Identifiable Waring subspaces over finite fields

Ferdinando Zullo
Università degli Studi della Campania "Luigi Vanvitelli"

(Joint work with Michel Lavrauw)

Finite Geometries - Sixth Irsee Conference

Waring's problem

For each natural number k, does there exist a positive integer s such that every natural number is the sum of at most s natural numbers raised to the power k?

Waring's problem

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Four-square theorem (Lagrange-1770)

Every natural number is the sum of at most 4 squares.

Waring's problem

For each natural number k, does there exist a positive integer s such that every natural number is the sum of at most s natural numbers raised to the power k?

Hilbert-Waring theorem (Hilbert-1909)

$s = g(k)$

Waring's problem

For each natural number k, does there exist a positive integer s such that every natural number is the sum of at most s natural numbers raised to the power k?

Hilbert-Waring theorem (Hilbert-1909)

$s = g(k)$

$g(2) = 4$ Lagrange (1770)
$g(3) = 9$ Wieferich and Kempner (1909/12)
$g(4) = 19$ Balasubramanian, Dress Deshhouillers (1986)
$g(5) = 37$ Chen (1964)
$g(6) = 73$ Pillai (1940)

Waring's problem for polynomials

$\mathbb{F} \rightarrow$ algebraically closed field of characteristic 0

$\mathbb{F}[X_0, \ldots, X_n]_d \rightarrow$ vector space of homogeneous polynomials of degree d in $n+1$ variables over $\mathbb{F}$

Waring's problem for polynomials

$\mathbb{F} \to$ algebraically closed field of characteristic 0

$\mathbb{F}[X_0, \ldots, X_n]_d \to$ vector space of homogeneous polynomials of degree d in $n+1$ variables over $\mathbb{F}$

Problem

Determine the minimum integer s such that the generic degree d homogeneous polynomial P in $\mathbb{F}[X_0, \ldots, X_n]_d$ is the sum of at most s d-th powers of linear forms?

$$P = L_1^d + \ldots + L_s^d$$

where $L_i = a_0 X_0 + \ldots + a_n X_n \in \mathbb{F}[X_0, \ldots, X_n]$

Waring's problem for polynomials

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Problem solved by Alexander and Hirschowitz in 1995

Waring's problem for polynomials

Problem (in terms of symmetric tensors)

$V \to$ vector space over $\mathbb{F}$

$\bigotimes^d V =$ vector space of multilinear functions $(V^\vee)^d \mapsto \mathbb{F}$

Pure tensors $\to$ $v_1, \ldots, v_d \in V^\vee$

$$v_1 \otimes \ldots \otimes v_d : (u_1, \ldots, u_d) \in V^d \mapsto \prod_{i=1}^{d} v_i(u_i) \in \mathbb{F}$$

Rank of a tensor T = minimum number of pure tensors required to express T as their sum

Waring's problem for polynomials

Problem (in terms of symmetric tensors)

$\sigma \in S_d$

$\sigma(v_1 \otimes \ldots \otimes v_d) = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)}$

Action extended to $\bigotimes^d V$

Waring's problem for polynomials

Problem (in terms of symmetric tensors)

$\sigma \in S_d$

$\sigma(v_1 \otimes \ldots \otimes v_d) = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)}$

Action extended to $\otimes^d V$

Symmetric tensor → tensor left invariant by this action of S_d

$\mathrm{Sym}^d(V)$ = vector space of symmetric tensors

Symmetric pure tensors → $v^d = v \otimes \ldots \otimes v$

Waring's problem for polynomials

Problem (in terms of symmetric tensors)

$$\mathrm{Sym}^d(V) \xleftrightarrow{\text{Polarization}} \mathbb{F}[X_0, \ldots, X_n]_d$$

Waring's problem for polynomials

Problem (in terms of symmetric tensors)

$$\mathrm{Sym}^d(V) \xleftrightarrow{\text{Polarization}} \mathbb{F}[X_0, \ldots, X_n]_d$$

Problem

Determine the minimum integer s such that the generic tensor T in $\mathrm{Sym}^d(V)$ can be written as the sum of at most s pure symmetric tensors?

$$T = v_1^d + \ldots + v_s^d$$

where $v_i \in V^\vee$

Waring's problem for polynomials

Problem (in terms of symmetric tensors)

$$\mathrm{Sym}^d(V) \xleftrightarrow[\text{Polarization}]{} \mathbb{F}[X_0, \ldots, X_n]_d$$

Problem

Determine the minimum integer s such that the generic tensor T in $\mathrm{Sym}^d(V)$ can be written as the sum of at most s pure symmetric tensors?

s=Waring rank

$$T = v_1^d + \ldots + v_s^d$$

where $v_i \in V^\vee$

Waring's problem for polynomials

Problem (in terms of symmetric tensors)

$$\boxed{\mathrm{Sym}^d(V) \xleftrightarrow{\text{Polarization}} \mathbb{F}[X_0, \ldots, X_n]_d}$$

Problem

Determine the minimum integer s such that the generic tensor T in $\mathrm{Sym}^d(V)$ can be written as the sum of at most s pure symmetric tensors?

s=Waring rank

$$T = \boxed{v_1^d + \ldots + v_s^d}$$

where $v_i \in V^\vee$

Waring decomposition

Waring's problem for polynomials

Problem (in terms of symmetric tensors)

$$\mathrm{Sym}^d(V) \xleftrightarrow{\text{Polarization}} \mathbb{F}[X_0, \ldots, X_n]_d$$

Problem

Determine the m[…] T is Waring identifiable if this decomposition is unique […]ne generic tensor T in Sym[…]n of at most s pure symmetric tensors?

s=Waring rank

$$T = \boxed{v_1^d + \ldots + v_s^d}$$

where $v_i \in V^\vee$

Waring decomposition

Waring's problem for polynomials

In terms of Veronese variety

Veronese map ν_d of degree d

$$\nu_d : (x_0, \ldots, x_n) \in \mathbb{P}^n \mapsto (\ldots, x^J, \ldots) \in \mathbb{P}^N$$

where x^J runs over all the monomials of degree d and

$$N = \binom{n+d}{d} - 1$$

Waring's problem for polynomials

In terms of Veronese variety

Veronese map ν_d of degree d

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where x^J runs over all the monomials of degree d and

$$N = \binom{n+d}{d} - 1$$

$\mathbb{V}_{n,d} = \nu_d(\mathbb{P}^n)$ is a projective algebraic variety

$\dim(\mathbb{V}_{n,d}) = n$

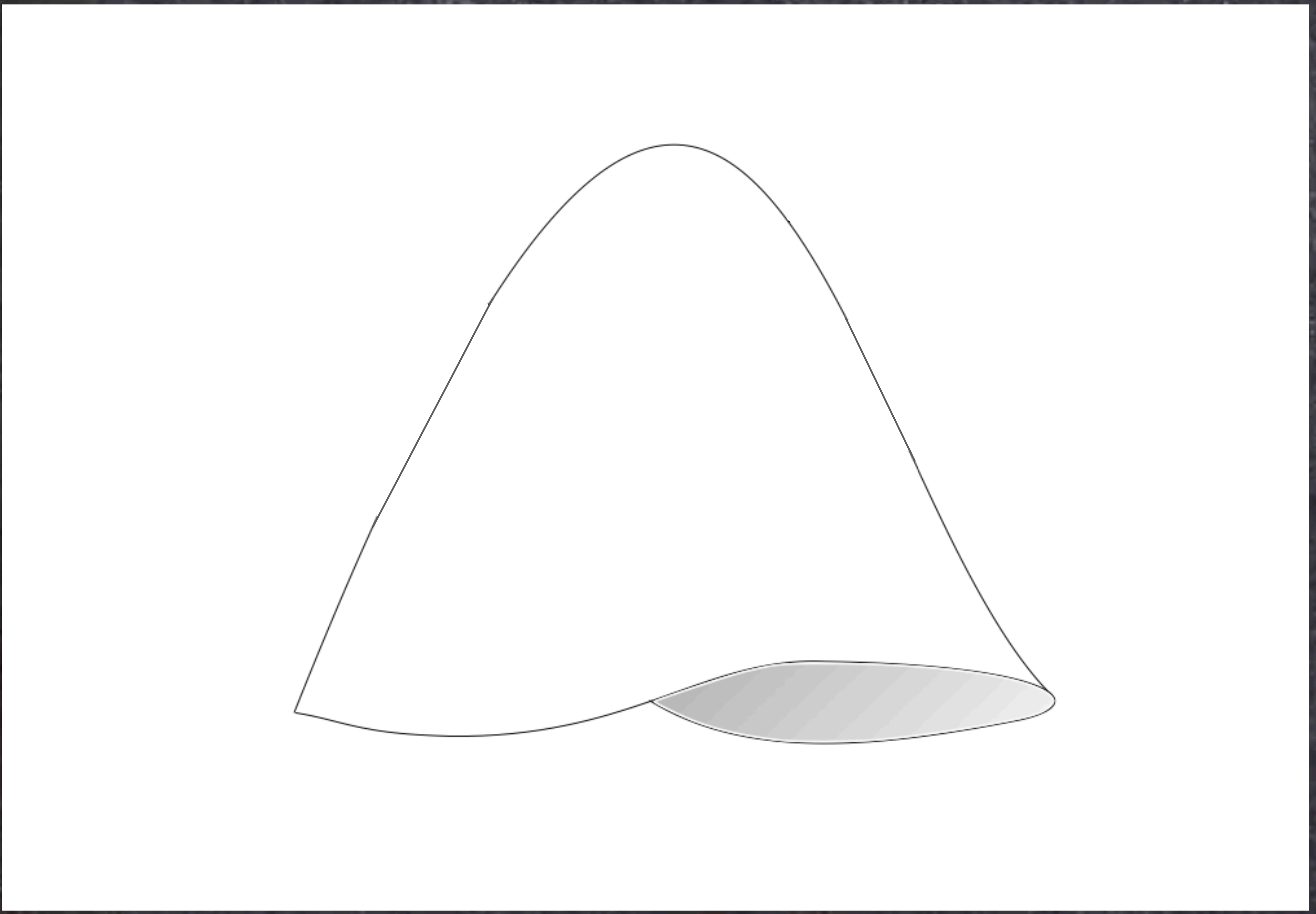
degree of $\mathbb{V}_{n,d} = d^n$

Veronese variety

Waring's problem for polynomials

In terms of Veronese variety

$$\mathbb{P}^N = \mathbb{P}(\mathbb{F}[x_0, \ldots, x_n]_d) \text{ or } \mathbb{P}(\mathrm{Sym}^d(\mathbb{F}^{n+1}))$$



