A standard form for scattered linearized polynomials and properties of the related translation planes

Corrado Zanella

Department of Management and Engineering

joint work with Giovanni Longobardi

UNIVERSITÀ DEGLI STUDI DI PADOVA

Scattered polynomials

Università degli Studi di Padova

Scattered polynomials

Università degli Studi di Padova

Definition.

A *scattered polynomial* in $\mathbb{F}_{q^n}[x]$ is an $\mathbb{F}_q$-linearized polynomial $f(x) = \sum_i a_i x^{q^i}$ such that

$$\frac{f(y)}{y} = \frac{f(z)}{z},\ y, z \in \mathbb{F}_{q^n}^* \Rightarrow \langle y \rangle_{\mathbb{F}_q} = \langle z \rangle_{\mathbb{F}_q}.$$

Scattered polynomials

Università degli Studi di Padova

Definition.

A *scattered polynomial* in $\mathbb{F}_{q^n}[x]$ is an $\mathbb{F}_q$-linearized polynomial $f(x) = \sum_i a_i x^{q^i}$ such that

$$\frac{f(y)}{y} = \frac{f(z)}{z},\ y, z \in \mathbb{F}_{q^n}^* \ \Rightarrow\ \langle y \rangle_{\mathbb{F}_q} = \langle z \rangle_{\mathbb{F}_q}.$$

Let $f(x) \in \mathbb{F}_{q^n}[x]$ be scattered. Then

Scattered polynomials

Definition.

A *scattered polynomial* in $\mathbb{F}_{q^n}[x]$ is an $\mathbb{F}_q$-linearized polynomial $f(x)=\sum_i a_i x^{q^i}$ such that

$$\frac{f(y)}{y}=\frac{f(z)}{z},\ y,z\in\mathbb{F}_{q^n}^* \ \Rightarrow\ \langle y\rangle_{\mathbb{F}_q}=\langle z\rangle_{\mathbb{F}_q}.$$

Let $f(x)\in\mathbb{F}_{q^n}[x]$ be scattered. Then

- $U_f=\{(y,f(y))\colon\ y\in\mathbb{F}_{q^n}\}$ is a scattered $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^n}^2$ with respect to the Desarguesian spread $\mathcal{D}=\{\langle v\rangle_{\mathbb{F}_{q^n}}:\ v\in\mathbb{F}_{q^n}^2,\ v\neq 0\}$; that is, $\dim_{\mathbb{F}_q}(U_f\cap\langle v\rangle_{\mathbb{F}_{q^n}})\leq 1$ for any $v\in\mathbb{F}_{q^n}^2$.

Scattered polynomials

Università degli Studi di Padova

Definition.

A *scattered polynomial* in $\mathbb{F}_{q^n}[x]$ is an $\mathbb{F}_q$-linearized polynomial $f(x) = \sum_i a_i x^{q^i}$ such that

$$\frac{f(y)}{y} = \frac{f(z)}{z},\ y, z \in \mathbb{F}^*_{q^n} \Rightarrow \langle y \rangle_{\mathbb{F}_q} = \langle z \rangle_{\mathbb{F}_q}.$$

Let $f(x) \in \mathbb{F}_{q^n}[x]$ be scattered. Then

- $U_f = \{(y, f(y)) \colon y \in \mathbb{F}_{q^n}\}$ is a scattered $\mathbb{F}_q$-subspace of $\mathbb{F}^2_{q^n}$ with respect to the Desarguesian spread $\mathcal{D} = \{\langle v \rangle_{\mathbb{F}_{q^n}} : v \in \mathbb{F}^2_{q^n},\ v \neq 0\}$; that is, $\dim_{\mathbb{F}_q}(U_f \cap \langle v \rangle_{\mathbb{F}_{q^n}}) \leq 1$ for any $v \in \mathbb{F}^2_{q^n}$.
- $L_f = \{\langle (y, f(y)) \rangle_{\mathbb{F}_{q^n}} : y \in \mathbb{F}^*_{q^n}\}$ is a scattered linear set of rank n in $\mathrm{PG}(1, q^n)$; $|L_f| = (q^n - 1)/(q - 1)$.

Scattered polynomials

Definition.

A *scattered polynomial* in $\mathbb{F}_{q^n}[x]$ is an $\mathbb{F}_q$-linearized polynomial $f(x) = \sum_i a_i x^{q^i}$ such that

$$\frac{f(y)}{y} = \frac{f(z)}{z},\ y, z \in \mathbb{F}^*_{q^n} \ \Rightarrow\ \langle y \rangle_{\mathbb{F}_q} = \langle z \rangle_{\mathbb{F}_q}.$$

Let $f(x) \in \mathbb{F}_{q^n}[x]$ be scattered. Then

- $U_f = \{(y, f(y)) \colon y \in \mathbb{F}_{q^n}\}$ is a scattered $\mathbb{F}_q$-subspace of $\mathbb{F}^2_{q^n}$ with respect to the Desarguesian spread $\mathcal{D} = \{\langle v \rangle_{\mathbb{F}_{q^n}} : v \in \mathbb{F}^2_{q^n},\ v \neq 0\}$; that is, $\dim_{\mathbb{F}_q}(U_f \cap \langle v \rangle_{\mathbb{F}_{q^n}}) \leq 1$ for any $v \in \mathbb{F}^2_{q^n}$.
- $L_f = \{\langle (y, f(y)) \rangle_{\mathbb{F}_{q^n}} : y \in \mathbb{F}^*_{q^n}\}$ is a scattered linear set of rank n in $\mathrm{PG}(1, q^n)$; $|L_f| = (q^n - 1)/(q - 1)$.

