGL(3,q)) = q^3 - q^2,\ d(PGL(3,q)) + \varepsilon(PGL(3,q)) = q^3 - q,$$

the DGZ curve is the unique $PGL(3,q)$-invariant irreducible plane curve of degree $q^3 - q^2$
the dual DGZ curve is an example for
$d(PGL(3,q)) + \varepsilon(PGL(3,q)) = q^3 - q$.
For $q \equiv 1 \pmod 3$, $PSL(3,q)$ is a maximal subgroup of $PGL(3,q)$ of index 3, but the same results hold.

Case $G = AGL(2, q)$, $|G| = q^2(q-1)(q^3-q)$

Case $G = AGL(2,q)$, $|G| = q^2(q-1)(q^3-q)$

$AGL(2,q)$ is viewed as the subgroup of $PGL(3,q)$ preserving the line of infinity.

Case $G = AGL(2,q)$, $|G| = q^2(q-1)(q^3-q)$

$AGL(2,q)$ is viewed as the subgroup of $PGL(3,q)$ preserving the line of infinity.
an $AGL(2,q)$-invariant pencil is:

$$\frac{(X^{q^3}-X)(Y^q-Y)-(Y^{q^3}-Y)(X^q-X)}{(X^{q^2}-X)(Y^q-Y)-(Y^{q^2}-Y)(X^q-X)}-\lambda=0.$$

Case $G = AGL(2,q)$, $|G| = q^2(q-1)(q^3-q)$

$AGL(2,q)$ is viewed as the subgroup of $PGL(3,q)$ preserving the line of infinity.
an $AGL(2,q)$-invariant pencil is:

$$\frac{(X^{q^3}-X)(Y^q-Y)-(Y^{q^3}-Y)(X^q-X)}{(X^{q^2}-X)(Y^q-Y)-(Y^{q^2}-Y)(X^q-X)}-\lambda=0.$$

$$d(AGL(2,q)) = q^3-q^2,\ d(AGL(2,q))+\varepsilon(AGL(2,q)) = q^3-q.$$

Case $G = AGL(2,q)$, $|G| = q^2(q-1)(q^3-q)$

$AGL(2,q)$ is viewed as the subgroup of $PGL(3,q)$ preserving the line of infinity.

an $AGL(2,q)$-invariant pencil is:

$$\frac{(X^{q^3}-X)(Y^q-Y)-(Y^{q^3}-Y)(X^q-X)}{(X^{q^2}-X)(Y^q-Y)-(Y^{q^2}-Y)(X^q-X)}-\lambda=0.$$

$$d(AGL(2,q)) = q^3-q^2,\ d(AGL(2,q))+\varepsilon(AGL(2,q)) = q^3-q.$$

the DGZ curve is the unique $AGL(2,q)$-invariant irreducible plane curve of degree q^3-q^2

Case $G = AGL(2,q)$, $|G| = q^2(q-1)(q^3-q)$

$AGL(2,q)$ is viewed as the subgroup of $PGL(3,q)$ preserving the line of infinity.
an $AGL(2,q)$-invariant pencil is:

$$\frac{(X^{q^3}-X)(Y^q-Y)-(Y^{q^3}-Y)(X^q-X)}{(X^{q^2}-X)(Y^q-Y)-(Y^{q^2}-Y)(X^q-X)}-\lambda=0.$$

$$d(AGL(2,q)) = q^3-q^2,\ d(AGL(2,q))+\varepsilon(AGL(2,q)) = q^3-q.$$

the DGZ curve is the unique $AGL(2,q)$-invariant irreducible plane curve of degree q^3-q^2
the dual DGZ curve is an example for
$d(AGL(2,q))+\varepsilon(AGL(2,q)) = q^3-q$.

Case $G = AGL(2,q)$, $|G| = q^2(q-1)(q^3-q)$

$AGL(2,q)$ is viewed as the subgroup of $PGL(3,q)$ preserving the line of infinity.
an $AGL(2,q)$-invariant pencil is:

$$\frac{(X^{q^3}-X)(Y^q-Y)-(Y^{q^3}-Y)(X^q-X)}{(X^{q^2}-X)(Y^q-Y)-(Y^{q^2}-Y)(X^q-X)}-\lambda=0.$$

$$d(AGL(2,q))=q^3-q^2,\ d(AGL(2,q))+\varepsilon(AGL(2,q))=q^3-q.$$

the DGZ curve is the unique $AGL(2,q)$-invariant irreducible plane curve of degree q^3-q^2
the dual DGZ curve is an example for $d(AGL(2,q))+\varepsilon(AGL(2,q))=q^3-q$.
All $AGL(2,q)$-invariant irreducible curves of degree q^3-q^2 belong, up to projectivity, to the above pencil

Case $G = \overline{AGL}(2, q)$, $|G| = q^2(q-1)(q^3-q)$

Case $G = \overline{AGL}(2, q)$, $|G| = q^2(q-1)(q^3-q)$

$\overline{AGL}(2, q)$ is viewed as the subgroup of $PGL(3, q)$ fixing a point.

$\overline{AGL}(2,q)$ is viewed as the subgroup of $PGL(3,q)$ fixing a point. an $\overline{AGL}(2,q)$-invariant pencil is:

$$\frac{(X^{q^3}-X)(Y^q-Y)-(Y^{q^3}-Y)(X^q-X)}{(X^{q^2}-X)(Y^q-Y)-(Y^{q^2}-Y)(X^q-X)}-$$

$$\lambda\frac{(X^{q^2}-X)(Y^q-Y)-(Y^{q^2}-Y)(X^q-X)}{(Y^q-Y)^{q+1}}=0.$$

Moreover,

$$d(\overline{AGL}(2,q)) = q^3-q^2,\ d(\overline{AGL}(2,q))+\varepsilon(\overline{AGL}(2,q)) = q^3-q.$$

Case $G = \overline{AGL}(2, q)$, $|G| = q^2(q-1)(q^3-q)$

$\overline{AGL}(2, q)$ is viewed as the subgroup of $PGL(3, q)$ fixing a point. an $\overline{AGL}(2, q)$-invariant pencil is:

$$\frac{(X^{q^3}-X)(Y^q-Y)-(Y^{q^3}-Y)(X^q-X)}{(X^{q^2}-X)(Y^q-Y)-(Y^{q^2}-Y)(X^q-X)}-$$

$$\lambda\frac{(X^{q^2}-X)(Y^q-Y)-(Y^{q^2}-Y)(X^q-X)}{(Y^q-Y)^{q+1}}=0.$$

Moreover,

$$d(\overline{AGL}(2, q)) = q^3-q^2,\ d(\overline{AGL}(2, q))+\varepsilon(\overline{AGL}(2, q)) = q^3-q.$$

the DGZ curve is the unique $\overline{AGL}(2, q)$-invariant irreducible plane curve of degree q^3-q^2

Case $G = \overline{AGL}(2,q)$, $|G| = q^2(q-1)(q^3-q)$

$\overline{AGL}(2,q)$ is viewed as the subgroup of $PGL(3,q)$ fixing a point.
an $\overline{AGL}(2,q)$-invariant pencil is:

$$\frac{(X^{q^3}-X)(Y^q-Y)-(Y^{q^3}-Y)(X^q-X)}{(X^{q^2}-X)(Y^q-Y)-(Y^{q^2}-Y)(X^q-X)}-$$

$$\lambda\frac{(X^{q^2}-X)(Y^q-Y)-(Y^{q^2}-Y)(X^q-X)}{(Y^q-Y)^{q+1}}=0.$$

Moreover,

$$d(\overline{AGL}(2,q)) = q^3-q^2,\ d(\overline{AGL}(2,q))+\varepsilon(\overline{AGL}(2,q)) = q^3-q.$$

the DGZ curve is the unique $\overline{AGL}(2,q)$-invariant irreducible plane curve of degree q^3-q^2
the dual DGZ curve is an example for $\varepsilon(\overline{AGL}(2,q)) = q^3-q$.

