Subcovers of generalized GK curves and their automorphism groups

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(joint work with Maria Montanucci and Guilherme Tizziotti)

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Outline

1. GK curve and generalizations
2. subcovers of the first generalized GK curve
3. their automorphism groups
4. subcovers of the second generalized GK curve
5. their automorphism groups
6. a characterization of the GK curve

Algebraic curves over finite fields

- $\mathcal{X} \subset \mathrm{PG}(r, \overline{\mathbb{F}_q})$

  projective, **absolutely irreducible**, non-singular algebraic curve

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Hasse-Weil bound:

$$q + 1 - 2g\sqrt{q} \;\leq\; |\mathcal{X}(\mathbb{F}_q)| \;\leq\; q + 1 + 2g\sqrt{q}$$

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- $\mathcal{P} = \mathcal{H}_{\sqrt{q}}(\mathbb{F}_q)$

  $\mathcal{L} = \{\mathcal{H}_{\sqrt{q}} \cap \ell \ : \ \ell$ is a $(\sqrt{q}+1)$-secant $\mathbb{F}_q$-rational line$\}$

  $\Longrightarrow$ **classical unital** $(\mathcal{P}, \mathcal{L})$

Maximal curves from subcovers

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Non-constant rational map:

$$\varphi : \mathcal{X} \to \mathcal{Y}, \quad \begin{cases} y_1 = \frac{F_1(x_1,\ldots,x_r)}{G_1(x_1,\ldots,x_r)} \\ \ldots \\ y_s = \frac{F_s(x_1,\ldots,x_r)}{G_s(x_1,\ldots,x_r)} \end{cases} \qquad F_i, G_j \in \mathbb{F}_q[x_1, \ldots, x_r]$$

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Theorem

*if $\mathcal{X}$ is $\mathbb{F}_q$-**maximal** and $\mathcal{Y}$ is an $\mathbb{F}_q$-subcover of $\mathcal{X}$ $\Longrightarrow$ $\mathcal{Y}$ is $\mathbb{F}_q$-**maximal***

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maximal curve $\mathcal{X}$ with $\mathrm{Aut}_{\mathbb{F}_q}(\mathcal{X})$ rich $\implies$ many maximal curves $\mathcal{X}/G$

Example:

$\mathbb{F}_q$-max. Hermitian curve $\mathcal{H}_{\sqrt{q}}$,
$\mathrm{Aut}(\mathcal{H}_{\sqrt{q}}) = \mathrm{Aut}_{\mathbb{F}_q}(\mathcal{H}_{\sqrt{q}}) \cong \mathrm{PGU}(3, \sqrt{q})$

GK curve

Giulietti-Korchmáros 2009:

$$\mathcal{GK}: \begin{cases} z^m = y^{q^2} - y \\ y^{q+1} = x^q + x \end{cases} \qquad m = \frac{q^3+1}{q+1}$$

- $\mathcal{GK}$ is $\mathbb{F}_{q^6}$-maximal
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- $\mathrm{Aut}(\mathcal{GK}) = \mathrm{PGU}(3,q) \cdot C_m$, contains $\mathrm{PGU}(3,q) \times C_{m/\mathrm{gcd}(3,m)}$

first generalized GK curve: GGS

Garcia-Güneri-Stichtenoth 2010:

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for $n \geq 5$: $\mathrm{Aut}(\mathcal{GGS}_n) = \mathrm{PGU}(3,q)_{P_\infty} \cdot C_m = S_{q^3} \rtimes C_{(q^2-1)m}$ fixes P_∞

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$$\mathrm{PGU}(3,q)_{P_\infty} = S_{q^3} \rtimes C_{q^2-1} = \{(x,y,z) \mapsto (a^{q+1}x + ab^q y + c,\ ay + b,\ z) \mid a,b,c \in \mathbb{F}_{q^2}, a \neq 0, c^q + c = b^{q+1}\}$$

$$C_m = \{(x,y,z) \mapsto (x,y,\lambda z) \mid \lambda^m = 1\}$$

Subcovers $\mathcal{Y}_{n,s}$ of the GGS curve

$n \geq 3$ odd, $m = \frac{q^n+1}{q+1}$, s divisor of m

Tafazolian, Teherán-Herrera, Torres (2016):

$$\mathcal{Y}_{n,s} : \begin{cases} z^{m/s} = y^{q^2} - y \\ y^{q+1} = x^q + x \end{cases}$$