Veronese variety $\mathbb{V}_{n,d}$

Waring's problem for polynomials

In terms of Veronese variety

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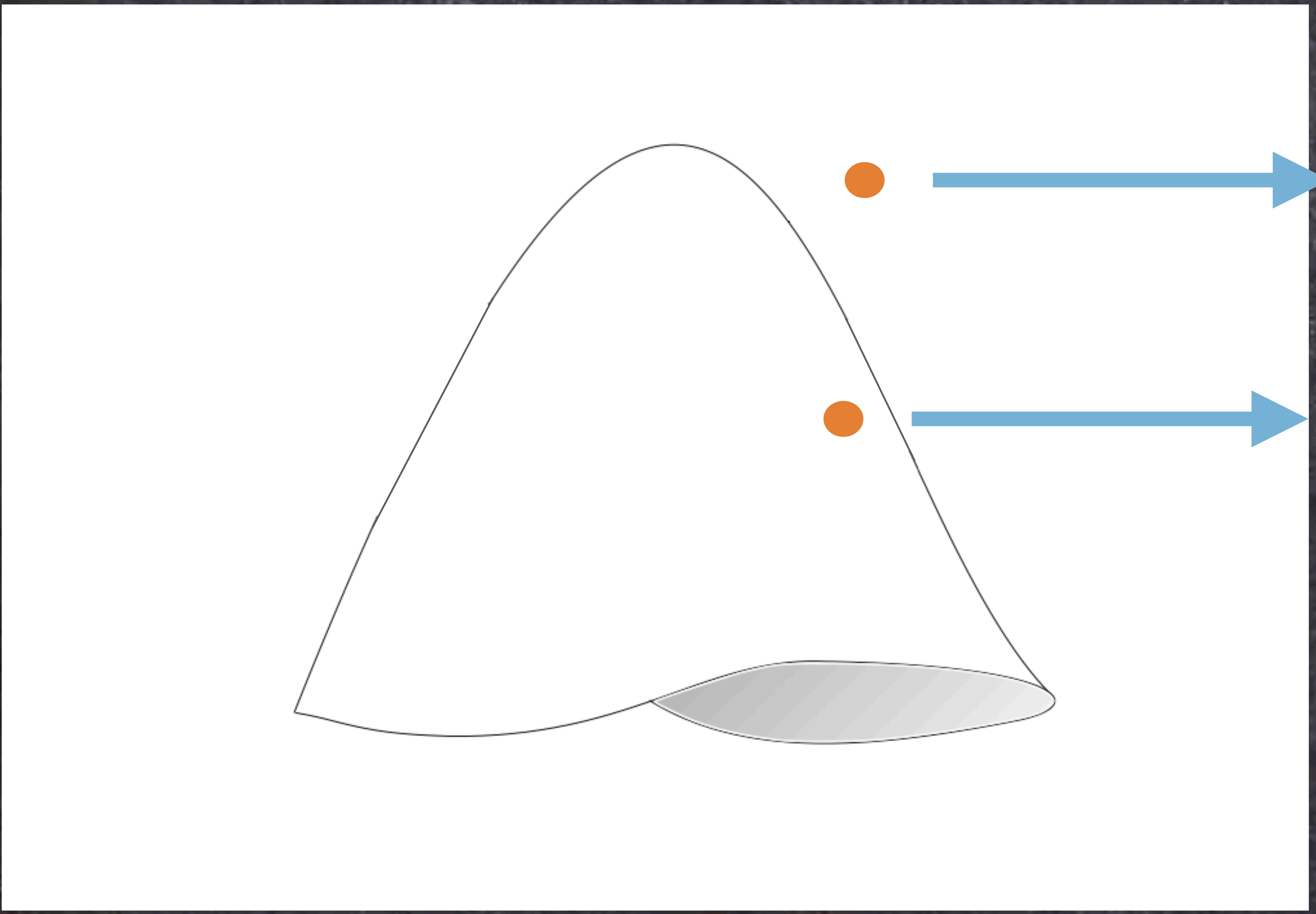


$[T]$ Symmetric tensor

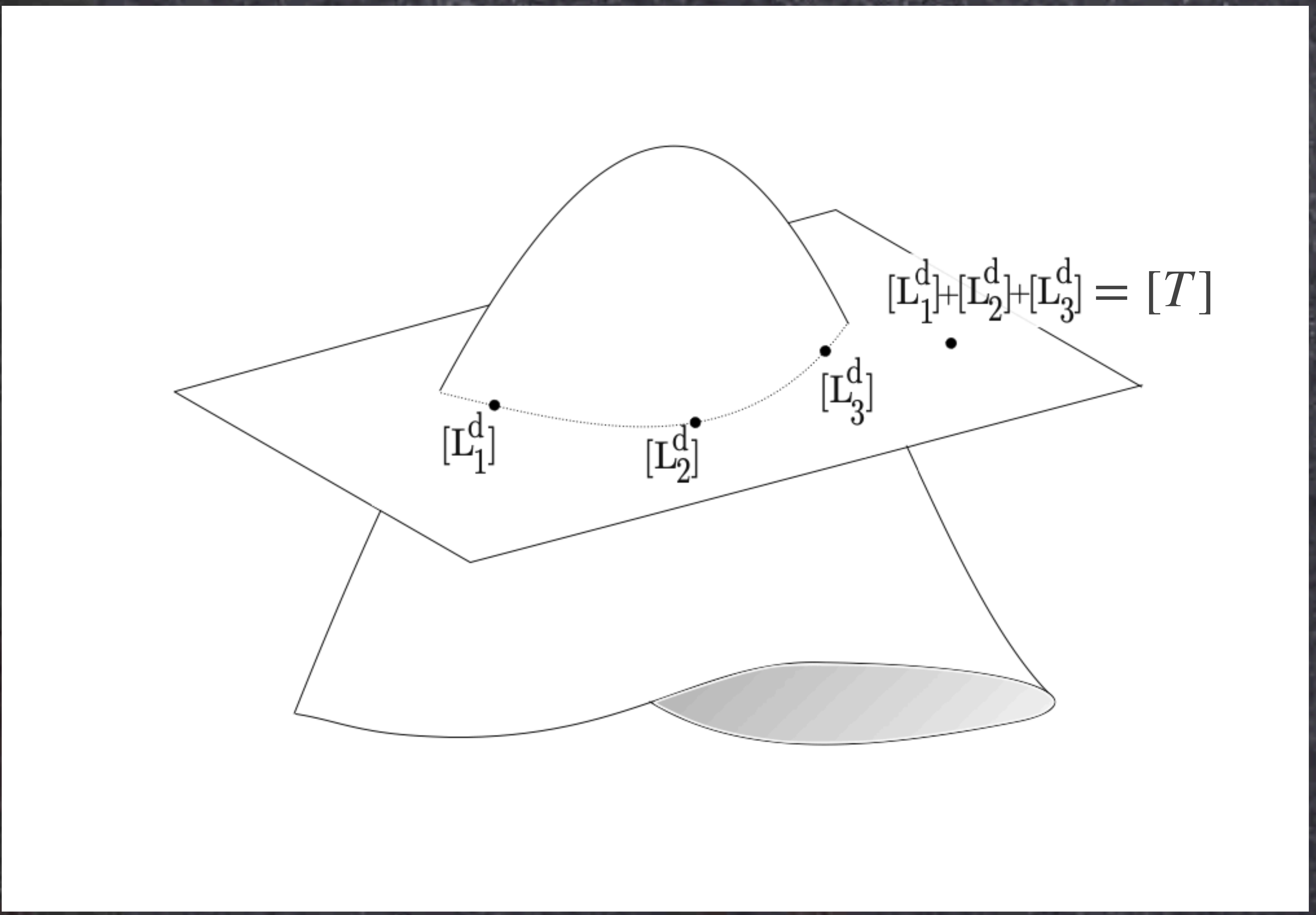
$[L^d]$ Pure symmetric tensor

Veronese variety $\mathbb{V}_{n,d}$

Waring's problem for polynomials

In terms of Veronese variety

$$\mathbb{P}^N = \mathbb{P}(\mathbb{F}[x_0, \ldots, x_n]_d) \text{ or } \mathbb{P}(\mathrm{Sym}^d(\mathbb{F}^{n+1}))$$