Case $G = \overline{AGL}(2,q)$, $|G| = q^2(q-1)(q^3-q)$

$\overline{AGL}(2,q)$ is viewed as the subgroup of $PGL(3,q)$ fixing a point.
an $\overline{AGL}(2,q)$-invariant pencil is:

$$\frac{(X^{q^3}-X)(Y^q-Y)-(Y^{q^3}-Y)(X^q-X)}{(X^{q^2}-X)(Y^q-Y)-(Y^{q^2}-Y)(X^q-X)}-$$

$$\lambda\frac{(X^{q^2}-X)(Y^q-Y)-(Y^{q^2}-Y)(X^q-X)}{(Y^q-Y)^{q+1}}=0.$$

Moreover,

$$d(\overline{AGL}(2,q)) = q^3-q^2,\ d(\overline{AGL}(2,q))+\varepsilon(\overline{AGL}(2,q)) = q^3-q.$$

the DGZ curve is the unique $\overline{AGL}(2,q)$-invariant irreducible plane curve of degree q^3-q^2
the dual DGZ curve is an example for $\varepsilon(\overline{AGL}(2,q)) = q^3-q$.
All $\overline{AGL}(2,q)$-invariant irreducible curves of degree q^3-q belong, up to projectivity, to the pencil with $\lambda \neq 1$.

Case $G = PGU(3, n)$, $q = n^2$, $|G| = (n^3 + 1)n^3(n^2 - 1) = (q^{3/2} + 1)q^{3/2}(q - 1)$

Case $G = PGU(3,n)$, $q = n^2$, $|G| = (n^3+1)n^3(n^2-1) = (q^{3/2}+1)q^{3/2}(q-1)$

well known example the Hermitian curve $\mathcal{H}_n$ of affine equation

$$Y^n + Y - X^{n+1} = 0.$$

$$d(PGU(3,n)) = n+1,\ d(PGU(3,n)) + \varepsilon(PGU(3,n)) = n^3+1,$$

Case $G = PGU(3,n)$, $q = n^2$, $|G| = (n^3+1)n^3(n^2-1) = (q^{3/2}+1)q^{3/2}(q-1)$

well known example the Hermitian curve $\mathcal{H}_n$ of affine equation

$$Y^n + Y - X^{n+1} = 0.$$

$$d(PGU(3,n)) = n+1,\ d(PGU(3,n)) + \varepsilon(PGU(3,n)) = n^3+1,$$

$\mathcal{H}_n$ is the unique $PGU(3,n)$-invariant irreducible plane curve of degree $n+1$.

Case $G = PGU(3,n)$, $q = n^2$, $|G| = (n^3+1)n^3(n^2-1) = (q^{3/2}+1)q^{3/2}(q-1)$

well known example the Hermitian curve $\mathcal{H}_n$ of affine equation

$$Y^n + Y - X^{n+1} = 0.$$

$$d(PGU(3,n)) = n+1,\ d(PGU(3,n)) + \varepsilon(PGU(3,n)) = n^3+1,$$

$\mathcal{H}_n$ is the unique $PGU(3,n)$-invariant irreducible plane curve of degree $n+1$.

Theorem *All $PGU(3,n)$-invariant irreducible plane curves of degree $d < nq(q-1) = n^3(n^2-1)$ other than the Hermitian curve have degree n^3+1 and belongs to the $PGU(3,n)$-fixed pencil*

Case $G = PGU(3,n)$, $q = n^2$, $|G| = (n^3+1)n^3(n^2-1) = (q^{3/2}+1)q^{3/2}(q-1)$

well known example the Hermitian curve $\mathcal{H}_n$ of affine equation

$$Y^n + Y - X^{n+1} = 0.$$

$$d(PGU(3,n)) = n+1,\ d(PGU(3,n)) + \varepsilon(PGU(3,n)) = n^3+1,$$

$\mathcal{H}_n$ is the unique $PGU(3,n)$-invariant irreducible plane curve of degree $n+1$.

Theorem *All $PGU(3,n)$-invariant irreducible plane curves of degree $d < nq(q-1) = n^3(n^2-1)$ other than the Hermitian curve have degree n^3+1 and belongs to the $PGU(3,n)$-fixed pencil*

$$Y^{n^3} + Y - X^{n^3+1} - \lambda(Y^n + Y - X^{n+1})^{q-n+1} = 0.$$

Case $G = PGU(3,n)$, $q = n^2$, $|G| = (n^3+1)n^3(n^2-1) = (q^{3/2}+1)q^{3/2}(q-1)$

well known example the Hermitian curve $\mathcal{H}_n$ of affine equation

$$Y^n + Y - X^{n+1} = 0.$$

$$d(PGU(3,n)) = n+1,\ d(PGU(3,n)) + \varepsilon(PGU(3,n)) = n^3+1,$$

$\mathcal{H}_n$ is the unique $PGU(3,n)$-invariant irreducible plane curve of degree $n+1$.

Theorem *All $PGU(3,n)$-invariant irreducible plane curves of degree $d < nq(q-1) = n^3(n^2-1)$ other than the Hermitian curve have degree n^3+1 and belongs to the $PGU(3,n)$-fixed pencil*

$$Y^{n^3} + Y - X^{n^3+1} - \lambda(Y^n + Y - X^{n+1})^{q-n+1} = 0.$$

For $\lambda = 1$, the curve splits into n^3+1 lines.

Case $G = \Delta_q$ preserves a triangle in $PG(2,q)$, $|G| = 6(q-1)^2$

Case $G = \Delta_q$ preserves a triangle in $PG(2,q)$, $|G| = 6(q-1)^2$

Remark $\Delta_q = (C_{q-1} \times C_{q-1}) \rtimes S_3$

$$d(\Delta_q) = q-1,\ d(\Delta_q) + \varepsilon(\Delta_q) = 2q-2,$$

Case $G = \Delta_q$ preserves a triangle in $PG(2,q)$, $|G| = 6(q-1)^2$

Remark $\Delta_q = (C_{q-1} \times C_{q-1}) \rtimes S_3$

$$d(\Delta_q) = q-1,\ d(\Delta_q) + \varepsilon(\Delta_q) = 2q-2,$$

unique Δ_q-invariant irreducible plane curve of degree $q-1$ is the Fermat curve of homogeneous equation.