Definition.

If U_f and U_g belong to the same orbit under the action of $\mathrm{GL}(2, q^n)$ [resp. $\Gamma\mathrm{L}(2, q^n)$], then f and g are GL-*equivalent* [resp. ΓL-*equivalent*].

Scattered polynomials

Definition.

A *scattered polynomial* in $\mathbb{F}_{q^n}[x]$ is an $\mathbb{F}_q$-linearized polynomial $f(x) = \sum_i a_i x^{q^i}$ such that

$$\frac{f(y)}{y} = \frac{f(z)}{z},\ y, z \in \mathbb{F}_{q^n}^* \Rightarrow \langle y \rangle_{\mathbb{F}_q} = \langle z \rangle_{\mathbb{F}_q}.$$

Let $f(x) \in \mathbb{F}_{q^n}[x]$ be scattered. Then

- $U_f = \{(y, f(y)) : y \in \mathbb{F}_{q^n}\}$ is a scattered $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^n}^2$ with respect to the Desarguesian spread $\mathcal{D} = \{\langle v \rangle_{\mathbb{F}_{q^n}} : v \in \mathbb{F}_{q^n}^2,\ v \neq 0\}$; that is, $\dim_{\mathbb{F}_q}(U_f \cap \langle v \rangle_{\mathbb{F}_{q^n}}) \leq 1$ for any $v \in \mathbb{F}_{q^n}^2$.
- $L_f = \{\langle (y, f(y)) \rangle_{\mathbb{F}_{q^n}} : y \in \mathbb{F}_{q^n}^*\}$ is a scattered linear set of rank n in $\mathrm{PG}(1, q^n)$; $|L_f| = (q^n - 1)/(q - 1)$.

Definition.

If U_f and U_g belong to the same orbit under the action of $\mathrm{GL}(2, q^n)$ [resp. $\Gamma\mathrm{L}(2, q^n)$], then f and g are GL-*equivalent* [resp. ΓL-*equivalent*].

Example.

$f(x) = x^q$ and $g(x) = ax + x^q$ are GL-equivalent scattered polynomials for any $a \in \mathbb{F}_{q^n}$.

Known scattered polynomials in $\mathbb{F}_{q^n}[x]$

- $f(x) = x^{q^s}$, $(s, n) = 1$ (pseudoregulus type) (Blokhuis-Lavrauw 2000)

Known scattered polynomials in $\mathbb{F}_{q^n}[x]$

- $f(x) = x^{q^s}$, $(s, n) = 1$ (pseudoregulus type) (Blokhuis-Lavrauw 2000)
- $f(x) = x^{q^s} + \delta x^{q^{n-s}}$, $n \geq 4$, $(s, n) = 1$, $\mathrm{N}_{q^n/q}(\delta) \neq 0, 1$ (Lunardon-Polverino 2001; Sheekey 2016)

- $f(x) = x^{q^s}$, $(s, n) = 1$ (pseudoregulus type) (Blokhuis-Lavrauw 2000)
- $f(x) = x^{q^s} + \delta x^{q^{n-s}}$, $n \geq 4$, $(s, n) = 1$, $\mathrm{N}_{q^n/q}(\delta) \neq 0, 1$ (Lunardon-Polverino 2001; Sheekey 2016)
- $f(x) = \delta x^{q^s} + x^{q^{s+n/2}}$, $n \in \{6, 8\}$, $(s, n/2) = 1$, some δ and q (Csajbók-Marino-Polverino-Z. 2018)

Known scattered polynomials in $\mathbb{F}_{q^n}[x]$

UNIVERSITÀ DEGLI STUDI DI PADOVA

- $f(x) = x^{q^s}$, $(s, n) = 1$ (pseudoregulus type) (Blokhuis-Lavrauw 2000)
- $f(x) = x^{q^s} + \delta x^{q^{n-s}}$, $n \geq 4$, $(s, n) = 1$, $\mathrm{N}_{q^n/q}(\delta) \neq 0, 1$ (Lunardon-Polverino 2001; Sheekey 2016)
- $f(x) = \delta x^{q^s} + x^{q^{s+n/2}}$, $n \in \{6, 8\}$, $(s, n/2) = 1$, some δ and q (Csajbók-Marino-Polverino-Z. 2018)
- $f(x) = x^q + x^{q^3} + \delta x^{q^5}$, $n = 6$, $\delta^2 + \delta = 1$, q odd (Csajbók, Marino, Montanucci, Zullo 2018, 2020)

Known scattered polynomials in $\mathbb{F}_{q^n}[x]$

UNIVERSITÀ DEGLI STUDI DI PADOVA

- $f(x) = x^{q^s}$, $(s, n) = 1$ (pseudoregulus type) (Blokhuis-Lavrauw 2000)
- $f(x) = x^{q^s} + \delta x^{q^{n-s}}$, $n \geq 4$, $(s, n) = 1$, $\mathrm{N}_{q^n/q}(\delta) \neq 0, 1$ (Lunardon-Polverino 2001; Sheekey 2016)
- $f(x) = \delta x^{q^s} + x^{q^{s+n/2}}$, $n \in \{6, 8\}$, $(s, n/2) = 1$, some δ and q (Csajbók-Marino-Polverino-Z. 2018)
- $f(x) = x^q + x^{q^3} + \delta x^{q^5}$, $n = 6$, $\delta^2 + \delta = 1$, q odd (Csajbók, Marino, Montanucci, Zullo 2018, 2020)
- $f(x) = x^{q^s} + x^{q^{s(t-1)}} + h^{1+q^s} x^{q^{s(t+1)}} + h^{1-q^{s(2t-1)}} x^{q^{s(2t-1)}}$, $n = 2t$, $t \geq 3$, $(s, n) = 1$, q odd, $\mathrm{N}_{q^n/q^t}(h) = -1$ (Bartoli, Longobardi, Marino, Neri, Santonastaso, Z., Zhou, Zullo 2020-202x)