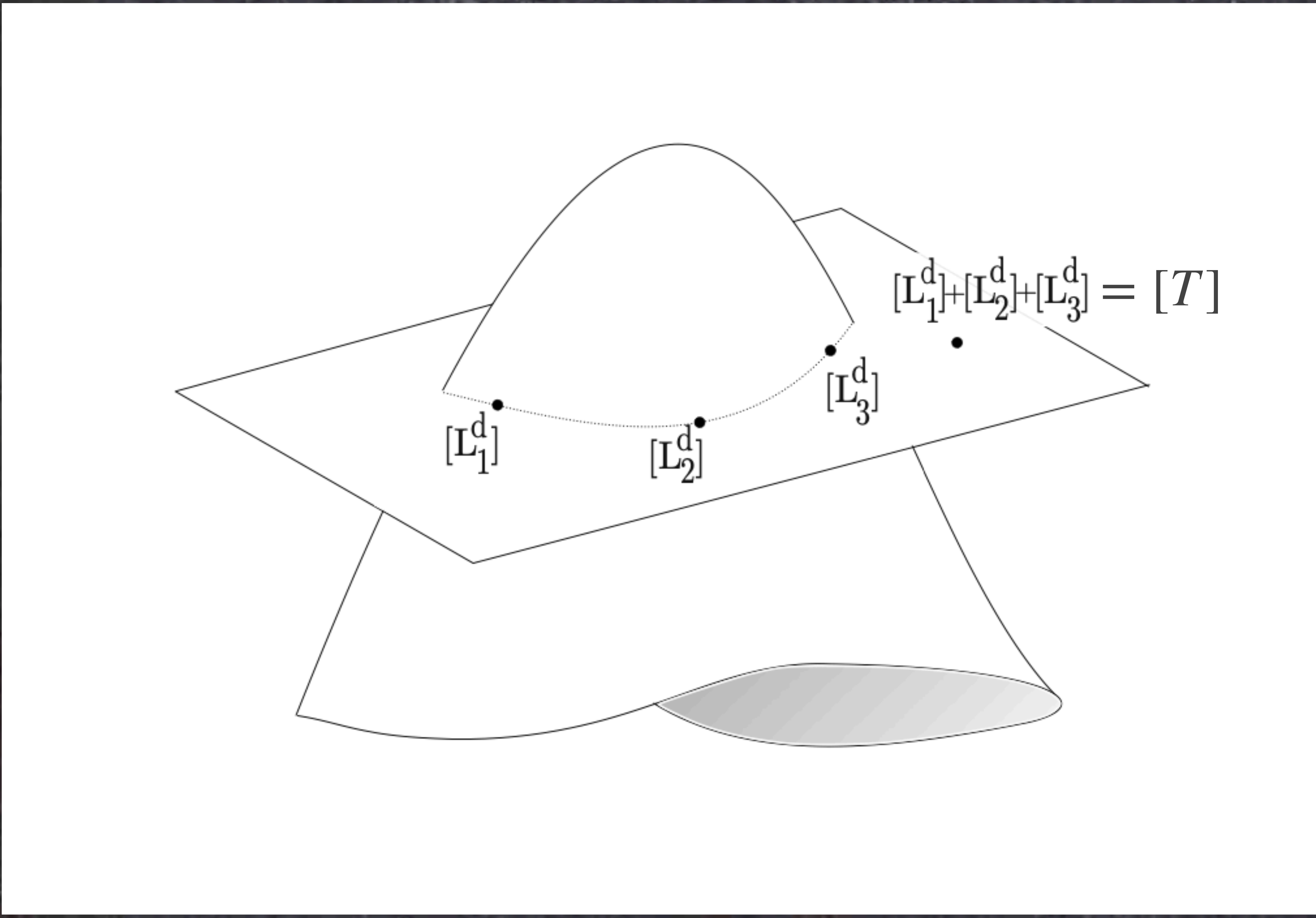
Veronese variety $\mathbb{V}_{n,d}$

Rank of T = $\mathbb{V}_{n,d}$-rank of $[T]$ = the minimum dimension of a subspace spanned by the points of $\mathbb{V}_{n,d}$ containing $[T]$

Waring's problem for polynomials

In terms of Veronese variety

$$\mathbb{P}^N = \mathbb{P}(\mathbb{F}[x_0, \ldots, x_n]_d) \text{ or } \mathbb{P}(\mathrm{Sym}^d(\mathbb{F}^{n+1}))$$



Veronese variety $\mathbb{V}_{n,d}$

T **Waring identifiable** of rank k =

$[T] \in \langle P_1, \ldots, P_k \rangle$ and

$\langle P_1, \ldots, P_k \rangle \cap \mathbb{V}_{n,d} = \{P_1, \ldots, P_k\}$

Waring subspaces over finite fields

M. Lavrauw and FZ: Waring identifiable subspaces over finite fields, arXiv:2207.1345

Waring subspaces over finite fields

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M. Lavrauw and FZ: Waring identifiable subspaces over finite fields, arXiv:2207.1345

$\mathscr{A} \subseteq \mathbb{P}^N(\mathbb{F})$ where $\mathbb{F}$ is any field

Waring subspaces over finite fields

M. Lavrauw and FZ: Waring identifiable subspaces over finite fields, arXiv:2207.1345

$\mathscr{A} \subseteq \mathbb{P}^N(\mathbb{F})$ where $\mathbb{F}$ is any field

$S \subseteq \mathbb{P}^N(\mathbb{F})$ subspace

- Waring subspace w.r.t. $\mathscr{A}$ if it is spanned by points of $\mathscr{A}$
- Waring identifiable w.r.t. $\mathscr{A}$ if it is contained in a unique Waring subspace of minimal dimension $\neq \mathbb{P}^N$
- Identifiable Waring w.r.t. $\mathscr{A}$ = Waring subspace + Waring identifiable

Waring subspaces over finite fields

M. Lavrauw and FZ: Waring identifiable subspaces over finite fields, arXiv:2207.1345

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E. Ballico, A. Bernardi, M.V. Catalisano and L. Chiantini: Grassmann secants, identifiability and linear systems of tensors, Linear Algebra Appl. 438 (2013), 121-135.

Waring subspaces over finite fields

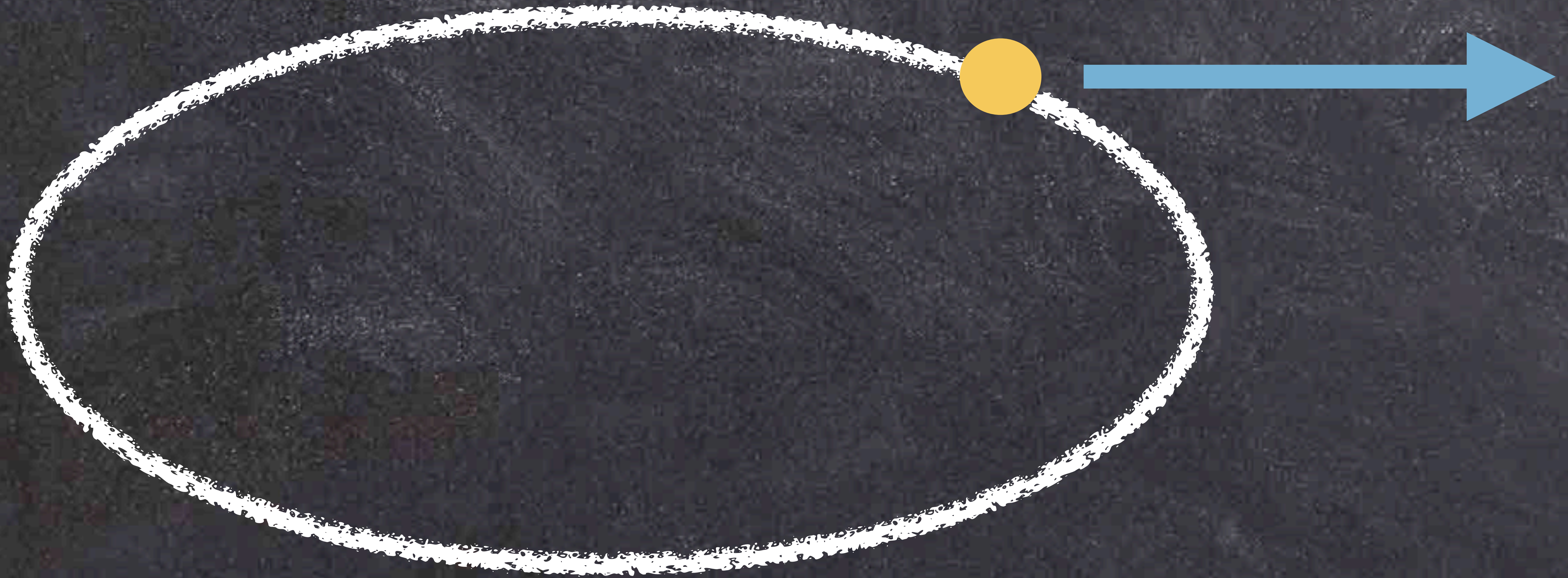
M. Lavrauw and FZ: Waring identifiable subspaces over finite fields, arXiv:2207.1345