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- $\mathcal{Y}_{n,s} = \mathcal{GGS}_n / C_s, \quad C_s = \{(x,y,z) \mapsto (x,y,\lambda z) \mid \lambda^s = 1\}$

$$\Rightarrow \mathcal{Y}_{n,s} \text{ is } \mathbb{F}_{q^{2n}}\text{-maximal}$$

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  $\Rightarrow \mathcal{Y}_{n,s}$ is $\mathbb{F}_{q^{2n}}$-maximal
- if $n = 3$ and $s(s+1) < q \implies \mathcal{Y}_{3,s}$ is not covered by $\mathcal{H}_{q^n}$

  Technique: find a contradiction to

$$\frac{|\mathcal{H}_{q^n}(\mathbb{F}_{q^{2n}})|}{|\mathcal{Y}_{n,s}(\mathbb{F}_{q^{2n}})|} \leq \deg(\varphi) \leq \frac{2g(\mathcal{H}_{q^n}) - 2}{2g(\mathcal{Y}_{n,s}) - 2}, \qquad \varphi : \mathcal{H}_{q^n} \rightarrow \mathcal{Y}_{n,s}$$

Automorphism group of $\mathcal{Y}_{n,s}$

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- Güneri-Özdemir-Stichtenoth:
  - determine the Weierstrass semigroup $H(P_\infty) = \langle\, q^3, qm, (q+1)m \,\rangle$
  - show $H(Q) \neq H(P_\infty)$ for all $Q \in \mathcal{GGS}_n(\mathbb{F}_{q^{2n}})$
- Malmskog-Guralnick-Pries:
  - structural results on groups with TI p-subgroups
  - use that $m = \frac{q^n+1}{q+1} >> q$

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Theorem (Montanucci-Tizziotti-Z.)

- *If* $3 \nmid n$ *or* $\frac{m}{s} \nmid \frac{q^3+1}{q+1} \implies \mathrm{Aut}(\mathcal{Y}_{n,s}) = S_{q^3} \rtimes C_{(q^2-1)\frac{m}{s}}$ *fixes* P_∞
- *If* $3 \mid n$ *and* $\frac{m}{s} \mid \frac{q^3+1}{q+1} \implies \mathcal{Y}_{n,s} \cong \mathcal{GK}/C_{\frac{q^2-q+1}{m/s}}$ *and*

  $\mathrm{Aut}(\mathcal{Y}_{n,s}) = \langle \, \mathrm{PGU}(3,q) \, , \, C_{m/s} \, \rangle$ *, of order* $(q^3+1)q^3(q^2-1)\frac{m}{s}$

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Notice: if $3 \mid n \implies \mathbb{F}_{q^{2n}} = \mathbb{F}_{q^{6d}}$ with d odd

$\implies$ the $\mathbb{F}_{q^6}$-maximal curve $\mathcal{GK}$ is also $\mathbb{F}_{q^{2n}}$-maximal

Automorphism group of $\mathcal{Y}_{n,s}$: steps

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$$\mathcal{O} := \left\{ P = (a, b, 0) \;\middle|\; a, b \in \mathbb{F}_{q^2}, b^{q+1} = a^q + a \right\}$$

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- Prove: $\mathrm{Aut}(\mathcal{Y}_{n,s})_{P_\infty} = S_{q^3} \rtimes C_{(q^2-1)\frac{m}{s}}$

Automorphism group of $\mathcal{Y}_{n,s}$: steps

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- Prove: $\frac{\mathrm{Aut}(\mathcal{Y})_{n,s}}{C_{m/s}} \cong \mathrm{PGU}(3,q) \iff 3 \mid n$ and $\frac{m}{s} \mid \frac{q^3+1}{q+1}$

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- $\frac{m}{s} \lesssim q^3$: use the structure of
  - **short orbits** of $\mathrm{Aut}(\mathcal{Y}_{n,s})$ when $|\mathrm{Aut}(\mathcal{Y}_{n,s})| > 84(g-1)$
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- lift of τ + fundamental equation $\rightarrow$ element of $H(P_\infty)$
- $H(P_\infty)$ is known (Tafazolian, Teherán-Herrera, Torres)

Subcovers $\mathcal{X}_{a,b,n,s}$ of the GGS curve

$n \geq 3$ odd, $m = \frac{q^n+1}{q+1}$, $s \mid m$, $q = p^a$, $\bar{q} = p^b$ with $b \mid a$, $c^{q-1} = -1$

Tafazolian, Teherán-Herrera, Torres (2016):

$$\mathcal{X}_{a,b,n,s} : \begin{cases} z^{m/s} = y^{q^2} - y \\ cy^{q+1} = x + x^{\bar{q}} + \cdots + x^{q/\bar{q}} \end{cases}$$

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- $\mathrm{Aut}(\mathcal{X}_{a,b,n,s})$?