Translation planes related to scattered polynomials

Translation planes related to scattered polynomials

Università degli Studi di Padova

Ingredients:

Translation planes related to scattered polynomials

Ingredients:

- $U_f = \{(y, f(y)) \colon y \in \mathbb{F}_{q^n}\}$

Translation planes related to scattered polynomials

Università degli Studi di Padova

Ingredients:

- $U_f = \{(y, f(y)) \colon y \in \mathbb{F}_{q^n}\}$
- $L_f = \{\langle (y, f(y)) \rangle_{\mathbb{F}_{q^n}} \colon y \in \mathbb{F}^*_{q^n}\}$

Translation planes related to scattered polynomials

Università degli Studi di Padova

Ingredients:

- $U_f = \{(y, f(y)) \colon y \in \mathbb{F}_{q^n}\}$
- $L_f = \{\langle (y, f(y)) \rangle_{\mathbb{F}_{q^n}} \colon y \in \mathbb{F}_{q^n}^*\}$ a subset of the Desarguesian spread
- $\mathcal{D} = \{\langle v \rangle_{\mathbb{F}_{q^n}} \colon v \in \mathbb{F}_{q^n}^2,\ v \neq 0\}$

Translation planes related to scattered polynomials

Ingredients:

- $U_f = \{(y, f(y)) \colon y \in \mathbb{F}_{q^n}\}$
- $L_f = \{\langle (y, f(y)) \rangle_{\mathbb{F}_{q^n}} \colon y \in \mathbb{F}^*_{q^n}\}$ a subset of the Desarguesian spread
- $\mathcal{D} = \{\langle v \rangle_{\mathbb{F}_{q^n}} \colon v \in \mathbb{F}^2_{q^n},\ v \neq 0\}$

If $f(x) \in \mathbb{F}_{q^n}[x]$ is scattered, then any two distinct subspaces hU_f, kU_f, $h, k \in \mathbb{F}^*_{q^n}$, intersect trivially $\left(\text{for: } h(y, f(y)) = k(z, f(z)) \Rightarrow \frac{f(y)}{y} = \frac{f(z)}{z}\right)$.
So, $\mathcal{B}_f = \left(\mathcal{D} \setminus L_f\right) \cup \{hU_f \colon h \in \mathbb{F}^*_{q^n}\}$ is a spread of $\mathbb{F}^2_{q^n}$ in n-dimensional $\mathbb{F}_q$-subspaces.

Translation planes related to scattered polynomials

Ingredients:

- $U_f = \{(y, f(y)) \colon y \in \mathbb{F}_{q^n}\}$
- $L_f = \{\langle (y, f(y)) \rangle_{\mathbb{F}_{q^n}} \colon y \in \mathbb{F}^*_{q^n}\}$ a subset of the Desarguesian spread
- $\mathcal{D} = \{\langle v \rangle_{\mathbb{F}_{q^n}} \colon v \in \mathbb{F}^2_{q^n},\ v \neq 0\}$

☞ If $f(x) \in \mathbb{F}_{q^n}[x]$ is scattered, then any two distinct subspaces hU_f, kU_f, $h, k \in \mathbb{F}^*_{q^n}$, intersect trivially $\left(\text{for: } h(y, f(y)) = k(z, f(z)) \Rightarrow \frac{f(y)}{y} = \frac{f(z)}{z}\right)$.
So, $\mathcal{B}_f = (\mathcal{D} \setminus L_f) \cup \{hU_f \colon h \in \mathbb{F}^*_{q^n}\}$ is a spread of $\mathbb{F}^2_{q^n}$ in n-dimensional $\mathbb{F}_q$-subspaces.

Definition.

$\mathcal{A}_f$ is the affine translation plane whose

Translation planes related to scattered polynomials

Ingredients:

- $U_f = \{(y, f(y)) \colon y \in \mathbb{F}_{q^n}\}$
- $L_f = \{\langle (y, f(y)) \rangle_{\mathbb{F}_{q^n}} \colon y \in \mathbb{F}_{q^n}^*\}$ a subset of the Desarguesian spread
- $\mathcal{D} = \{\langle v \rangle_{\mathbb{F}_{q^n}} \colon v \in \mathbb{F}_{q^n}^2,\ v \neq 0\}$

If $f(x) \in \mathbb{F}_{q^n}[x]$ is scattered, then any two distinct subspaces hU_f, kU_f, $h, k \in \mathbb{F}_{q^n}^*$, intersect trivially $\left(\text{for: } h(y, f(y)) = k(z, f(z)) \Rightarrow \frac{f(y)}{y} = \frac{f(z)}{z}\right)$.
So, $\mathcal{B}_f = (\mathcal{D} \setminus L_f) \cup \{hU_f \colon h \in \mathbb{F}_{q^n}^*\}$ is a spread of $\mathbb{F}_{q^n}^2$ in n-dimensional $\mathbb{F}_q$-subspaces.

Definition.