Case $G = \Delta_q$ preserves a triangle in $PG(2,q)$, $|G| = 6(q-1)^2$

Remark $\Delta_q = (C_{q-1} \times C_{q-1}) \rtimes S_3$

$$d(\Delta_q) = q-1,\ d(\Delta_q) + \varepsilon(\Delta_q) = 2q-2,$$

unique Δ_q-invariant irreducible plane curve of degree $q-1$ is the Fermat curve of homogeneous equation.

$$X^{q-1} + Y^{q-1} + Z^{q-1} = 0.$$

Case $G = \Delta_q$ preserves a triangle in $PG(2,q)$, $|G| = 6(q-1)^2$

Remark $\Delta_q = (C_{q-1} \times C_{q-1}) \rtimes S_3$

$$d(\Delta_q) = q-1,\ d(\Delta_q) + \varepsilon(\Delta_q) = 2q-2,$$

unique Δ_q-invariant irreducible plane curve of degree $q-1$ is the Fermat curve of homogeneous equation.

$$X^{q-1} + Y^{q-1} + Z^{q-1} = 0.$$

All Δ_q-invariant curves with $d(\Delta_q) + \varepsilon(\Delta_q) = 2q-2$ belong to the pencil

$$\lambda(X^{q-1} + Y^{q-1} + Z^{q-1})^2 + (XY)^{q-1} + (YZ)^{q-1} + (ZX)^{q-1}) = 0.$$

Case $G = NS_q$ preserves a triangle in $PG(2,q^3) \setminus PG(2,q)$, $|G| = 3(q^2+q+1)$

Case $G = NS_q$ preserves a triangle in $PG(2,q^3) \setminus PG(2,q)$, $|G| = 3(q^2+q+1)$

$NS_q :=$ normalizer of the Singer subgroup of $PG(2,q)$
$NS_q = C_{q^2+q+1} \rtimes C_3$.
Remark

Case $G = NS_q$ preserves a triangle in $PG(2,q^3) \setminus PG(2,q)$, $|G| = 3(q^2+q+1)$

$NS_q :=$ normalizer of the Singer subgroup of $PG(2,q)$
$NS_q = C_{q^2+q+1} \rtimes C_3$.
Remark
All irreducible plane curves of degree $d \leq 2q+2$ invariant by the Singer subgroup are known, (Cossidente, Siciliano, Pellikaan).

$$d(NS_q) = q+2,\ d(NS_q) + \varepsilon(NS_q) = 2q+1,$$

Case $G = NS_q$ preserves a triangle in $PG(2,q^3) \setminus PG(2,q)$, $|G| = 3(q^2+q+1)$

$NS_q :=$ normalizer of the Singer subgroup of $PG(2,q)$
$NS_q = C_{q^2+q+1} \rtimes C_3$.
Remark
All irreducible plane curves of degree $d \leq 2q+2$ invariant by the Singer subgroup are known, (Cossidente, Siciliano, Pellikaan).

$$d(NS_q) = q+2,\ d(NS_q) + \varepsilon(NS_q) = 2q+1,$$

the Pellikaan curve of homogenous equation

$$X^{q+1}Y + Y^{q+1}Z + Z^{q+1}X = 0$$

is the unique NS_q-invariant curve of degree $q+2$.

Case $G = NS_q$ preserves a triangle in $PG(2,q^3) \setminus PG(2,q)$, $|G| = 3(q^2+q+1)$

$NS_q :=$ normalizer of the Singer subgroup of $PG(2,q)$
$NS_q = C_{q^2+q+1} \rtimes C_3$.

Remark

All irreducible plane curves of degree $d \leq 2q+2$ invariant by the Singer subgroup are known, (Cossidente, Siciliano, Pellikaan).

$$d(NS_q) = q+2,\ d(NS_q) + \varepsilon(NS_q) = 2q+1,$$

the Pellikaan curve of homogenous equation

$$X^{q+1}Y + Y^{q+1}Z + Z^{q+1}X = 0$$

is the unique NS_q-invariant curve of degree $q+2$.
Example of an NS_q-invariant curve of degree $2q+1$:

$$X^{q+1}Y^q + Y^{q+1}Z^q + Z^{q+1}X^q = 0$$

Case $G = NS_q$ preserves a triangle in $PG(2, q^3) \setminus PG(2, q)$, $|G| = 3(q^2 + q + 1)$

$NS_q :=$ normalizer of the Singer subgroup of $PG(2, q)$
$NS_q = C_{q^2+q+1} \rtimes C_3$.
Remark
All irreducible plane curves of degree $d \leq 2q + 2$ invariant by the Singer subgroup are known, (Cossidente, Siciliano, Pellikaan).

$$d(NS_q) = q + 2,\ d(NS_q) + \varepsilon(NS_q) = 2q + 1,$$

the Pellikaan curve of homogenous equation

$$X^{q+1}Y + Y^{q+1}Z + Z^{q+1}X = 0$$

is the unique NS_q-invariant curve of degree $q + 2$.
Example of an NS_q-invariant curve of degree $2q + 1$:

$$X^{q+1}Y^q + Y^{q+1}Z^q + Z^{q+1}X^q = 0$$

Case q odd, $G = PGL(2, q)$ preserves an irreducible conic in $PG(2, q)$, $|G| = q^3 - q$

Case q odd, $G = PGL(2,q)$ preserves an irreducible conic in $PG(2,q)$, $|G| = q^3 - q$

Apart from the unique $PGL(2,q)$-invariant conic $\mathcal{C}^2$, those of minimum degree $q+1$ belong to the $PGL(2,q)$-fixed pencil

$$Y^{q+1} - (X^q Z + XZ^q) - \lambda(Y^2 - 2XZ)^{(q+1)/2} = 0. \qquad (1)$$

Case q odd, $G = PGL(2,q)$ preserves an irreducible conic in $PG(2,q)$, $|G| = q^3 - q$

Apart from the unique $PGL(2,q)$-invariant conic $\mathcal{C}^2$, those of minimum degree $q+1$ belong to the $PGL(2,q)$-fixed pencil

$$Y^{q+1} - (X^q Z + XZ^q) - \lambda(Y^2 - 2XZ)^{(q+1)/2} = 0. \qquad (1)$$

Therefore,

$$d(PGL(2,q)) = 2,\ d(PGL(2,q)) + \varepsilon(PGL(2,q)) = q+1.$$

Case q odd, $G = PGL(2, q)$ preserves an irreducible conic in $PG(2, q)$, $|G| = q^3 - q$

Apart from the unique $PGL(2, q)$-invariant conic $\mathcal{C}^2$, those of minimum degree $q + 1$ belong to the $PGL(2, q)$-fixed pencil

$$Y^{q+1} - (X^q Z + XZ^q) - \lambda(Y^2 - 2XZ)^{(q+1)/2} = 0. \qquad (1)$$

Therefore,

$$d(PGL(2, q)) = 2,\ d(PGL(2, q)) + \varepsilon(PGL(2, q)) = q + 1.$$

$\mathcal{C}_1$ is completely reducible, product of the tangents to $\mathcal{C}^2$ at its points in $PG(2, q)$;

Case q odd, $G = PGL(2,q)$ preserves an irreducible conic in $PG(2,q)$, $|G| = q^3 - q$