Automorphism group of $\mathcal{X}_{a,b,n,s}$

$$\mathcal{Y}_{n,s} : \begin{cases} w^{m/s} = v^{q^2} - v \\ v^{q+1} = u^q + u \end{cases} \qquad \mathcal{X}_{a,b,n,s} : \begin{cases} z^{m/s} = y^{q^2} - y \\ cy^{q+1} = x + x^{\bar{q}} + \cdots + x^{q/\bar{q}} \end{cases}$$

- $\mathcal{X}_{a,b,n,s} = \mathcal{Y}_{n,s}/E_{\bar{q}}$ with $E_{\bar{q}} = \left\{(u,v,w) \mapsto (u + \frac{\alpha}{c}, v, w) \mid \alpha \in \mathbb{F}_{\bar{q}}\right\}$

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- $C_{m/s} = \left\{(x,y,z) \mapsto (x,y,\lambda z) \mid \lambda^{m/s} = 1\right\} \leq \mathrm{Aut}(\mathcal{X}_{a,b,n,s})$

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Theorem (Montanucci-Tizziotti-Z.)

$$\mathrm{Aut}(\mathcal{X}_{a,b,n,s}) \cong \frac{S_{q^3}}{E_{\bar{q}}} \rtimes C_{(q+1)(\bar{q}-1)\frac{m}{s}}$$

Second generalized GK curve: BM

Beelen-Montanucci 2018:

$$\mathcal{BM}_n : \begin{cases} z^m = y\frac{x^{q^2}-x}{x^{q+1}-1} \\ y^{q+1} = x^{q+1} - 1 \end{cases} \qquad m = \frac{q^n+1}{q+1}, \quad n \geq 3 \text{ odd}$$

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- for $n \geq 5$:

$$\mathrm{Aut}(\mathcal{BM}_n) = \langle\, \mathrm{PGU}(3,q)_\ell\,,\ C_{m/s}\,\rangle \cong \mathrm{SL}(2,q) \rtimes C_{q^n+1}$$

$\mathrm{Aut}(\mathcal{BM}_n)$ is the lift of the stabilizer $\mathrm{PGU}(3,q)_\ell$

of a $(q+1)$-secant ℓ to $\tilde{\mathcal{H}}_q :\ y^{q+1} = x^{q+1} - 1$

Subcovers $\tilde{\mathcal{Y}}_{n,s}$ of the BM curve

$$\tilde{\mathcal{Y}}_{n,s} : \begin{cases} z^{m/s} = y\,\frac{x^{q^2}-x}{x^{q+1}-1} \\ y^{q+1} = x^{q+1} - 1 \end{cases} \qquad m = \frac{q^n+1}{q+1}, \quad s \mid m$$

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Theorem (Montanucci-Tizziotti-Z.)

- *If* $3 \nmid n$ *or* $\frac{m}{s} \nmid \frac{q^3+1}{q+1} \implies \mathrm{Aut}(\tilde{\mathcal{Y}}_{n,s}) \cong \mathrm{SL}(2,q) \rtimes C_{(q^n+1)/s}$
- *If* $3 \mid n$ *and* $\frac{m}{s} \mid \frac{q^3+1}{q+1} \implies \mathcal{Y}_{n,s} \cong \mathcal{GK}/C_{\frac{q^2-q+1}{m/s}}$ *and*

$\mathrm{Aut}(\tilde{\mathcal{Y}}_{n,s}) = \langle \mathrm{PGU}(3,q), C_{m/s} \rangle$, *of order* $(q^3+1)q^3(q^2-1)\frac{m}{s}$

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In the first case: $g(\tilde{\mathcal{Y}}_{n,s}) = g(\mathcal{Y}_{n,s})$ but $\tilde{\mathcal{Y}}_{n,s} \not\cong \mathcal{Y}_{n,s}$

$\implies$ new $\mathbb{F}_{q^{2n}}$-maximal curves not covered by $\mathcal{H}_{q^n}$

Subcovers $\tilde{\mathcal{X}}_{a,b,n,s}$ of the BM curve

$n \geq 3$ odd, $m = \frac{q^n+1}{q+1}$, $s \mid m$, $\quad q = p^a$, $\bar{q} = p^b$, $b \mid a$

$$\tilde{\mathcal{Y}}_{n,s} : \begin{cases} z^{m/s} = y \frac{x^{q^2}-x}{x^{q+1}-1} \\ y^{q+1} = x^{q+1} - 1 \end{cases} \qquad \tilde{\mathcal{X}}_{a,b,n,s} := \tilde{\mathcal{Y}}_{n,s}/E_{\bar{q}}$$

where $E_{\bar{q}} \leq \mathrm{Aut}(\tilde{\mathcal{Y}}_{n,s})$ is the lift to $\tilde{\mathcal{Y}}_{n,s}$ with $E_{\bar{q}}(z) = z$
of an elementary abelian group of elations
fixing a point $P \in \tilde{\mathcal{H}}_q(\mathbb{F}_{q^2})$, $\tilde{\mathcal{H}}_q : \; y^{q+1} = x^{q+1} - 1$