$\mathscr{A} \subseteq \mathbb{P}^N(\mathbb{F})$ where $\mathbb{F}$ is any field

Waring subspaces over finite fields

Example - non-degenerate conic

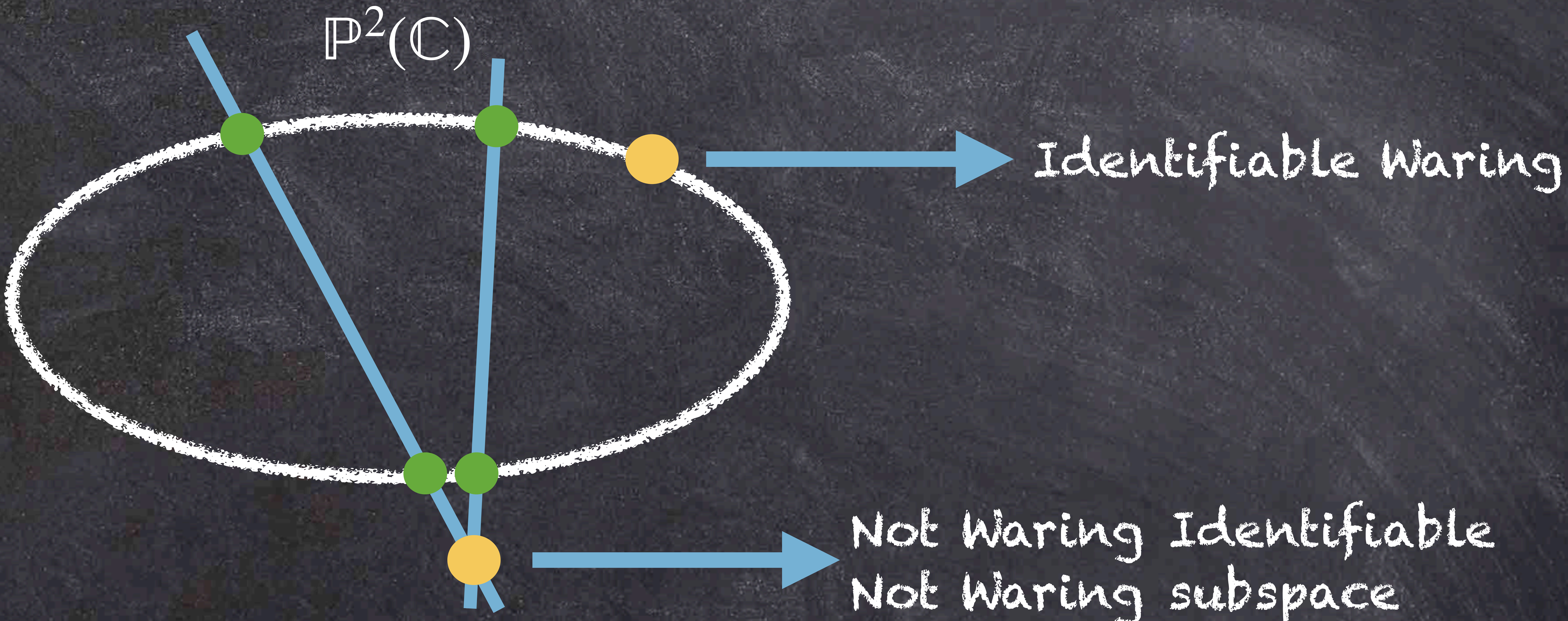
$\mathbb{P}^2(\mathbb{C})$



Identifiable Waring

Waring subspaces over finite fields

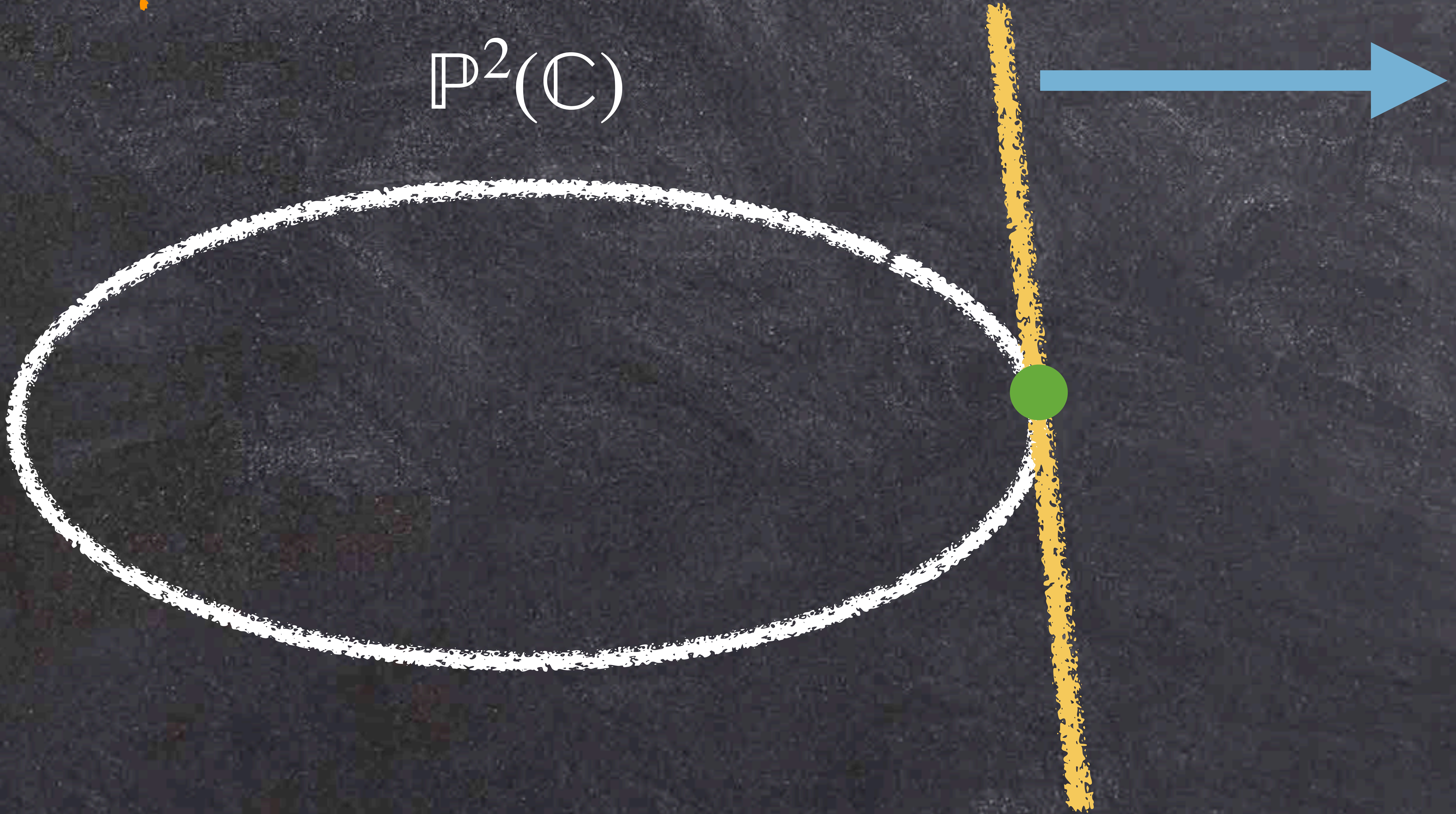
Example - non-degenerate conic



Waring subspaces over finite fields

Example - non-degenerate conic

$\mathbb{P}^2(\mathbb{C})$



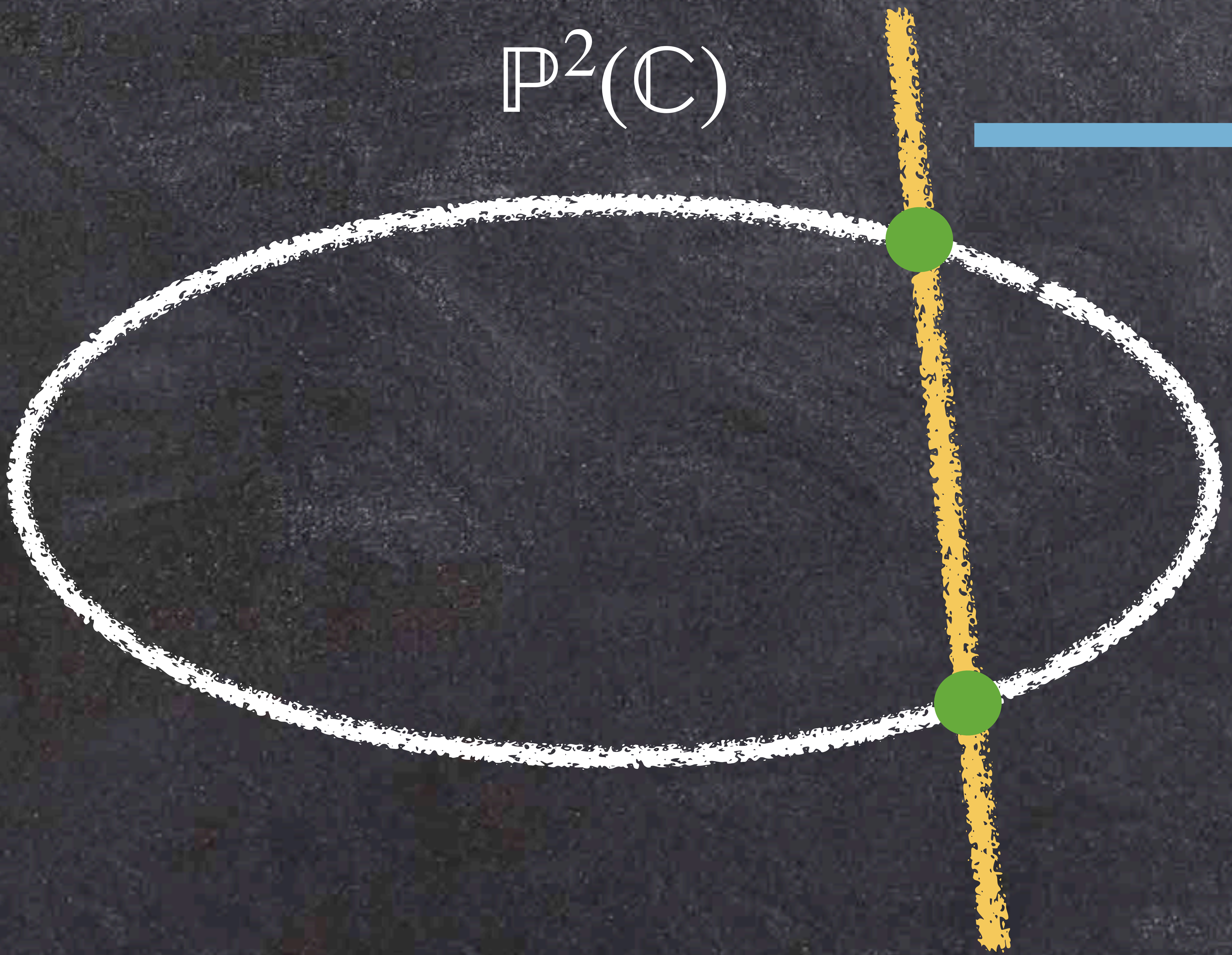
Not Waring subspace

Not Waring identifiable

Waring subspaces over finite fields

Example - non-degenerate conic

$\mathbb{P}^2(\mathbb{C})$

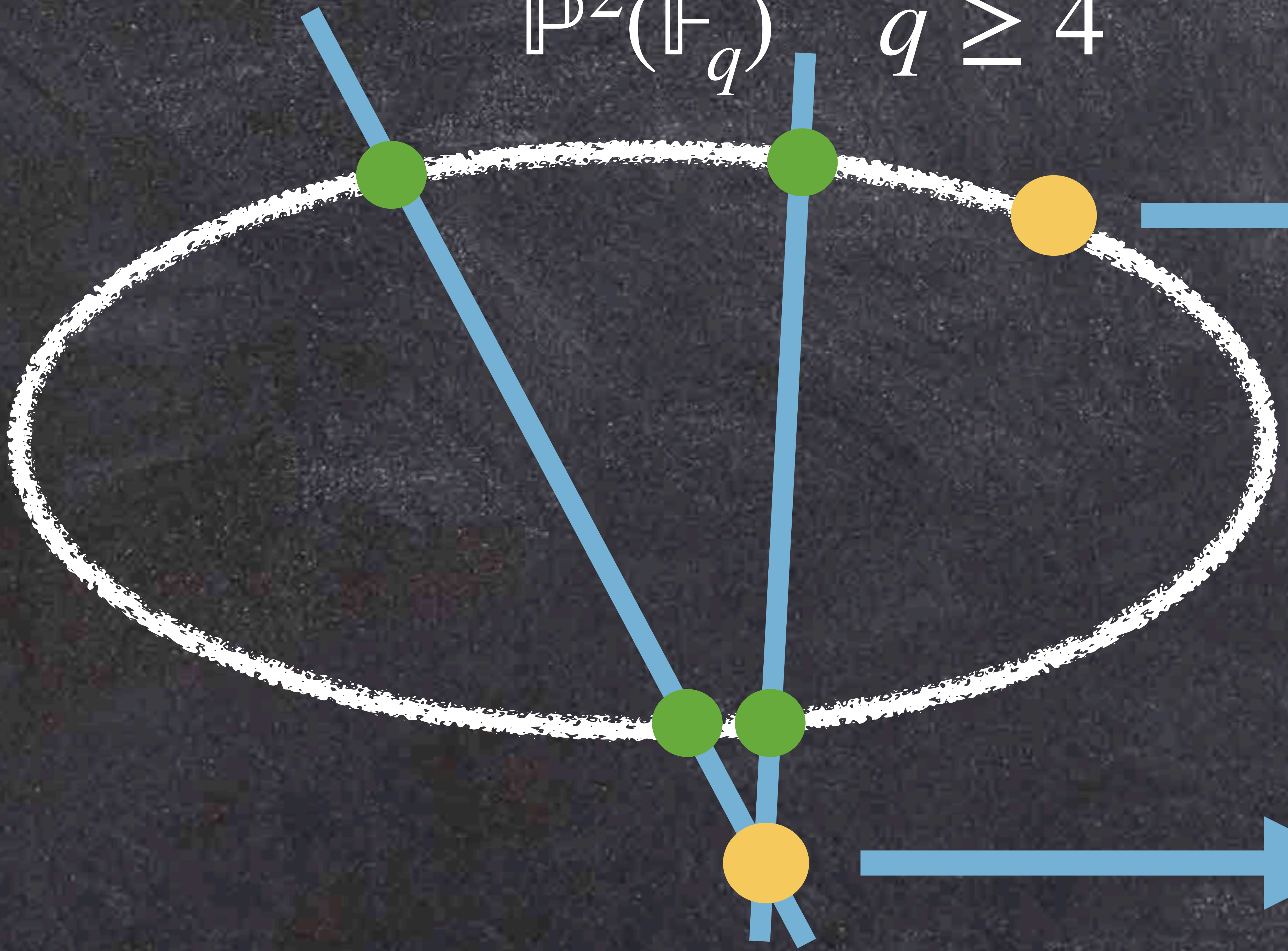


Identifiable Waring

Waring subspaces over finite fields

Example - non-degenerate conic

$\mathbb{P}^2(\mathbb{F}_q)$ $q \geq 4$



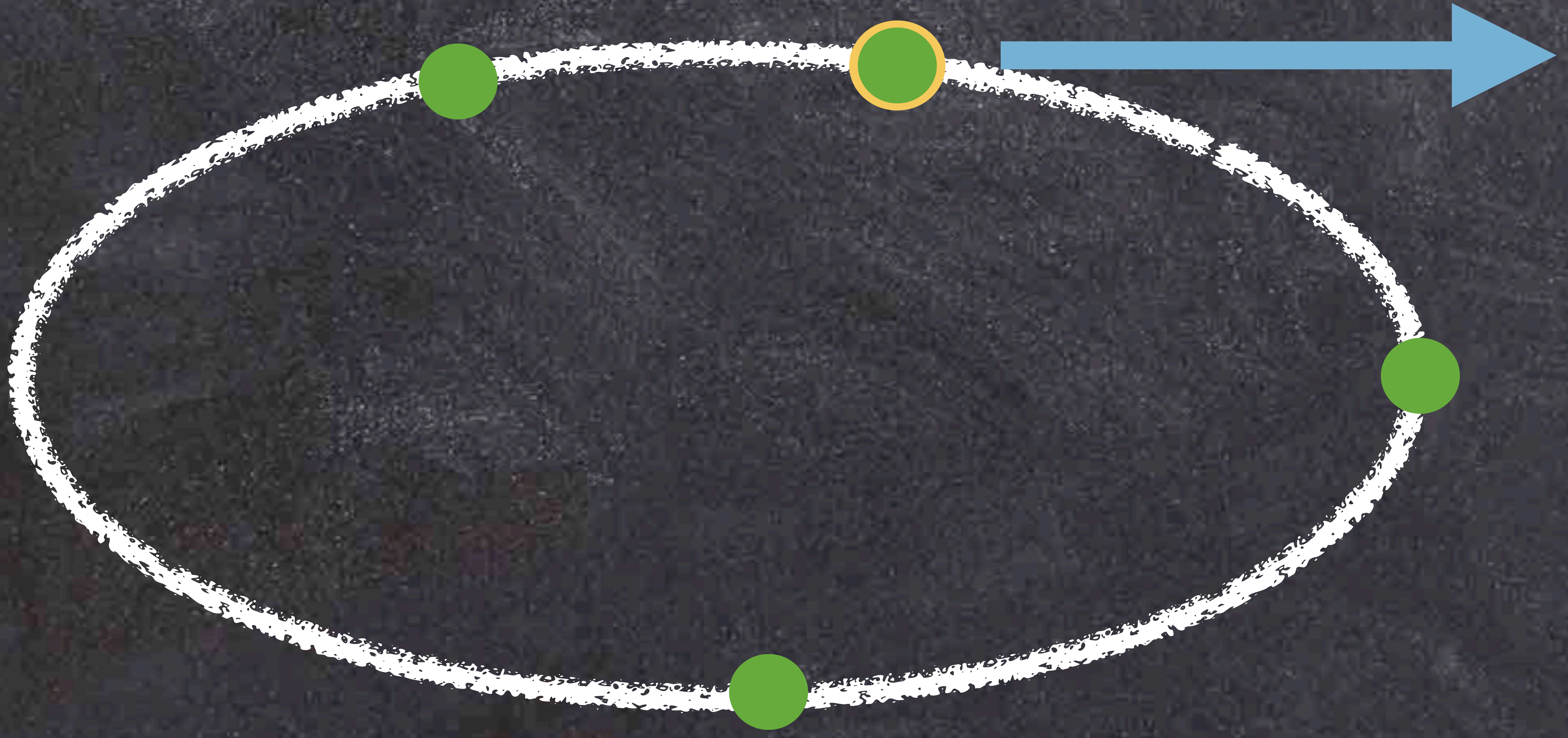
Identifiable Waring

Not Waring Identifiable
Not Waring subspace

Waring subspaces over finite fields

Example - non-degenerate conic

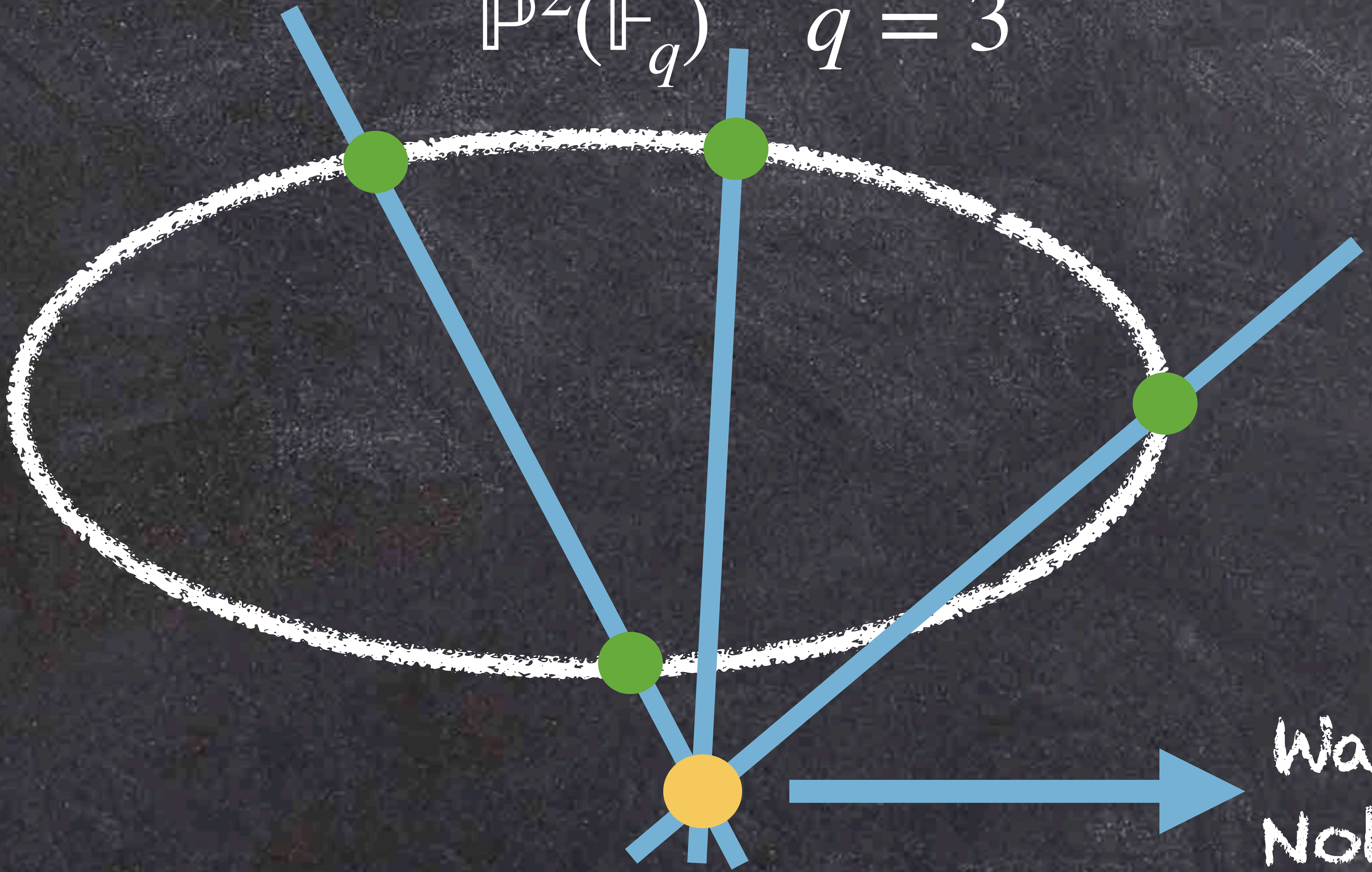
$\mathbb{P}^2(\mathbb{F}_q)$ $\quad q = 3$



Identifiable Waring

Waring subspaces over finite fields

Example - non-degenerate conic

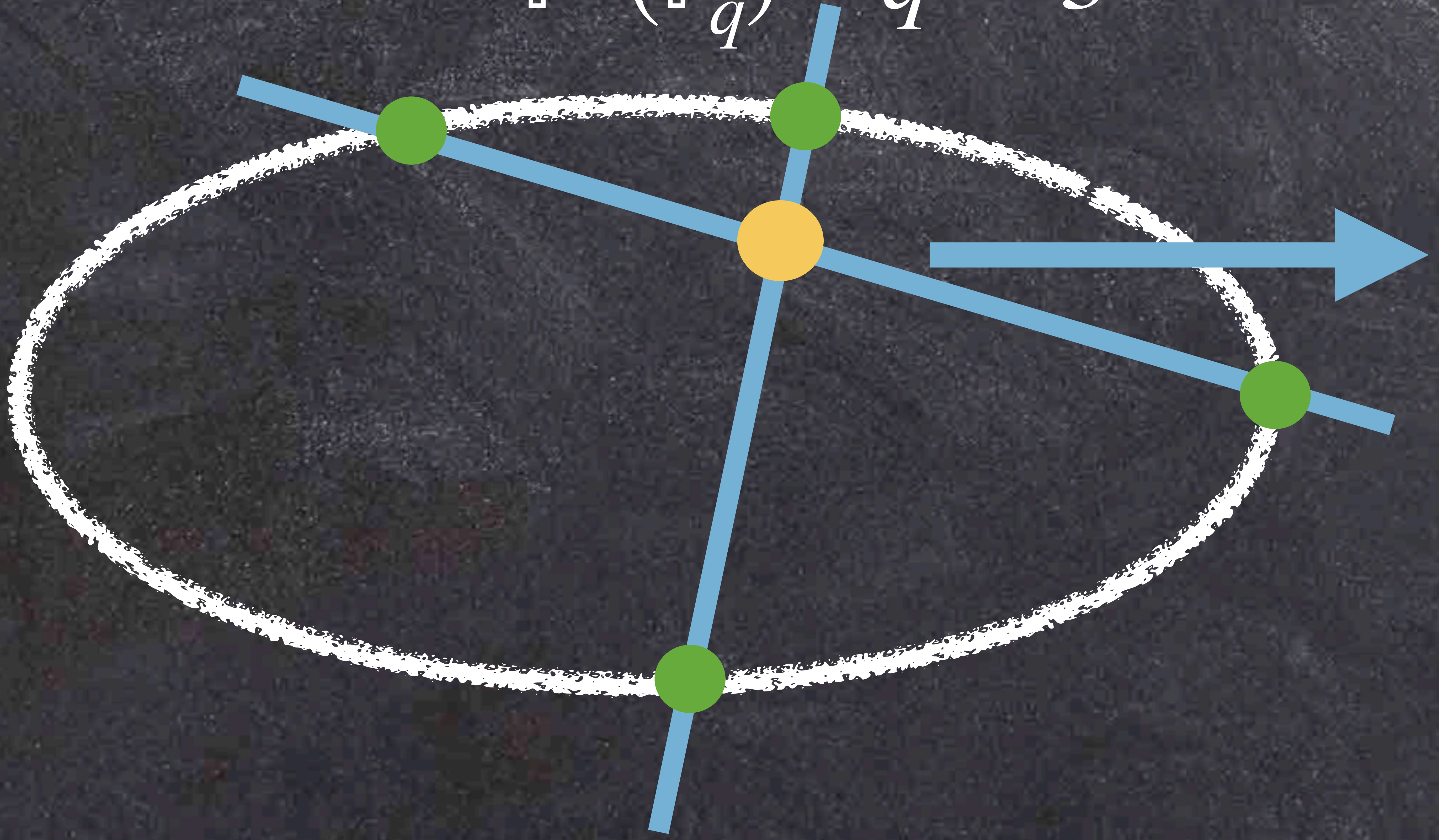