$\mathcal{A}_f$ is the affine translation plane whose

- *point set* is $\mathbb{F}_{q^n}^2$

Translation planes related to scattered polynomials

UNIVERSITÀ DEGLI STUDI DI PADOVA

Ingredients:

- $U_f = \{(y, f(y)) : y \in \mathbb{F}_{q^n}\}$
- $L_f = \{\langle (y, f(y)) \rangle_{\mathbb{F}_{q^n}} : y \in \mathbb{F}^*_{q^n}\}$ a subset of the Desarguesian spread
- $\mathcal{D} = \{\langle v \rangle_{\mathbb{F}_{q^n}} : v \in \mathbb{F}^2_{q^n},\ v \neq 0\}$

If $f(x) \in \mathbb{F}_{q^n}[x]$ is scattered, then any two distinct subspaces hU_f, kU_f, $h, k \in \mathbb{F}^*_{q^n}$, intersect trivially $\left(\text{for: } h(y, f(y)) = k(z, f(z)) \ \Rightarrow\ \frac{f(y)}{y} = \frac{f(z)}{z}\right)$.
So, $\mathcal{B}_f = (\mathcal{D} \setminus L_f) \cup \{hU_f : h \in \mathbb{F}^*_{q^n}\}$ is a spread of $\mathbb{F}^2_{q^n}$ in n-dimensional $\mathbb{F}_q$-subspaces.

DEFINITION.

$\mathcal{A}_f$ is the affine translation plane whose

- *point set* is $\mathbb{F}^2_{q^n}$
- *lines* are $v + U$, $v \in \mathbb{F}^2_{q^n}$, $U \in \mathcal{B}_f$

Translation planes related to scattered polynomials

Universitá degli Studi di Padova

Ingredients:

- $U_f = \{(y, f(y)) \colon y \in \mathbb{F}_{q^n}\}$
- $L_f = \{\langle (y, f(y)) \rangle_{\mathbb{F}_{q^n}} \colon y \in \mathbb{F}^*_{q^n}\}$ a subset of the Desarguesian spread
- $\mathcal{D} = \{\langle v \rangle_{\mathbb{F}_{q^n}} \colon v \in \mathbb{F}^2_{q^n},\ v \neq 0\}$

If $f(x) \in \mathbb{F}_{q^n}[x]$ is scattered, then any two distinct subspaces hU_f, kU_f, $h, k \in \mathbb{F}^*_{q^n}$, intersect trivially $\left(\text{for: } h(y, f(y)) = k(z, f(z)) \ \Rightarrow \ \frac{f(y)}{y} = \frac{f(z)}{z}\right)$.
So, $\mathcal{B}_f = \left(\mathcal{D} \setminus L_f\right) \cup \{hU_f \colon h \in \mathbb{F}^*_{q^n}\}$ is a spread of $\mathbb{F}^2_{q^n}$ in n-dimensional $\mathbb{F}_q$-subspaces.

Definition.

$\mathcal{A}_f$ is the affine translation plane whose

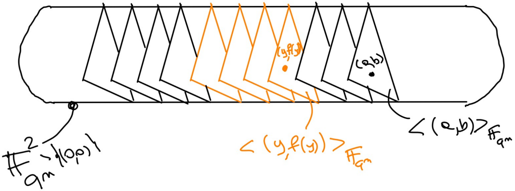
- *point set* is $\mathbb{F}^2_{q^n}$
- *lines* are $v + U$, $v \in \mathbb{F}^2_{q^n}$, $U \in \mathcal{B}_f$

The elements of $\mathcal{B}_f$ are lines through the origin, and will also be considered as points of the *line at infinity* $L_\infty = (L_\infty)_f$.

L_∞ and $\mathrm{PG}(1, q^n)$ share some points

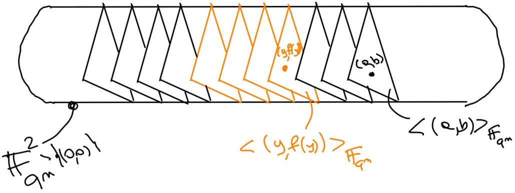
L_∞ and PG$(1, q^n)$ share some points

Università degli Studi di Padova



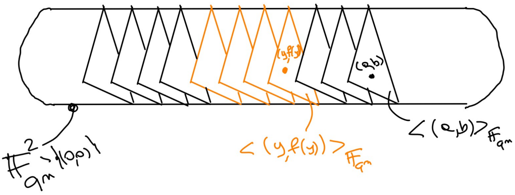
L_∞ and $\mathrm{PG}(1, q^n)$ share some points

Università degli Studi di Padova

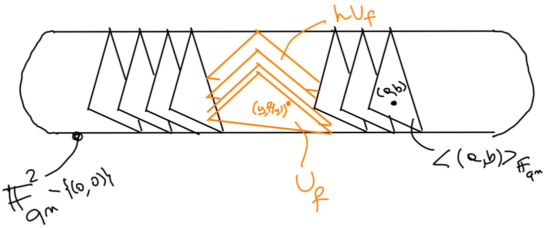


$\mathrm{PG}(1, q^n)$ is a spread $\mathcal{D}$ of $\mathbb{F}^2_{q^n}$ containing L_f