Apart from the unique $PGL(2,q)$-invariant conic $\mathcal{C}^2$, those of minimum degree $q+1$ belong to the $PGL(2,q)$-fixed pencil

$$Y^{q+1} - (X^q Z + XZ^q) - \lambda(Y^2 - 2XZ)^{(q+1)/2} = 0. \qquad (1)$$

Therefore,

$$d(PGL(2,q)) = 2, \; d(PGL(2,q)) + \varepsilon(PGL(2,q)) = q+1.$$

$\mathcal{C}_1$ is completely reducible, product of the tangents to $\mathcal{C}^2$ at its points in $PG(2,q)$;
$\mathcal{C}_{-1}$ is rational and has interesting combinatorial properties:

Case q odd, $G = PGL(2,q)$ preserves an irreducible conic in $PG(2,q)$, $|G| = q^3 - q$

Apart from the unique $PGL(2,q)$-invariant conic $\mathcal{C}^2$, those of minimum degree $q+1$ belong to the $PGL(2,q)$-fixed pencil

$$Y^{q+1} - (X^q Z + XZ^q) - \lambda(Y^2 - 2XZ)^{(q+1)/2} = 0. \qquad (1)$$

Therefore,

$$d(PGL(2,q)) = 2,\ d(PGL(2,q)) + \varepsilon(PGL(2,q)) = q+1.$$

$\mathcal{C}_1$ is completely reducible, product of the tangents to $\mathcal{C}^2$ at its points in $PG(2,q)$;
$\mathcal{C}_{-1}$ is rational and has interesting combinatorial properties: the $q+1$ points of $\mathcal{C}^2$ in $PG(2,q)$ are simple points of $\mathcal{C}$, the $\frac{1}{2}q(q-1)$ internal points to $\mathcal{C}^2$ are double points of $\mathcal{C}$.

Plane (k, n)-arcs from algebraic curves

Plane (k, n)-arcs from algebraic curves

Natural candidate for a plane (k, n)-arc is the set of the points of a plane algebraic curve of degree n.

Plane (k, n)-arcs from algebraic curves

Natural candidate for a plane (k, n)-arc is the set of the points of a plane algebraic curve of degree n.
Well known example of such a (k, n)-arc is the Hermitian unital.

Plane (k, n)-arcs from algebraic curves

Natural candidate for a plane (k, n)-arc is the set of the points of a plane algebraic curve of degree n.
Well known example of such a (k, n)-arc is the Hermitian unital.
$\mathcal{C}$:=plane algebraic curve (naturally defined) of $PG(2, q^m)$ and viewed as a curve in $PG(2, q^{rm})$, $r \geq 1$.

Plane (k, n)-arcs from algebraic curves

Natural candidate for a plane (k, n)-arc is the set of the points of a plane algebraic curve of degree n.
Well known example of such a (k, n)-arc is the Hermitian unital.
$\mathcal{C}$:=plane algebraic curve (naturally defined) of $PG(2, q^m)$ and viewed as a curve in $PG(2, q^{rm})$, $r \geq 1$.
(k, n)-arc arising from $\mathcal{C}$ in $PG(2, q^{rm})$:= set of all points of $\mathcal{C}$ in $PG(2, q^{rm})$.

Plane (k,n)-arcs from algebraic curves

Natural candidate for a plane (k,n)-arc is the set of the points of a plane algebraic curve of degree n.
Well known example of such a (k,n)-arc is the Hermitian unital.
$\mathcal{C}$:=plane algebraic curve (naturally defined) of $PG(2,q^m)$ and viewed as a curve in $PG(2,q^{rm})$, $r \geq 1$.
(k,n)-arc arising from $\mathcal{C}$ in $PG(2,q^{rm})$:= set of all points of $\mathcal{C}$ in $PG(2,q^{rm})$.
Remark For r big enough, $k \approx q^{rm}$, i.e. the (k,n)-arc is small.

Plane (k, n)-arcs from algebraic curves

Natural candidate for a plane (k, n)-arc is the set of the points of a plane algebraic curve of degree n.
Well known example of such a (k, n)-arc is the Hermitian unital.
$\mathcal{C}$:=plane algebraic curve (naturally defined) of $PG(2, q^m)$ and viewed as a curve in $PG(2, q^{rm})$, $r \geq 1$.
(k, n)-arc arising from $\mathcal{C}$ in $PG(2, q^{rm})$:= set of all points of $\mathcal{C}$ in $PG(2, q^{rm})$.
Remark For r big enough, $k \approx q^{rm}$, i.e. the (k, n)-arc is small.
Remark Complete (k, n)-arcs in $PG(2, q^{rm}) \Leftrightarrow$ non-extendible $[k, n, k-n]_{q^{rm}}$ Almost-MDS codes.

Plane (k, n)-arcs from algebraic curves

Natural candidate for a plane (k, n)-arc is the set of the points of a plane algebraic curve of degree n.

Well known example of such a (k, n)-arc is the Hermitian unital.

$\mathcal{C}$:=plane algebraic curve (naturally defined) of $PG(2, q^m)$ and viewed as a curve in $PG(2, q^{rm})$, $r \geq 1$.

(k, n)-arc arising from $\mathcal{C}$ in $PG(2, q^{rm})$:= set of all points of $\mathcal{C}$ in $PG(2, q^{rm})$.

Remark For r big enough, $k \approx q^{rm}$, i.e. the (k, n)-arc is small.

Remark Complete (k, n)-arcs in $PG(2, q^{rm}) \Leftrightarrow$ non-extendible $[k, n, k-n]_{q^{rm}}$ Almost-MDS codes.

Examples of complete (k, n)-arcs (from Frobenius non-classical curves) due to Giulietti, Pambianco, Ughi and Torres (2008).

Plane (k, n)-arcs from algebraic curves

Natural candidate for a plane (k, n)-arc is the set of the points of a plane algebraic curve of degree n.

Well known example of such a (k, n)-arc is the Hermitian unital.

$\mathcal{C}$:=plane algebraic curve (naturally defined) of $PG(2, q^m)$ and viewed as a curve in $PG(2, q^{rm})$, $r \geq 1$.

(k, n)-arc arising from $\mathcal{C}$ in $PG(2, q^{rm})$:= set of all points of $\mathcal{C}$ in $PG(2, q^{rm})$.

Remark For r big enough, $k \approx q^{rm}$, i.e. the (k, n)-arc is small.

Remark Complete (k, n)-arcs in $PG(2, q^{rm}) \Leftrightarrow$ non-extendible $[k, n, k-n]_{q^{rm}}$ Almost-MDS codes.

Examples of complete (k, n)-arcs (from Frobenius non-classical curves) due to Giulietti, Pambianco, Ughi and Torres (2008).

Their work was the first important step towards an algebraic theory of complete (k, n)-arcs, based on Galois theory (and results of van der Waerden (1933) and Abhyankar (1992))

The approach from Galois theory

Basic idea is in the papers of Guralnick, Zieve, Möller (2010) (and some others) on permutation polynomials

The approach from Galois theory

Basic idea is in the papers of Guralnick, Zieve, Möller (2010) (and some others) on permutation polynomials
Adaption for (k, n)-arcs is due to Bartoli and Micheli (2021).