Waring Identifiable
Not Waring subspace

Waring subspaces over finite fields

Example - non-degenerate conic

$\mathbb{P}^2(\mathbb{F}_q)$ $q = 3$



Not Waring Identifiable
Not Waring subspace

Waring subspaces over finite fields

M. Lavrauw and FZ: Waring identifiable subspaces over finite fields, arXiv:2207.1345

$\mathscr{A} \subseteq \mathbb{P}^N(\mathbb{F})$ where $\mathbb{F}$ is any field

$\mathrm{Aut}(\mathscr{A}) =$ group of collineations of $\mathbb{P}^N(\mathbb{F})$ fixing $\mathscr{A}$

Classification of subspaces w.r.t. $\mathrm{Aut}(\mathscr{A})$

Waring subspaces over finite fields

M. Lavrauw and FZ: Waring identifiable subspaces over finite fields, arXiv:2207.1345

$$\mathscr{A} \subseteq \mathbb{P}^N(\mathbb{F}_q)$$

$\mathrm{Aut}(\mathscr{A})$ = group of collineations of $\mathbb{P}^N(\mathbb{F}_q)$ fixing $\mathscr{A}$

Waring polynomial of $\mathscr{A}$

$$\mathscr{W}_{\mathscr{A}}(X) = \sum_{i=0}^{N-1} \lambda_i(\mathscr{A}) X^i$$

$\lambda_i(\mathscr{A})$ = number of $\mathrm{Aut}(\mathscr{A})$-orbits of i-dimensional Waring subspaces

Waring subspaces over finite fields

M. Lavrauw and FZ: Waring identifiable subspaces over finite fields, arXiv:2207.1345