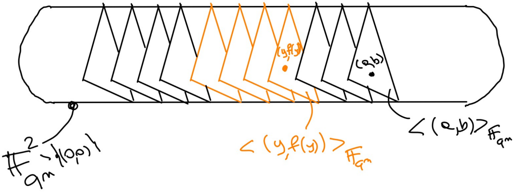
L_∞ and PG$(1, q^n)$ share some points



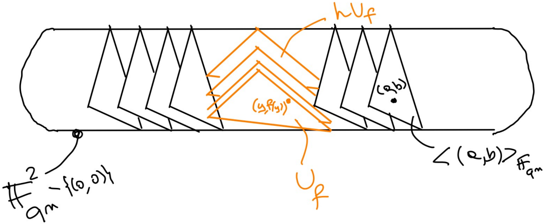
PG$(1, q^n)$ is a spread $\mathcal{D}$ of $\mathbb{F}_{q^n}^2$ containing L_f



UNIVERSITÀ DEGLI STUDI DI PADOVA



$PG(1, q^n)$ is a spread $\mathcal{D}$ of $\mathbb{F}^2_{q^n}$ containing L_f



$(L_\infty)_f$ is a spread $\mathcal{B}_f = (\mathcal{D} \setminus L_f) \cup \{hU_f \colon h \in \mathbb{F}^*_{q^n}\}$ of $\mathbb{F}^2_{q^n}$ obtained from $\mathcal{D}$ by replacing L_f

Equivalence of translation planes

Always: $q > 3$.

Equivalence of translation planes

Università degli Studi di Padova

Always: $q > 3$.

Theorem (Casarino-Longobardi-Z. 2022).

Let $f(x), g(x) \in \mathbb{F}_{q^n}[x]$ be scattered, and let $\Psi : \mathcal{A}_f \to \mathcal{A}_g$ be a collineation. Then

Equivalence of translation planes

Always: $q > 3$.

THEOREM (CASARINO-LONGOBARDI-Z. 2022).

Let $f(x), g(x) \in \mathbb{F}_{q^n}[x]$ be scattered, and let $\Psi : \mathcal{A}_f \to \mathcal{A}_g$ be a collineation. Then

1. $\Psi : x \mapsto v + \varphi(x)$ with $v \in \mathbb{F}_{q^n}^2$, $\varphi \in \Gamma\mathrm{L}(2, q^n)$;

Equivalence of translation planes

Always: $q > 3$.

Theorem (Casarino-Longobardi-Z. 2022).

Let $f(x), g(x) \in \mathbb{F}_{q^n}[x]$ be scattered, and let $\Psi : \mathcal{A}_f \to \mathcal{A}_g$ be a collineation. Then

1. $\Psi : x \mapsto v + \varphi(x)$ with $v \in \mathbb{F}_{q^n}^2$, $\varphi \in \Gamma\mathrm{L}(2, q^n)$;
2. φ stabilizes the partial spreads $\mathcal{D} \setminus L_f$ and $\{hU_f : h \in \mathbb{F}_{q^n}^*\}$.

Equivalence of translation planes

Always: $q > 3$.

Theorem (Casarino-Longobardi-Z. 2022).

Let $f(x), g(x) \in \mathbb{F}_{q^n}[x]$ be scattered, and let $\Psi : \mathcal{A}_f \to \mathcal{A}_g$ be a collineation. Then

1. $\Psi : x \mapsto v + \varphi(x)$ with $v \in \mathbb{F}_{q^n}^2$, $\varphi \in \Gamma\mathrm{L}(2, q^n)$;
2. φ stabilizes the partial spreads $\mathcal{D} \setminus L_f$ and $\{hU_f : h \in \mathbb{F}_{q^n}^*\}$.

The converse holds.

Always: $q > 3$.

Theorem (Casarino-Longobardi-Z. 2022).

Let $f(x), g(x) \in \mathbb{F}_{q^n}[x]$ be scattered, and let $\Psi : \mathcal{A}_f \to \mathcal{A}_g$ be a collineation. Then

1. $\Psi : x \mapsto v + \varphi(x)$ with $v \in \mathbb{F}_{q^n}^2$, $\varphi \in \Gamma\mathrm{L}(2, q^n)$;
2. φ stabilizes the partial spreads $\mathcal{D} \setminus L_f$ and $\{hU_f : h \in \mathbb{F}_{q^n}^*\}$.

The converse holds.

Corollary 1.

$\mathcal{A}_f \cong \mathcal{A}_g$ iff $f(x)$ and $g(x)$ are $\Gamma\mathrm{L}$-equivalent.

Equivalence of translation planes

UNIVERSITÀ DEGLI STUDI DI PADOVA

Always: $q > 3$.

THEOREM (CASARINO-LONGOBARDI-Z. 2022).

Let $f(x), g(x) \in \mathbb{F}_{q^n}[x]$ be scattered, and let $\Psi : \mathcal{A}_f \to \mathcal{A}_g$ be a collineation. Then

1. $\Psi : x \mapsto v + \varphi(x)$ with $v \in \mathbb{F}_{q^n}^2$, $\varphi \in \Gamma\mathrm{L}(2, q^n)$;
2. φ stabilizes the partial spreads $\mathcal{D} \setminus L_f$ and $\{hU_f : h \in \mathbb{F}_{q^n}^*\}$.

The converse holds.

COROLLARY 1.

$\mathcal{A}_f \cong \mathcal{A}_g$ iff $f(x)$ and $g(x)$ are $\Gamma\mathrm{L}$-equivalent.

COROLLARY 2.

Any linear set L_f of $\Gamma\mathrm{L}$-class c (Csajbók-Marino-Polverino 2018) gives rise to c pairwise nonisomorphic translation planes.

André planes

André planes

Definition.