The approach from Galois theory

Basic idea is in the papers of Guralnick, Zieve, Möller (2010) (and some others) on permutation polynomials

Adaption for (k, n)-arcs is due to Bartoli and Micheli (2021).

Complete (k, n)-arcs in $PG(2, q^{rm})$ with $r \ggg n$ from rational and hyperelliptic curves defined over $\mathbb{F}_{q^m}$ (Bartoli-Micheli 2021)

The approach from Galois theory

Basic idea is in the papers of Guralnick, Zieve, Möller (2010) (and some others) on permutation polynomials

Adaption for (k, n)-arcs is due to Bartoli and Micheli (2021).

Complete (k, n)-arcs in $PG(2, q^{rm})$ with $r \ggg n$ from rational and hyperelliptic curves defined over $\mathbb{F}_{q^m}$ (Bartoli-Micheli 2021)

Problem *Construction of complete (k, n)-arcs in $PG(2, q^{rm})$ for almost all r (using other curves defined over $\mathbb{F}_{q^m}$)*

The approach from Galois theory

Basic idea is in the papers of Guralnick, Zieve, Möller (2010) (and some others) on permutation polynomials

Adaption for (k,n)-arcs is due to Bartoli and Micheli (2021).

Complete (k,n)-arcs in $PG(2,q^{rm})$ with $r \ggg n$ from rational and hyperelliptic curves defined over $\mathbb{F}_{q^m}$ (Bartoli-Micheli 2021)

Problem *Construction of complete (k,n)-arcs in $PG(2,q^{rm})$ for almost all r (using other curves defined over $\mathbb{F}_{q^m}$)*

Question *Among the curves arising from maximal subgroups we have come across, which provide a solution for the above Problem?*

Basic idea is in the papers of Guralnick, Zieve, Möller (2010) (and some others) on permutation polynomials

Adaption for (k, n)-arcs is due to Bartoli and Micheli (2021).

Complete (k, n)-arcs in $PG(2, q^{rm})$ with $r \ggg n$ from rational and hyperelliptic curves defined over $\mathbb{F}_{q^m}$ (Bartoli-Micheli 2021)

Problem *Construction of complete (k, n)-arcs in $PG(2, q^{rm})$ for almost all r (using other curves defined over $\mathbb{F}_{q^m}$)*

Question *Among the curves arising from maximal subgroups we have come across, which provide a solution for the above Problem?*

So far we have solved positively this problem for the Hermitian curve and for the $PGL(2, q)$-invariant curve $\mathcal{C}_{-1}$.

$$Y^{q+1} - (X^q + X) + (Y^2 - 2X)^{(q+1)/2} = 0.$$

The approach from Galois theory

Basic idea is in the papers of Guralnick, Zieve, Möller (2010) (and some others) on permutation polynomials

Adaption for (k, n)-arcs is due to Bartoli and Micheli (2021).

Complete (k, n)-arcs in $PG(2, q^{rm})$ with $r \ggg n$ from rational and hyperelliptic curves defined over $\mathbb{F}_{q^m}$ (Bartoli-Micheli 2021)

Problem *Construction of complete (k, n)-arcs in $PG(2, q^{rm})$ for almost all r (using other curves defined over $\mathbb{F}_{q^m}$)*

Question *Among the curves arising from maximal subgroups we have come across, which provide a solution for the above Problem?*

So far we have solved positively this problem for the Hermitian curve and for the $PGL(2, q)$-invariant curve $\mathcal{C}_{-1}$.

$$Y^{q+1} - (X^q + X) + (Y^2 - 2X)^{(q+1)/2} = 0.$$

The other cases are open.

The approach from Galois theory

Basic idea is in the papers of Guralnick, Zieve, Möller (2010) (and some others) on permutation polynomials

Adaption for (k, n)-arcs is due to Bartoli and Micheli (2021).

Complete (k, n)-arcs in $PG(2, q^{rm})$ with $r \ggg n$ from rational and hyperelliptic curves defined over $\mathbb{F}_{q^m}$ (Bartoli-Micheli 2021)

Problem *Construction of complete (k, n)-arcs in $PG(2, q^{rm})$ for almost all r (using other curves defined over $\mathbb{F}_{q^m}$)*

Question *Among the curves arising from maximal subgroups we have come across, which provide a solution for the above Problem?*

So far we have solved positively this problem for the Hermitian curve and for the $PGL(2, q)$-invariant curve $\mathcal{C}_{-1}$.

$$Y^{q+1} - (X^q + X) + (Y^2 - 2X)^{(q+1)/2} = 0.$$

The other cases are open.

Here we deal with the Hermitian curve. Our method also applies to $\mathcal{C}_{-1}$ (proofs are even simpler).

The case of the Hermitian curve

The case of the Hermitian curve

$r \geq 3$ integer

The case of the Hermitian curve

$r \geq 3$ integer
$\mathcal{H}_q :=$ Hermitian curve of degree $q+1$, regarded as a curve in $PG(2, q^{2r})$

$r \geq 3$ integer
$\mathcal{H}_q :=$ Hermitian curve of degree $q+1$, regarded as a curve in $PG(2, q^{2r})$
$\Omega :=$ set of all points of $\mathcal{H}_q$ in $PG(2, q^{2r})$, i.e. $\Omega = \mathcal{H}_q(\mathbb{F}_{q^{2r}})$

The case of the Hermitian curve

$r \geq 3$ integer
$\mathcal{H}_q :=$ Hermitian curve of degree $q+1$, regarded as a curve in $PG(2, q^{2r})$
$\Omega :=$ set of all points of $\mathcal{H}_q$ in $PG(2, q^{2r})$, i.e. $\Omega = \mathcal{H}_q(\mathbb{F}_{q^{2r}})$
$k := |\Omega|$ where $k = q^{2r} + 1 \pm q^{r+1}(q-1)$ according as r is odd or even

The case of the Hermitian curve

$r \geq 3$ integer
$\mathcal{H}_q :=$ Hermitian curve of degree $q+1$, regarded as a curve in $PG(2, q^{2r})$
$\Omega :=$ set of all points of $\mathcal{H}_q$ in $PG(2, q^{2r})$, i.e. $\Omega = \mathcal{H}_q(\mathbb{F}_{q^{2r}})$
$k := |\Omega|$ where $k = q^{2r} + 1 \pm q^{r+1}(q-1)$ according as r is odd or even
Ω is a small $(k, q+1)$-arc in $PG(2, q^{2r})$.