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$n \geq 3$ odd, $m = \frac{q^n+1}{q+1}$, $s \mid m$, $q = p^a$, $\bar{q} = p^b$, $b \mid a$

$$\tilde{\mathcal{Y}}_{n,s} : \begin{cases} z^{m/s} = y\,\frac{x^{q^2}-x}{x^{q+1}-1} \\ y^{q+1} = x^{q+1} - 1 \end{cases} \qquad \tilde{\mathcal{X}}_{a,b,n,s} := \tilde{\mathcal{Y}}_{n,s}/E_{\bar{q}}$$

where $E_{\bar{q}} \leq \mathrm{Aut}(\tilde{\mathcal{Y}}_{n,s})$ is the lift to $\tilde{\mathcal{Y}}_{n,s}$ with $E_{\bar{q}}(z) = z$ of an elementary abelian group of elations fixing a point $P \in \tilde{\mathcal{H}}_q(\mathbb{F}_{q^2})$, $\tilde{\mathcal{H}}_q : \ y^{q+1} = x^{q+1} - 1$

Theorem (Montanucci-Tizziotti-Z.)

$$\mathrm{Aut}(\tilde{\mathcal{X}}_{a,b,n,s}) \;=\; \mathrm{N}_{\mathrm{Aut}(\tilde{\mathcal{Y}}_{n,s})}(E_{\bar{q}})/E_{\bar{q}} \;\cong\; (E_q/E_{\bar{q}}) \rtimes C_{(q+1)(\bar{q}-1)\frac{m}{s}}$$

Subcovers $\tilde{\mathcal{X}}_{a,b,n,s}$ of the BM curve

$n \geq 3$ odd, $m = \frac{q^n+1}{q+1}$, $s \mid m$, $q = p^a$, $\bar{q} = p^b$, $b \mid a$

$$\tilde{\mathcal{Y}}_{n,s} : \begin{cases} z^{m/s} = y\,\frac{x^{q^2}-x}{x^{q+1}-1} \\ y^{q+1} = x^{q+1} - 1 \end{cases} \qquad \tilde{\mathcal{X}}_{a,b,n,s} := \tilde{\mathcal{Y}}_{n,s}/E_{\bar{q}}$$

where $E_{\bar{q}} \leq \mathrm{Aut}(\tilde{\mathcal{Y}}_{n,s})$ is the lift to $\tilde{\mathcal{Y}}_{n,s}$ with $E_{\bar{q}}(z) = z$
of an elementary abelian group of elations
fixing a point $P \in \tilde{\mathcal{H}}_q(\mathbb{F}_{q^2})$, $\tilde{\mathcal{H}}_q : \; y^{q+1} = x^{q+1} - 1$

Theorem (Montanucci-Tizziotti-Z.)

$$\mathrm{Aut}(\tilde{\mathcal{X}}_{a,b,n,s}) \;=\; \mathrm{N}_{\mathrm{Aut}(\tilde{\mathcal{Y}}_{n,s})}(E_{\bar{q}})/E_{\bar{q}} \;\cong\; (E_q/E_{\bar{q}}) \rtimes C_{(q+1)(\bar{q}-1)\frac{m}{s}}$$

$g(\tilde{\mathcal{X}}_{a,b,n,s}) = g(\mathcal{X}_{a,b,n,s})$, $\tilde{\mathcal{X}}_{a,b,n,s} \not\cong \mathcal{X}_{a,b,n,s}$

$\Longrightarrow$ new $\mathbb{F}_{q^{2n}}$-maximal curves not covered by $\mathcal{H}_{q^n}$

Conclusion

- Subcovers $\mathcal{Y}_{n,s}$, $\mathcal{X}_{a,b,n,s}$, $\tilde{\mathcal{Y}}_{n,s}$, $\tilde{\mathcal{X}}_{a,b,n,s}$
  of the first (GGS) and second (BM) generalized GK curve
- their automorphism groups
- new maximal curves not covered by the Hermitian curve
- a characterization of the curves

$$\mathcal{GK}/C_s \in \{ \mathcal{Y}_{n,s} , \mathcal{X}_{a,b,n,s} , \tilde{\mathcal{Y}}_{n,s} , \tilde{\mathcal{X}}_{a,b,n,s} \}$$

by

$$\mathrm{PGU}(3,q) \leq \mathrm{Aut}(\mathcal{GK}/C_s)$$

Thank you for your attention!

Guten Appetit!