Let $\alpha : \mathbb{F}_q^* \to \{0, 1, \ldots, n-1\}$ be any mapping, and

$$B_{m,\alpha} = \{(x, mx^{q^{\alpha\left(N_{q^n/q}(m)\right)}}) \colon x \in \mathbb{F}_{q^n}^*\}, \quad m \in \mathbb{F}_{q^n}^*.$$

Then $\mathcal{B}_\alpha = \{B_{m,\alpha} \colon m \in \mathbb{F}_{q^n}^*\} \cup \{\langle(1,0)\rangle_{\mathbb{F}_{q^n}}, \langle(0,1)\rangle_{\mathbb{F}_{q^n}}\}$ is a spread of $\mathbb{F}_{q^n}^2$, and the related translation plane is an *André plane*.

André planes

Università degli Studi di Padova

Definition.

Let $\alpha : \mathbb{F}_q^* \to \{0, 1, \ldots, n-1\}$ be any mapping, and

$$B_{m,\alpha} = \{(x, mx^{q^{\alpha\left(N_{q^n/q}(m)\right)}}) : x \in \mathbb{F}_{q^n}^*\}, \quad m \in \mathbb{F}_{q^n}^*.$$

Then $\mathcal{B}_\alpha = \{B_{m,\alpha} : m \in \mathbb{F}_{q^n}^*\} \cup \{\langle(1,0)\rangle_{\mathbb{F}_{q^n}}, \langle(0,1)\rangle_{\mathbb{F}_{q^n}}\}$ is a spread of $\mathbb{F}_{q^n}^2$, and the related translation plane is an *André plane*.

Recall: $\mathcal{B}_f = \left(\mathcal{D} \setminus L_f\right) \cup \{hU_f : h \in \mathbb{F}_{q^n}^*\}$

André planes

Università degli Studi di Padova

Definition.

Let $\alpha : \mathbb{F}_q^* \to \{0, 1, \ldots, n-1\}$ be any mapping, and

$$B_{m,\alpha} = \{(x, mx^{q^{\alpha\left(N_{q^n/q}(m)\right)}}) : x \in \mathbb{F}_{q^n}^*\}, \quad m \in \mathbb{F}_{q^n}^*.$$

Then $\mathcal{B}_\alpha = \{B_{m,\alpha} : m \in \mathbb{F}_{q^n}^*\} \cup \{\langle(1,0)\rangle_{\mathbb{F}_{q^n}}, \langle(0,1)\rangle_{\mathbb{F}_{q^n}}\}$ is a spread of $\mathbb{F}_{q^n}^2$, and the related translation plane is an *André plane*.

Recall: $\mathcal{B}_f = \left(\mathcal{D} \setminus L_f\right) \cup \{hU_f : h \in \mathbb{F}_{q^n}^*\}$

Example.

If $f(x) = x^{q^s}$, then $\mathcal{B}_f = \mathcal{B}_\alpha$ where

$$\alpha(\nu) = \begin{cases} s & \text{if } \nu = 1, \\ 0 & \text{otherwise,} \end{cases}$$

hence $\mathcal{A}_f$ is an André plane (Lunardon-Polverino 2001).

Definition.

Let $\alpha : \mathbb{F}_q^* \to \{0, 1, \dots, n-1\}$ be any mapping, and

$$B_{m,\alpha} = \{(x, mx^{q^{\alpha(N_{q^n/q}(m))}}) : x \in \mathbb{F}_{q^n}^*\}, \quad m \in \mathbb{F}_{q^n}^*.$$

Then $\mathcal{B}_\alpha = \{B_{m,\alpha} : m \in \mathbb{F}_{q^n}^*\} \cup \{\langle(1,0)\rangle_{\mathbb{F}_{q^n}}, \langle(0,1)\rangle_{\mathbb{F}_{q^n}}\}$ is a spread of $\mathbb{F}_{q^n}^2$, and the related translation plane is an *André plane*.

Recall: $\mathcal{B}_f = (\mathcal{D} \setminus L_f) \cup \{hU_f : h \in \mathbb{F}_{q^n}^*\}$

Example.

If $f(x) = x^{q^s}$, then $\mathcal{B}_f = \mathcal{B}_\alpha$ where

$$\alpha(\nu) = \begin{cases} s & \text{if } \nu = 1, \\ 0 & \text{otherwise,} \end{cases}$$

hence $\mathcal{A}_f$ is an André plane (Lunardon-Polverino 2001).

Question.

For which $f(x)$ is $\mathcal{A}_f$ an André plane, or a *generalized* André plane?

A pattern for scattered polynomials

A pattern for scattered polynomials

Many (but not all) scattered polynomials, e.g.

A pattern for scattered polynomials

Many (but not all) scattered polynomials, e.g.

(1) $x^{q^s} + \delta x^{q^{n-s}}, \quad n$ even

A pattern for scattered polynomials

Many (but not all) scattered polynomials, e.g.

(1) $x^{q^s} + \delta x^{q^{n-s}}, \quad n$ even
(2) $\delta x^{q^s} + x^{q^{s+n/2}}, \quad n \in \{6, 8\}$

A pattern for scattered polynomials

Many (but not all) scattered polynomials, e.g.

(1) $x^{q^s} + \delta x^{q^{n-s}}, \quad n$ even
(2) $\delta x^{q^s} + x^{q^{s+n/2}}, \quad n \in \{6, 8\}$
(3) $x^q + x^{q^3} + \delta x^{q^5}, \quad n = 6$

A pattern for scattered polynomials

Many (but not all) scattered polynomials, e.g.