The case of the Hermitian curve

$r \geq 3$ integer
$\mathcal{H}_q :=$ Hermitian curve of degree $q+1$, regarded as a curve in $PG(2, q^{2r})$
$\Omega :=$ set of all points of $\mathcal{H}_q$ in $PG(2, q^{2r})$, i.e. $\Omega = \mathcal{H}_q(\mathbb{F}_{q^{2r}})$
$k := |\Omega|$ where $k = q^{2r} + 1 \pm q^{r+1}(q-1)$ according as r is odd or even
Ω is a small $(k, q+1)$-arc in $PG(2, q^{2r})$.
Theorem *(K. Szőnyi, G.P. Nagy, 2022) Ω is complete for $r \geq 5$.*

Sketch of the proof, set up

Sketch of the proof, set up

$\mathcal{H}_q :=$ Hermitian curve of affine equation $Y^q + Y + X^{q+1} = 0$

Sketch of the proof, set up

$\mathcal{H}_q :=$ Hermitian curve of affine equation $Y^q + Y + X^{q+1} = 0$
$P = P(a,b)$, $a^{q+1} + b^q + b \neq 0$, $P(a,b) \in PG(2,q^{2r}) \setminus PG(2,q^2)$

Sketch of the proof, set up

$\mathcal{H}_q :=$ Hermitian curve of affine equation $Y^q + Y + X^{q+1} = 0$
$P = P(a, b)$, $a^{q+1} + b^q + b \neq 0$, $P(a, b) \in PG(2, q^{2r}) \setminus PG(2, q^2)$
$\ell_t :=$ the (non vertical) line through P with slope t, i.e. ℓ_t:
$Y = t(X - a) + b$

Sketch of the proof, set up

$\mathcal{H}_q :=$ Hermitian curve of affine equation $Y^q + Y + X^{q+1} = 0$
$P = P(a,b)$, $a^{q+1} + b^q + b \neq 0$, $P(a,b) \in PG(2,q^{2r}) \setminus PG(2,q^2)$
$\ell_t :=$ the (non vertical) line through P with slope t, i.e. ℓ_t:
$Y = t(X-a) + b$
$F(X) = X^{q+1} + X^q(a+t^q) + X(a^q+t) + a^{q+1} + b^q + b$

Sketch of the proof, set up

$\mathcal{H}_q :=$ Hermitian curve of affine equation $Y^q + Y + X^{q+1} = 0$
$P = P(a,b)$, $a^{q+1} + b^q + b \neq 0$, $P(a,b) \in PG(2,q^{2r}) \setminus PG(2,q^2)$
$\ell_t :=$ the (non vertical) line through P with slope t, i.e. ℓ_t:
$Y = t(X-a) + b$
$F(X) = X^{q+1} + X^q(a + t^q) + X(a^q + t) + a^{q+1} + b^q + b$
ℓ_t is a $(q+1)$-secant of $\mathcal{H}_q$ in $PG(2,q^{2r}) \Leftrightarrow F(X)$ has $q+1$ pairwise distinct roots in $\mathbb{F}_{q^{2r}}$

Sketch of the proof, set up

$\mathcal{H}_q :=$ Hermitian curve of affine equation $Y^q + Y + X^{q+1} = 0$
$P = P(a,b)$, $a^{q+1} + b^q + b \neq 0$, $P(a,b) \in PG(2,q^{2r}) \setminus PG(2,q^2)$
$\ell_t :=$ the (non vertical) line through P with slope t, i.e. ℓ_t:
$Y = t(X-a) + b$
$F(X) = X^{q+1} + X^q(a + t^q) + X(a^q + t) + a^{q+1} + b^q + b$
ℓ_t is a $(q+1)$-secant of $\mathcal{H}_q$ in $PG(2,q^{2r}) \Leftrightarrow F(X)$ has $q+1$
pairwise distinct roots in $\mathbb{F}_{q^{2r}}$
$K :=$ the rational field $\bar{\mathbb{F}}_{q^2}(t)$

Sketch of the proof, set up

$\mathcal{H}_q$:= Hermitian curve of affine equation $Y^q + Y + X^{q+1} = 0$
$P = P(a,b)$, $a^{q+1} + b^q + b \neq 0$, $P(a,b) \in PG(2,q^{2r}) \setminus PG(2,q^2)$
ℓ_t := the (non vertical) line through P with slope t, i.e. ℓ_t:
$Y = t(X-a) + b$
$F(X) = X^{q+1} + X^q(a + t^q) + X(a^q + t) + a^{q+1} + b^q + b$
ℓ_t is a $(q+1)$-secant of $\mathcal{H}_q$ in $PG(2,q^{2r}) \Leftrightarrow F(X)$ has $q+1$ pairwise distinct roots in $\mathbb{F}_{q^{2r}}$
K := the rational field $\bar{\mathbb{F}}_{q^2}(t)$
$F(X) = X^{q+1} + X^q(a + t^q) + X(a^q + t) + a^{q+1} + b^q + b \in \mathbb{F}_{q^{2r}}(t)[X] \subset K[X]$.

Sketch of the proof, set up

$\mathcal{H}_q :=$ Hermitian curve of affine equation $Y^q + Y + X^{q+1} = 0$
$P = P(a,b)$, $a^{q+1} + b^q + b \neq 0$, $P(a,b) \in PG(2,q^{2r}) \setminus PG(2,q^2)$
$\ell_t :=$ the (non vertical) line through P with slope t, i.e. ℓ_t:
$Y = t(X-a) + b$
$F(X) = X^{q+1} + X^q(a + t^q) + X(a^q + t) + a^{q+1} + b^q + b$
ℓ_t is a $(q+1)$-secant of $\mathcal{H}_q$ in $PG(2,q^{2r}) \Leftrightarrow F(X)$ has $q+1$ pairwise distinct roots in $\mathbb{F}_{q^{2r}}$
$K :=$ the rational field $\bar{\mathbb{F}}_{q^2}(t)$
$F(X) = X^{q+1} + X^q(a + t^q) + X(a^q + t) + a^{q+1} + b^q + b \in \mathbb{F}_{q^{2r}}(t)[X] \subset K[X]$.
$F(X)$ is irreducible

Sketch of the proof, set up

$\mathcal{H}_q :=$ Hermitian curve of affine equation $Y^q + Y + X^{q+1} = 0$
$P = P(a,b)$, $a^{q+1} + b^q + b \neq 0$, $P(a,b) \in PG(2,q^{2r}) \setminus PG(2,q^2)$
$\ell_t :=$ the (non vertical) line through P with slope t, i.e. ℓ_t:
$Y = t(X-a) + b$
$F(X) = X^{q+1} + X^q(a+t^q) + X(a^q+t) + a^{q+1} + b^q + b$
ℓ_t is a $(q+1)$-secant of $\mathcal{H}_q$ in $PG(2,q^{2r}) \Leftrightarrow F(X)$ has $q+1$ pairwise distinct roots in $\mathbb{F}_{q^{2r}}$
$K :=$ the rational field $\bar{\mathbb{F}}_{q^2}(t)$
$F(X) = X^{q+1} + X^q(a+t^q) + X(a^q+t) + a^{q+1} + b^q + b \in \mathbb{F}_{q^{2r}}(t)[X] \subset K[X]$.
$F(X)$ is irreducible
$L := K(u)$ with $F(u) = 0$ the field extension $L : K$; it is not Galois