$$\mathscr{A} \subseteq \mathbb{P}^N(\mathbb{F}_q)$$

$\mathrm{Aut}(\mathscr{A})$ = group of collineations of $\mathbb{P}^N(\mathbb{F}_q)$ fixing $\mathscr{A}$

Waring Identifiable polynomial of $\mathscr{A}$

$$\mathscr{W}\mathscr{I}_{\mathscr{A}}(X) = \sum_{i=0}^{N-1} \mu_i(\mathscr{A}) X^i$$

$\mu_i(\mathscr{A})$ = number of $\mathrm{Aut}(\mathscr{A})$-orbits of i-dimensional Waring identifiable subspaces

Waring subspaces over finite fields

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$$\mathscr{A} \subseteq \mathbb{P}^N(\mathbb{F}_q)$$

$\mathrm{Aut}(\mathscr{A})$ = group of collineations of $\mathbb{P}^N(\mathbb{F}_q)$ fixing $\mathscr{A}$

Identifiable Waring polynomial of $\mathscr{A}$

$$\mathscr{I}\mathscr{W}_{\mathscr{A}}(X) = \sum_{i=0}^{N-1} \eta_i(\mathscr{A}) X^i$$

$\eta_i(\mathscr{A})$ = number of $\mathrm{Aut}(\mathscr{A})$-orbits of i-dimensional identifiable Waring subspaces

Waring subspaces over finite fields

M. Lavrauw and FZ: Waring identifiable subspaces over finite fields, arXiv:2207.1345

$$\mathscr{A} \subseteq \mathbb{P}^N(\mathbb{F}_q)$$

$\mathrm{Aut}(\mathscr{A})$ = group of collineations of $\mathbb{P}^N(\mathbb{F}_q)$ fixing $\mathscr{A}$