(1) $x^{q^s} + \delta x^{q^{n-s}}, \quad n$ even
(2) $\delta x^{q^s} + x^{q^{s+n/2}}, \quad n \in \{6, 8\}$
(3) $x^q + x^{q^3} + \delta x^{q^5}, \quad n = 6$
(4) $x^q + x^{q^{t-1}} + h^{1+q} x^{q^{t+1}} + h^{1-q^{2t-1}} x^{q^{2t-1}}, \quad n = 2t,\ t$ even

A pattern for scattered polynomials

Università degli Studi di Padova

Many (but not all) scattered polynomials, e.g.

(1) $x^{q^s} + \delta x^{q^{n-s}}, \quad n$ even
(2) $\delta x^{q^s} + x^{q^{s+n/2}}, \quad n \in \{6, 8\}$
(3) $x^q + x^{q^3} + \delta x^{q^5}, \quad n = 6$
(4) $x^q + x^{q^{t-1}} + h^{1+q} x^{q^{t+1}} + h^{1-q^{2t-1}} x^{q^{2t-1}}, \quad n = 2t,\ t$ even

are of type $F(x^{q^s})$ where $F(x)$ is $\mathbb{F}_{q^r}$-linear for some $1 < r \mid n$, and $(s, r) = 1$:

A pattern for scattered polynomials

Many (but not all) scattered polynomials, e.g.

(1) $x^{q^s} + \delta x^{q^{n-s}}, \quad n$ even
(2) $\delta x^{q^s} + x^{q^{s+n/2}}, \quad n \in \{6, 8\}$
(3) $x^q + x^{q^3} + \delta x^{q^5}, \quad n = 6$
(4) $x^q + x^{q^{t-1}} + h^{1+q} x^{q^{t+1}} + h^{1-q^{2t-1}} x^{q^{2t-1}}, \quad n = 2t, \ t$ even

are of type $F(x^{q^s})$ where $F(x)$ is $\mathbb{F}_{q^r}$-linear for some $1 < r \mid n$, and $(s, r) = 1$:

(1) $F(x) = x + \delta x^{q^{n-2s}}, \qquad r = 2$

A pattern for scattered polynomials

Many (but not all) scattered polynomials, e.g.

(1) $x^{q^s} + \delta x^{q^{n-s}}, \quad n$ even
(2) $\delta x^{q^s} + x^{q^{s+n/2}}, \quad n \in \{6, 8\}$
(3) $x^q + x^{q^3} + \delta x^{q^5}, \quad n = 6$
(4) $x^q + x^{q^{t-1}} + h^{1+q} x^{q^{t+1}} + h^{1-q^{2t-1}} x^{q^{2t-1}}, \quad n = 2t, \ t$ even

are of type $F(x^{q^s})$ where $F(x)$ is $\mathbb{F}_{q^r}$-linear for some $1 < r \mid n$, and $(s, r) = 1$:

(1) $F(x) = x + \delta x^{q^{n-2s}}, \qquad r = 2$
(2) $F(x) = \delta x + x^{q^{n/2}}, \qquad r = n/2$

A pattern for scattered polynomials

Many (but not all) scattered polynomials, e.g.

(1) $x^{q^s} + \delta x^{q^{n-s}}$, $\quad n$ even
(2) $\delta x^{q^s} + x^{q^{s+n/2}}$, $\quad n \in \{6, 8\}$
(3) $x^q + x^{q^3} + \delta x^{q^5}$, $\quad n = 6$
(4) $x^q + x^{q^{t-1}} + h^{1+q} x^{q^{t+1}} + h^{1-q^{2t-1}} x^{q^{2t-1}}$, $\quad n = 2t,\ t$ even

are of type $F(x^{q^s})$ where $F(x)$ is $\mathbb{F}_{q^r}$-linear for some $1 < r \mid n$, and $(s, r) = 1$:

(1) $F(x) = x + \delta x^{q^{n-2s}}$, $\qquad r = 2$
(2) $F(x) = \delta x + x^{q^{n/2}}$, $\qquad r = n/2$
(3) $F(x) = x + x^{q^2} + \delta x^{q^4}$, $\qquad r = 2,\ s = 1$

A pattern for scattered polynomials

Many (but not all) scattered polynomials, e.g.

(1) $x^{q^s} + \delta x^{q^{n-s}}, \quad n$ even
(2) $\delta x^{q^s} + x^{q^{s+n/2}}, \quad n \in \{6, 8\}$
(3) $x^q + x^{q^3} + \delta x^{q^5}, \quad n = 6$
(4) $x^q + x^{q^{t-1}} + h^{1+q} x^{q^{t+1}} + h^{1-q^{2t-1}} x^{q^{2t-1}}, \quad n = 2t,\ t$ even

are of type $F(x^{q^s})$ where $F(x)$ is $\mathbb{F}_{q^r}$-linear for some $1 < r \mid n$, and $(s, r) = 1$:

(1) $F(x) = x + \delta x^{q^{n-2s}}, \qquad r = 2$
(2) $F(x) = \delta x + x^{q^{n/2}}, \qquad r = n/2$
(3) $F(x) = x + x^{q^2} + \delta x^{q^4}, \qquad r = 2,\ s = 1$
(4) $F(x) = x + x^{q^{t-2}} + h^{1+q} x^{q^t} + h^{1-q^{2t-1}} x^{q^{2t-2}}, \qquad r = 2,\ s = 1$

A pattern for scattered polynomials

Many (but not all) scattered polynomials, e.g.