Sketch of the proof, set up

$\mathcal{H}_q :=$ Hermitian curve of affine equation $Y^q + Y + X^{q+1} = 0$
$P = P(a,b)$, $a^{q+1} + b^q + b \neq 0$, $P(a,b) \in PG(2,q^{2r}) \setminus PG(2,q^2)$
$\ell_t :=$ the (non vertical) line through P with slope t, i.e. ℓ_t:
$Y = t(X-a) + b$
$F(X) = X^{q+1} + X^q(a+t^q) + X(a^q+t) + a^{q+1} + b^q + b$
ℓ_t is a $(q+1)$-secant of $\mathcal{H}_q$ in $PG(2,q^{2r}) \Leftrightarrow F(X)$ has $q+1$ pairwise distinct roots in $\mathbb{F}_{q^{2r}}$
$K :=$ the rational field $\bar{\mathbb{F}}_{q^2}(t)$
$F(X) = X^{q+1} + X^q(a+t^q) + X(a^q+t) + a^{q+1} + b^q + b \in \mathbb{F}_{q^{2r}}(t)[X] \subset K[X]$.
$F(X)$ is irreducible
$L := K(u)$ with $F(u) = 0$ the field extension $L : K$; it is not Galois
$M :=$ Galois closure of $L : K$, i.e. M is the splitting field of $F(X)$ over K

Sketch of the proof, set up

$\mathcal{H}_q :=$ Hermitian curve of affine equation $Y^q + Y + X^{q+1} = 0$
$P = P(a, b)$, $a^{q+1} + b^q + b \neq 0$, $P(a, b) \in PG(2, q^{2r}) \setminus PG(2, q^2)$
$\ell_t :=$ the (non vertical) line through P with slope t, i.e. ℓ_t:
$Y = t(X - a) + b$
$F(X) = X^{q+1} + X^q(a + t^q) + X(a^q + t) + a^{q+1} + b^q + b$
ℓ_t is a $(q + 1)$-secant of $\mathcal{H}_q$ in $PG(2, q^{2r}) \Leftrightarrow F(X)$ has $q + 1$ pairwise distinct roots in $\mathbb{F}_{q^{2r}}$
$K :=$ the rational field $\bar{\mathbb{F}}_{q^2}(t)$
$F(X) = X^{q+1} + X^q(a + t^q) + X(a^q + t) + a^{q+1} + b^q + b \in \mathbb{F}_{q^{2r}}(t)[X] \subset K[X]$.
$F(X)$ is irreducible
$L := K(u)$ with $F(u) = 0$ the field extension $L : K$; it is not Galois
$M :=$ Galois closure of $L : K$, i.e. M is the splitting field of $F(X)$ over K
$G := Gal(M : K)$ Galois group, i.e. the geometric monodromy group of $F(X)$ over K

Sketch of the proof, results

Sketch of the proof, results

$G \cong PGL(2, q)$

Sketch of the proof, results

$G \cong PGL(2, q)$ (*tool is Abyhankar's twisted derivative*)

Sketch of the proof, results

$G \cong PGL(2, q)$ (*tool is Abyhankar's twisted derivative*)
M has as many as $(q+1)^2$ ramified places in the Galois extension
$M : K$ (*depends on the geometry of* $\mathcal{H}_q$)

Sketch of the proof, results

$G \cong PGL(2, q)$ (*tool is Abyhankar's twisted derivative*)
M has as many as $(q+1)^2$ ramified places in the Galois extension
$M : K$ (*depends on the geometry of* $\mathcal{H}_q$)
G has $q+1$ short orbits on the set of places of M

Sketch of the proof, results

$G \cong PGL(2,q)$ (*tool is Abyhankar's twisted derivative*)
M has as many as $(q+1)^2$ ramified places in the Galois extension $M:K$ (*depends on the geometry of* $\mathcal{H}_q$)
G has $q+1$ short orbits on the set of places of M
G acts on each short orbit as $PGL(2,q)$ in its 3-transitive permutation representation (*tool is van der Waerden's result*)

Sketch of the proof, results

$G \cong PGL(2,q)$ (*tool is Abyhankar's twisted derivative*)
M has as many as $(q+1)^2$ ramified places in the Galois extension
$M : K$ (*depends on the geometry of* $\mathcal{H}_q$)
G has $q+1$ short orbits on the set of places of M
G acts on each short orbit as $PGL(2,q)$ in its 3-transitive permutation representation (*tool is van der Waerden's result*)
The genus $\mathfrak{g}(M)$ of M is given by

$$2\mathfrak{g}(M) - 2 = q^4 + q^3 - 4q^2 - 3q + 1.$$

Sketch of the proof, results

$G \cong PGL(2,q)$ (*tool is Abyhankar's twisted derivative*)
M has as many as $(q+1)^2$ ramified places in the Galois extension $M : K$ (*depends on the geometry of $\mathcal{H}_q$*)
G has $q+1$ short orbits on the set of places of M
G acts on each short orbit as $PGL(2,q)$ in its 3-transitive permutation representation (*tool is van der Waerden's result*)
The genus $\mathfrak{g}(M)$ of M is given by

$$2\mathfrak{g}(M) - 2 = q^4 + q^3 - 4q^2 - 3q + 1.$$

(*tool is Serre's ramification theory*)

Sketch of the proof, results

$G \cong PGL(2,q)$ (*tool is Abyhankar's twisted derivative*)
M has as many as $(q+1)^2$ ramified places in the Galois extension $M:K$ (*depends on the geometry of $\mathcal{H}_q$*)
G has $q+1$ short orbits on the set of places of M
G acts on each short orbit as $PGL(2,q)$ in its 3-transitive permutation representation (*tool is van der Waerden's result*)
The genus $\mathfrak{g}(M)$ of M is given by

$$2\mathfrak{g}(M) - 2 = q^4 + q^3 - 4q^2 - 3q + 1.$$

(*tool is Serre's ramification theory*)

Conclusion of the proof

P is covered by at least one (non-vertical) $(q+1)$-secant $\Leftrightarrow$ M has at least one $\mathbb{F}_{q^{2r}}$-rational place which is unramified in the Galois extension $M : K$

Conclusion of the proof

P is covered by at least one (non-vertical) $(q+1)$-secant $\Leftrightarrow$ M has at least one $\mathbb{F}_{q^{2r}}$-rational place which is unramified in the Galois extension $M : K$

The Hasse-Weil lower bound $\Rightarrow$:

Conclusion of the proof

P is covered by at least one (non-vertical) $(q+1)$-secant $\Leftrightarrow$ M has at least one $\mathbb{F}_{q^{2r}}$-rational place which is unramified in the Galois extension $M:K$

The Hasse-Weil lower bound $\Rightarrow$: such an unramified place exists whenever

$q^{2r}+1 > q^{r+4}+q^{3+r}-4q^{2+r}-3q^{1+r}+3q^r+q^2+2q+1$

Conclusion of the proof

P is covered by at least one (non-vertical) $(q+1)$-secant $\Leftrightarrow$ M has at least one $\mathbb{F}_{q^{2r}}$-rational place which is unramified in the Galois extension $M:K$

The Hasse-Weil lower bound $\Rightarrow$: such an unramified place exists whenever

$$q^{2r}+1 > q^{r+4}+q^{3+r}-4q^{2+r}-3q^{1+r}+3q^r+q^2+2q+1$$

in particular for $r \geq 5$

Conclusion of the proof

P is covered by at least one (non-vertical) $(q+1)$-secant $\Leftrightarrow$ M has at least one $\mathbb{F}_{q^{2r}}$-rational place which is unramified in the Galois extension $M : K$

The Hasse-Weil lower bound $\Rightarrow$: such an unramified place exists whenever

$q^{2r}+1 > q^{r+4}+q^{3+r}-4q^{2+r}-3q^{1+r}+3q^r+q^2+2q+1$

in particular for $r \geq 5$

If $r \geq 5$ *then*

Conclusion of the proof

P is covered by at least one (non-vertical) $(q+1)$-secant $\Leftrightarrow$ M has at least one $\mathbb{F}_{q^{2r}}$-rational place which is unramified in the Galois extension $M : K$

The Hasse-Weil lower bound $\Rightarrow$: such an unramified place exists whenever

$q^{2r}+1 > q^{r+4}+q^{3+r}-4q^{2+r}-3q^{1+r}+3q^{r}+q^{2}+2q+1$

in particular for $r \geq 5$

If $r \geq 5$ then the set of all points of $\mathcal{H}_q$ in $PG(2, q^{2r})$ is a complete $(k, q+1)$-arc.