Big Problem

For any $\mathscr{A}$, find $\mathscr{W}_{\mathscr{A}}(X), \mathscr{W}\mathscr{I}_{\mathscr{A}}(X), \mathscr{I}\mathscr{W}_{\mathscr{A}}(X)$

Waring subspaces over finite fields

M. Lavrauw and FZ: Waring identifiable subspaces over finite fields, arXiv:2207.1345

$$\mathscr{A} \subseteq \mathbb{P}^N(\mathbb{F}_q)$$

$\mathrm{Aut}(\mathscr{A})$ = group of collineations of $\mathbb{P}^N(\mathbb{F}_q)$ fixing $\mathscr{A}$

Big Problem

For any $\mathscr{A}$, find $\mathscr{W}_{\mathscr{A}}(X), \mathscr{W}\mathscr{I}_{\mathscr{A}}(X), \mathscr{I}\mathscr{W}_{\mathscr{A}}(X)$

$$\mathscr{A} = \mathbb{V}_{n,2}(\mathbb{F}_q)$$

$$\mathrm{Aut}(\mathscr{A}) \simeq \mathrm{PGL}(n+1,q), q \neq 2$$

$$\mathrm{Aut}(\mathscr{A}) \simeq S_7, q = 2$$

Waring subspaces over finite fields

$$\mathscr{A} = \mathbb{V}_{n,2}(\mathbb{F}_q) \subseteq \mathbb{P}^N(\mathbb{F}_q) \text{ where } N = \binom{n+2}{2} - 1$$

Theorem (Lavrauw-FZ)

If $\mathscr{S}$ is a subspace in $\mathbb{P}^N(\mathbb{F}_q)$ spanned by the images of a frame $\Rightarrow$ $\mathscr{S}$ is an identifiable Waring subspace

Waring subspaces over finite fields

$$\mathscr{A} = \mathbb{V}_{n,2}(\mathbb{F}_q) \subseteq \mathbb{P}^N(\mathbb{F}_q) \text{ where } N = \binom{n+2}{2} - 1$$

Theorem (Lavrauw-FZ)

If $\mathscr{S}$ is a subspace in $\mathbb{P}^N(\mathbb{F}_q)$ spanned by the images of a frame $\Rightarrow$ $\mathscr{S}$ is an identifiable Waring subspace

Corollary

There exist identifiable Waring subspaces of dimension $\ell \leq n+1$, equivalently $\eta_i(\mathscr{A}) \geq 1$ for any $1 \leq i \leq n+1$

Waring subspaces over finite fields

$$\mathscr{A} = \mathbb{V}_{2,2}(\mathbb{F}_q) \subseteq \mathbb{P}^5(\mathbb{F}_q)$$

Theorem (Lavrauw-FZ)

If $q \neq 2{,}4 \Rightarrow \mathscr{IW}_{\mathscr{A}}(X) = 1 + X + X^2 + X^3$

If $q = 2 \Rightarrow \mathscr{IW}_{\mathscr{A}}(X) = 1 + X + 2X^2 + 2X^3 + X^4$

If $q = 4 \Rightarrow \mathscr{IW}_{\mathscr{A}}(X) = 1 + X + X^2 + X^3 + X^4$

Waring subspaces over finite fields

$$\mathscr{A} = \mathbb{V}_{2,2}(\mathbb{F}_q) \subseteq \mathbb{P}^5(\mathbb{F}_q)$$

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If $q \neq 2,4 \Rightarrow \mathscr{IW}_{\mathscr{A}}(X) = 1 + X + X^2 + X^3$

If $q = 2 \Rightarrow \mathscr{IW}_{\mathscr{A}}(X) = 1 + X + 2X^2 + 2X^3 + X$

If $q = 4 \Rightarrow \mathscr{IW}_{\mathscr{A}}(X) = 1 + X + X^2 + X^3 + X^4$

See Michel's talk on Friday!

- N. Alnajjarine, M. Lavrauw, and T. Popiel: Solids in the space of the Veronese surface in even characteristic. Finite Fields Appl. 83 (2022)
- N. Alnajjarine, M. Lavrauw: Planes intersecting the Veronese surface in PG(5, q), q even, in preparation
- M. Lavrauw and T. Popiel: The symmetric representation of lines in PG(F^3⊗ F^3), Discrete Math. 343(4) (2020)
- M. Lavrauw, T. Popiel and J. Sheekey: Nets of conics of rank one in PG(2,q), q odd, J. Geom. 111 (2020)

Waring subspaces over finite fields

$$\mathscr{A} = \mathbb{V}_{3,2}(\mathbb{F}_q) \subseteq \mathbb{P}^9(\mathbb{F}_q)$$

$\eta_i(\mathscr{A}) \geq 1$ for any $1 \leq i \leq 4$

Waring subspaces over finite fields

$$\mathscr{A} = \mathbb{V}_{3,2}(\mathbb{F}_q) \subseteq \mathbb{P}^9(\mathbb{F}_q) = \mathbb{P}(\mathrm{Sym}^2(\mathbb{F}_q^4))$$

$\eta_i(\mathscr{A}) \geq 1$ for any $1 \leq i \leq 4$

Theorem (Lavrauw-FZ)

Let $\omega \in \mathbb{F}_q \backslash \{0, \pm 1\}$, then

$\mathscr{S} = \langle \mathbf{e}_1^{\otimes 2}, \mathbf{e}_2^{\otimes 2}, \mathbf{e}_3^{\otimes 2}, \mathbf{e}_4^{\otimes 2}, (\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4)^{\otimes 2}, \mathbf{v}, \mathbf{w} \rangle$

with $\mathbf{v} = (\mathbf{e}_1 + \omega \mathbf{e}_2 + \omega \mathbf{e}_3 + \omega^2 \mathbf{e}_4)^{\otimes 2}$ and

$\mathbf{w} = (\mathbf{e}_1 + \omega \mathbf{e}_2 + \mathbf{e}_3 + \omega \mathbf{e}_4)^{\otimes 2}$

is an identifiable Waring subspace of dimension 6

Waring subspaces over finite fields

$$\mathscr{A} = \mathbb{V}_{3,2}(\mathbb{F}_q) \subseteq \mathbb{P}^9(\mathbb{F}_q) = \mathbb{P}(\mathrm{Sym}^2(\mathbb{F}_q^4))$$

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with $\mathbf{v} = (\mathbf{e}_1 + \omega \mathbf{e}_2 + \omega \mathbf{e}_3 + \omega^2 \mathbf{e}_4)^{\otimes 2}$ and

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$\eta_i(\mathscr{A}) \geq 1$ for any $1 \leq i \leq 6$

Waring subspaces over finite fields

$$\mathscr{A} = \mathbb{V}_{3,2}(\mathbb{F}_q) \subseteq \mathbb{P}^9(\mathbb{F}_q) = \mathbb{P}(\mathrm{Sym}^2(\mathbb{F}_q^4))$$

$\eta_i(\mathscr{A}) \geq 1$ for any $1 \leq i \leq 6$

Theorem (Lavrauw-FZ)

Let $\omega \in \mathbb{F}_q \setminus \{0,1,2\}$ a square in $\mathbb{F}_q$, then

$\mathscr{S} = \langle \mathbf{e}_1^{\otimes 2}, \mathbf{e}_2^{\otimes 2}, \mathbf{e}_3^{\otimes 2}, \mathbf{e}_4^{\otimes 2}, \mathbf{e}^{\otimes 2}, (\omega \mathbf{e}_1 + \mathbf{e}_2 + \omega \mathbf{e}_3)^{\otimes 2}, (\sqrt{\omega}\mathbf{e}_1 + \sqrt{\omega}\mathbf{e}_2 + \sqrt{\omega^3}\mathbf{e}_4)^{\otimes 2} \rangle$

with $\mathbf{e} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4)^{\otimes 2}$

is an identifiable Waring subspace of dimension 6

Waring subspaces over finite fields

$$\mathscr{A} = \mathbb{V}_{3,2}(\mathbb{F}_q) \subseteq \mathbb{P}^9(\mathbb{F}_q) = \mathbb{P}(\mathrm{Sym}^2(\mathbb{F}_q^4))$$

$\eta_i(\mathscr{A}) \geq 1$ for any $1 \leq i \leq 6$

Theorem (Lavrauw-FZ)
Let $\omega \in \mathbb{F}_q \setminus \{0, \pm 1\}$, then
$\mathscr{S} = \langle \mathbf{e}_1^{\otimes 2}, \mathbf{e}_2^{\otimes 2}, \mathbf{e}_3^{\otimes 2}, \mathbf{e}_4^{\otimes 2}, (\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4)^{\otimes 2}, \mathbf{v}, \mathbf{w} \rangle$
with $\mathbf{v} = (\mathbf{e}_1 + \omega \mathbf{e}_2 + \omega \mathbf{e}_3 + \omega^2 \mathbf{e}_4)^{\otimes 2}$ and $\mathbf{w} = (\mathbf{e}_1 + \omega \mathbf{e}_2 + \mathbf{e}_3 + \omega \mathbf{e}_4)^{\otimes 2}$
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Theorem (Lavrauw-FZ)
Let $\omega \in \mathbb{F}_q \setminus \{0,1,2\}$ a square in $\mathbb{F}_q$, then
$\mathscr{S} = \langle \mathbf{e}_1^{\otimes 2}, \mathbf{e}_2^{\otimes 2}, \mathbf{e}_3^{\otimes 2}, \mathbf{e}_4^{\otimes 2}, \mathbf{e}^{\otimes 2}, (\omega \mathbf{e}_1 + \mathbf{e}_2 + \omega \mathbf{e}_3)^{\otimes 2}, (\sqrt{\omega}\mathbf{e}_1 + \sqrt{\omega}\mathbf{e}_2 + \sqrt{\omega^3}\mathbf{e}_4)^{\otimes 2} \rangle$
with $\mathbf{e} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4)^{\otimes 2}$
is an identifiable Waring subspace of dimension 6

These two examples are inequivalent!