Conclusion of the proof

P is covered by at least one (non-vertical) $(q+1)$-secant $\Leftrightarrow$ M has at least one $\mathbb{F}_{q^{2r}}$-rational place which is unramified in the Galois extension $M : K$

The Hasse-Weil lower bound $\Rightarrow$: such an unramified place exists whenever

$q^{2r}+1 > q^{r+4}+q^{3+r}-4q^{2+r}-3q^{1+r}+3q^r+q^2+2q+1$

in particular for $r \geq 5$

If $r \geq 5$ then the set of all points of $\mathcal{H}_q$ in $PG(2, q^{2r})$ is a complete $(k, q+1)$-arc.

Remark

Cases $r = 3, 4$ are open

P is covered by at least one (non-vertical) $(q+1)$-secant $\Leftrightarrow$ M has at least one $\mathbb{F}_{q^{2r}}$-rational place which is unramified in the Galois extension $M:K$

The Hasse-Weil lower bound $\Rightarrow$: such an unramified place exists whenever

$q^{2r}+1>q^{r+4}+q^{3+r}-4q^{2+r}-3q^{1+r}+3q^r+q^2+2q+1$

in particular for $r \geq 5$

If $r \geq 5$ then the set of all points of $\mathcal{H}_q$ in $PG(2,q^{2r})$ is a complete $(k,q+1)$-arc.

Remark

Cases $r=3,4$ are open

The case $r = 3$

The case $r = 3$

- What do we know on case $r = 3$ for the Hermitian curve?

The case $r = 3$

- What do we know on case $r = 3$ for the Hermitian curve?

Let $P(a, b) \in \mathcal{H}_q$ and $P(a, b) \in PG(2, q^{2r}) \setminus PG(2, q^2)$

The case $r = 3$

- What do we know on case $r = 3$ for the Hermitian curve?

Let $P(a, b) \in \mathcal{H}_q$ and $P(a, b) \in PG(2, q^{2r}) \setminus PG(2, q^2)$

Proposition Through $P(a, b)$, we have as many as $2q^4 + q^2 + q + 1$ $q + 1$-secants to $\mathcal{H}_q$.

- What do we know on case $r = 3$ for the Hermitian curve?

Let $P(a, b) \in \mathcal{H}_q$ and $P(a, b) \in PG(2, q^{2r}) \setminus PG(2, q^2)$

Proposition Through $P(a, b)$, we have as many as $2q^4 + q^2 + q + 1$ $q + 1$-secants to $\mathcal{H}_q$.

Proof The above method is used with some variations.

The case $r = 3$

- What do we know on case $r = 3$ for the Hermitian curve?

Let $P(a, b) \in \mathcal{H}_q$ and $P(a, b) \in PG(2, q^{2r}) \setminus PG(2, q^2)$

Proposition Through $P(a, b)$, we have as many as $2q^4 + q^2 + q + 1$ $q + 1$-secants to $\mathcal{H}_q$.
Proof The above method is used with some variations.
The Galois group $G = Gal(M : K)$ has order $q(q - 1)$ $(\cong AGL(1, q))$,

The case $r = 3$

- What do we know on case $r = 3$ for the Hermitian curve?

Let $P(a, b) \in \mathcal{H}_q$ and $P(a, b) \in PG(2, q^{2r}) \setminus PG(2, q^2)$

Proposition Through $P(a, b)$, we have as many as $2q^4 + q^2 + q + 1$ $q + 1$-secants to $\mathcal{H}_q$.
Proof The above method is used with some variations.
The Galois group $G = Gal(M : K)$ has order $q(q - 1)$ $(\cong AGL(1, q))$,
M is a maximal function field over $\mathbb{F}_q^6$.
Remark B. Csajbók gave an elementary proof for the Proposition (Norm functions + highly non trivial computation).

The case $r = 3$

- What do we know on case $r = 3$ for the Hermitian curve?

Let $P(a, b) \in \mathcal{H}_q$ and $P(a, b) \in PG(2, q^{2r}) \setminus PG(2, q^2)$

Proposition Through $P(a, b)$, we have as many as
$2q^4 + q^2 + q + 1$ $q + 1$-secants to $\mathcal{H}_q$.
Proof The above method is used with some variations.
The Galois group $G = Gal(M : K)$ has order $q(q-1)$
$(\cong AGL(1, q))$,
M is a maximal function field over $\mathbb{F}_q^6$.
Remark B. Csajbók gave an elementary proof for the Proposition
(Norm functions + highly non trivial computation).
Magma computation shows for $q = 3$ that the above $(892, 4)$-arc
in $PG(2, 3^6)$ is complete.

The case $r = 3$

- What do we know on case $r = 3$ for the Hermitian curve?

Let $P(a, b) \in \mathcal{H}_q$ and $P(a, b) \in PG(2, q^{2r}) \setminus PG(2, q^2)$

Proposition Through $P(a, b)$, we have as many as $2q^4 + q^2 + q + 1$ $q + 1$-secants to $\mathcal{H}_q$.
Proof The above method is used with some variations.
The Galois group $G = Gal(M : K)$ has order $q(q - 1)$ $(\cong AGL(1, q))$,
M is a maximal function field over $\mathbb{F}_q^6$.
Remark B. Csajbók gave an elementary proof for the Proposition (Norm functions + highly non trivial computation).
Magma computation shows for $q = 3$ that the above $(892, 4)$-arc in $PG(2, 3^6)$ is complete.

Application to the Galois-inverse problems

Theorem

Let $K = \bar{\mathbb{F}}_{q^2}(t)$ *and* $L := K(u)$ *where* $u^{q+1} + u^q t^q + ut - ((ta-b)^q + ta - b)$. *Then the (geometric) monodromy group of* $L : K$ *is isomorphic to* $PGL(2, q)$*, and the Galois closure* M *of* $L : K$ *is* $M = \bar{\mathbb{F}}_{q^2}(t, u, v, w)$ *where*

$$\begin{cases} u^{q+1} + u^q t^q + ut - ((ta-b)^q + ta - b); \\ v^q + (u + t^q)v^{q-1} + u^q + t = 0; \\ v + u + t^q - (u + t^q)w^{q-1} = 0. \end{cases}$$