$\Downarrow$

$\eta_6(\mathscr{A}) \geq 2$

Waring subspaces over finite fields

$$\mathscr{A} = \mathbb{V}_{3,2}(\mathbb{F}_q) \subseteq \mathbb{P}^9(\mathbb{F}_q) = \mathbb{P}(\mathrm{Sym}^2(\mathbb{F}_q^4))$$

Theorem (Lavrauw-FZ)

Let $\omega \in \mathbb{F}_q \setminus \{0, \pm 1\}$, then

$$\mathcal{S} = \langle \mathbf{e}_1^{\otimes 2}, \mathbf{e}_2^{\otimes 2}, \mathbf{e}_3^{\otimes 2}, \mathbf{e}_4^{\otimes 2}, (\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4)^{\otimes 2}, \mathbf{v}, \mathbf{w}, (\mathbf{e}_1 + \mathbf{e}_3 + \omega^2 \mathbf{e}_4)^{\otimes 2} \rangle$$

with $\mathbf{v} = (\mathbf{e}_1 + \omega \mathbf{e}_2 + \omega \mathbf{e}_3 + \omega^2 \mathbf{e}_4)^{\otimes 2}$ and $\mathbf{w} = (\mathbf{e}_1 + \omega \mathbf{e}_2 + \mathbf{e}_3 + \omega \mathbf{e}_4)^{\otimes 2}$

is an identifiable Waring subspace of dimension 7

$\Updownarrow$

* $q \in \{4,5,7,8,9\}$ and ω primitive element
* $q = 11$ and $\omega \in \{7,8\}$
* $q = 13$ and $\omega \in \{2,7\}$

Waring subspaces over finite fields

$$\mathscr{A} = \mathbb{V}_{3,2}(\mathbb{F}_q) \subseteq \mathbb{P}^9(\mathbb{F}_q) = \mathbb{P}(\mathrm{Sym}^2(\mathbb{F}_q^4))$$

Idea

$$\mathscr{S} = \langle \mathbf{e}_1^{\otimes 2}, \mathbf{e}_2^{\otimes 2}, \mathbf{e}_3^{\otimes 2}, \mathbf{e}_4^{\otimes 2}, (\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4)^{\otimes 2}, \mathbf{v}, \mathbf{w}, (\mathbf{e}_1 + \mathbf{e}_3 + \omega^2 \mathbf{e}_4)^{\otimes 2} \rangle$$

$$\begin{pmatrix} a+f+g+h & \omega f+\omega g & \omega f+g+h & \omega^2 f+\omega g+\omega^2 h \\ \omega f+\omega g & b+e+\omega^2 f+\omega^2 g & e+\omega^2 f+\omega g & e+\omega^3 f+\omega^2 g \\ \omega f+g+h & e+\omega^2 f+\omega g & c+e+\omega^2 f+g+h & e+\omega^3 f+\omega g+\omega^2 h \\ \omega^2 f+\omega g+\omega^2 h & e+\omega^3 f+\omega^2 g & e+\omega^3 f+\omega g+\omega^2 h & d+e+\omega^4 f+\omega^2 g+\omega^4 h \end{pmatrix}$$

with $(a,b,c,d,e,f,g,h) \in \mathbb{F}_q^8$

Waring subspaces over finite fields

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Idea

$\mathscr{S} = \langle \mathbf{e}_1^{\otimes 2}, \mathbf{e}_2^{\otimes 2}, \mathbf{e}_3^{\otimes 2}, \mathbf{e}_4^{\otimes 2}, (\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4)^{\otimes 2}, \mathbf{v}, \mathbf{w}, (\mathbf{e}_1 + \mathbf{e}_3 + \omega^2 \mathbf{e}_4)^{\otimes 2} \rangle$

is an identifiable Waring subspace $\Leftrightarrow$

$\mathscr{C}: (\omega - 1)^2 X^2 Y + \omega Y + (\omega - 1)^2 \omega X Y^2 + \omega^2 X + (\omega - 1)^2 (\omega + 1) XY = 0$

has no $\mathbb{F}_q$-rational points (x, y) such that

$$x \neq y,\ x \neq \frac{\omega}{\omega - 1},\ x \neq \frac{1}{\omega - 1},\ 1 + x + \omega y \neq 0$$

Waring subspaces over finite fields

$$\mathscr{A} = \mathbb{V}_{3,2}(\mathbb{F}_q) \subseteq \mathbb{P}^9(\mathbb{F}_q) = \mathbb{P}(\mathrm{Sym}^2(\mathbb{F}_q^4))$$

Idea

$\mathscr{S} = \langle \mathbf{e}_1^{\otimes 2}, \mathbf{e}_2^{\otimes 2}, \mathbf{e}_3^{\otimes 2}, \mathbf{e}_4^{\otimes 2}, (\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4)^{\otimes 2}, \mathbf{v}, \mathbf{w}, (\mathbf{e}_1 + \mathbf{e}_3 + \omega^2 \mathbf{e}_4)^{\otimes 2} \rangle$

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Hasse-Weil bound

Orange conditions hold!

Waring subspaces over finite fields

$$\mathscr{A} = \mathbb{V}_{3,2}(\mathbb{F}_q) \subseteq \mathbb{P}^9(\mathbb{F}_q) = \mathbb{P}(\mathrm{Sym}^2(\mathbb{F}_q^4))$$

Identifiable Waring subspaces of dimension 7

$\Updownarrow$

Pencils of quadrics whose base consist of 8 points not contained in any net of quadrics of $\mathbb{P}^3$

Waring subspaces over finite fields

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Identifiable Waring subspaces of dimension 7

$\Updownarrow$

Pencils of quadrics whose base consist of 8 points not contained in any net of quadrics of $\mathbb{P}^3$

A.A. Bruen and J.W.P. Hirschfeld: Intersections in projective space II: pencils of quadrics, Europ. J. Combinatorics 9 (1988), 255-270.

Classification of non-degenerate pencils of quadrics

Waring subspaces over finite fields

$$\mathscr{A} = \mathbb{V}_{3,2}(\mathbb{F}_q) \subseteq \mathbb{P}^9(\mathbb{F}_q) = \mathbb{P}(\mathrm{Sym}^2(\mathbb{F}_q^4))$$

Theorem (Lavrauw-FZ)

$\eta_7(\mathscr{A}) = 0$ if $q \geq 53$

If $q \leq 13$ an identifiable Waring subspace of dimension 7 is spanned by the images under ν_2 of the $\mathbb{F}_q$-rational points of:

* Four distinct lines and $q = 2$
* Two skew lines over $\mathbb{F}_q$ and two conjugate lines over $\mathbb{F}_{q^2}$ and $q = 3$
* Two disjoint conics over $\mathbb{F}_q$ and $q = 3$
* Two conics intersecting in two points and $q = 4$
* An elliptic curve for $q \in \{4,5,7,8,9,11,13\}$
* Some rational curves and $q = 7$

Waring subspaces over finite fields

$$\mathscr{A} = \mathbb{V}_{3,2}(\mathbb{F}_q) \subseteq \mathbb{P}^9(\mathbb{F}_q) = \mathbb{P}(\mathrm{Sym}^2(\mathbb{F}_q^4))$$

Corollary (Lavrauw-FZ)

* $\eta_7(\mathscr{A}) = 0$ if $q \geq 53$
* $\eta_7(\mathscr{A}) = 1$ if $q = 2$
* $\eta_7(\mathscr{A}) = 3$ if $q = 3$
* $\eta_7(\mathscr{A}) \geq 2$ if $q = 4$
* $\eta_7(\mathscr{A}) \geq 1$ if $q \in \{5,7,8,9,11,13\}$

Waring subspaces over finite fields

$$\mathscr{A} = \mathbb{V}_{3,2}(\mathbb{F}_q) \subseteq \mathbb{P}^9(\mathbb{F}_q) = \mathbb{P}(\mathrm{Sym}^2(\mathbb{F}_q^4))$$

Corollary (Lavrauw-FZ)

* $\eta_7(\mathscr{A}) = 0$ if $q \geq 53$
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* $\eta_7(\mathscr{A}) = 3$ if $q = 3$
* $\eta_7(\mathscr{A}) \geq 2$ if $q = 4$
* $\eta_7(\mathscr{A}) \geq 1$ if $q \in \{5,7,8,9,11,13\}$

What does it happen when $16 \leq q \leq 49$?

Waring subspaces over finite fields

$$\mathscr{A} = \mathbb{V}_{3,2}(\mathbb{F}_q) \subseteq \mathbb{P}^9(\mathbb{F}_q) = \mathbb{P}(\mathrm{Sym}^2(\mathbb{F}_q^4))$$

Identifiable Waring subspaces of dimension 8

$\Updownarrow$

Quadrics of $\mathbb{P}^3$ having 9 $\mathbb{F}_q$-rational points whose images (via ν_2) span a hyperplane of $\mathbb{P}^9$

Waring subspaces over finite fields

$$\mathscr{A} = \mathbb{V}_{3,2}(\mathbb{F}_q) \subseteq \mathbb{P}^9(\mathbb{F}_q) = \mathbb{P}(\mathrm{Sym}^2(\mathbb{F}_q^4))$$

Identifiable Waring subspaces of dimension 8

$\Updownarrow$

Quadrics of $\mathbb{P}^3$ having 9 $\mathbb{F}_q$-rational points whose images (via ν_2) span a hyperplane of $\mathbb{P}^9$

Theorem (Lavrauw-FZ)

$\eta_8(\mathscr{A}) = 0$ if $q \geq 3$

$\eta_8(\mathscr{A}) = 1$ if $q = 2$ (images under ν_2 of $\mathbb{F}_q$-rational points of a hyperbolic quadric)

Thank you for your attention!