(1) $x^{q^s} + \delta x^{q^{n-s}}, \quad n$ even
(2) $\delta x^{q^s} + x^{q^{s+n/2}}, \quad n \in \{6, 8\}$
(3) $x^q + x^{q^3} + \delta x^{q^5}, \quad n = 6$
(4) $x^q + x^{q^{t-1}} + h^{1+q} x^{q^{t+1}} + h^{1-q^{2t-1}} x^{q^{2t-1}}, \quad n = 2t,\ t$ even

are of type $F(x^{q^s})$ where $F(x)$ is $\mathbb{F}_{q^r}$-linear for some $1 < r \mid n$, and $(s, r) = 1$:

(1) $F(x) = x + \delta x^{q^{n-2s}}, \qquad r = 2$
(2) $F(x) = \delta x + x^{q^{n/2}}, \qquad r = n/2$
(3) $F(x) = x + x^{q^2} + \delta x^{q^4}, \qquad r = 2,\ s = 1$
(4) $F(x) = x + x^{q^{t-2}} + h^{1+q} x^{q^t} + h^{1-q^{2t-1}} x^{q^{2t-2}}, \qquad r = 2,\ s = 1$

We call a polynomial in standard form any polynomial of this type.

A pattern for scattered polynomials

Università degli Studi di Padova

Definition (Longobardi-Z. 202x).

Let $f(x) = \sum_i a_i x^{q^i} \in \mathbb{F}_{q^n}[x]$ be scattered and

$$\Delta_f = \{(i-j) \mod n \colon a_i a_j \neq 0\} \cup \{n\}.$$

Let $r = r_f = \gcd \Delta_f$. If $r > 1$, then $f(x)$ is in *standard form.*

A pattern for scattered polynomials

Università degli Studi di Padova

Definition (Longobardi-Z. 202x).

Let $f(x)=\sum_i a_i x^{q^i}\in\mathbb{F}_{q^n}[x]$ be scattered and

$$\Delta_f=\{(i-j)\mod n\colon a_ia_j\neq 0\}\cup\{n\}.$$

Let $r=r_f=\gcd\Delta_f$. If $r>1$, then $f(x)$ is in *standard form.*

☞ If $f(x)=F(x^{q^s})$ is in standard form, then $f(x)=\sum_k b_k x^{q^{s+kr}}$ where $r>1$, $(s,r)=1$.

A pattern for scattered polynomials

Università degli Studi di Padova

Definition (Longobardi-Z. 202x).

Let $f(x)=\sum_i a_i x^{q^i} \in \mathbb{F}_{q^n}[x]$ be scattered and

$$\Delta_f = \{(i-j) \mod n \colon a_i a_j \neq 0\} \cup \{n\}.$$

Let $r = r_f = \gcd \Delta_f$. If $r>1$, then $f(x)$ is in *standard form*.

If $f(x)=F(x^{q^s})$ is in standard form, then $f(x)=\sum_k b_k x^{q^{s+kr}}$ where $r>1$, $(s,r)=1$.

The *stabilizer* of $U_f=\{(y,f(y))\colon y\in\mathbb{F}_{q^n}\}$ is
$G_f = \mathrm{GL}(2,q^n)_{\{U_f\}}$ (i.e. setwise stabilizer).

A pattern for scattered polynomials

Università degli Studi di Padova

Definition (Longobardi-Z. 202x).

Let $f(x) = \sum_i a_i x^{q^i} \in \mathbb{F}_{q^n}[x]$ be scattered and

$$\Delta_f = \{(i-j) \mod n \colon a_i a_j \neq 0\} \cup \{n\}.$$

Let $r = r_f = \gcd \Delta_f$. If $r > 1$, then $f(x)$ is in *standard form*.

If $f(x) = F(x^{q^s})$ is in standard form, then $f(x) = \sum_k b_k x^{q^{s+kr}}$ where $r > 1$, $(s, r) = 1$.

The *stabilizer* of $U_f = \{(y, f(y)) \colon y \in \mathbb{F}_{q^n}\}$ is

$$G_f = \mathrm{GL}(2, q^n)_{\{U_f\}} \text{ (i.e. setwise stabilizer).}$$

$$|G_f| \geq q - 1 \text{ since } \mathrm{diag}(\alpha, \alpha) \in G_f \text{ for all } \alpha \in \mathbb{F}_q^*.$$

A pattern for scattered polynomials

Università degli Studi di Padova

Definition (Longobardi-Z. 202x).

Let $f(x)=\sum_i a_i x^{q^i} \in \mathbb{F}_{q^n}[x]$ be scattered and

$$\Delta_f = \{(i-j) \mod n \colon a_i a_j \neq 0\} \cup \{n\}.$$

Let $r = r_f = \gcd \Delta_f$. If $r>1$, then $f(x)$ is in *standard form.*

☞ If $f(x)=F(x^{q^s})$ is in standard form, then $f(x)=\sum_k b_k x^{q^{s+kr}}$ where $r>1$, $(s,r)=1$.

The *stabilizer* of $U_f=\{(y,f(y))\colon y\in\mathbb{F}_{q^n}\}$ is

$$G_f = \mathrm{GL}(2,q^n)_{\{U_f\}} \text{ (i.e. setwise stabilizer).}$$

$|G_f|\geq q-1$ since $\mathrm{diag}(\alpha,\alpha)\in G_f$ for all $\alpha\in\mathbb{F}_q^*$.
If $|G_f|=q-1$, then G_f is trivial.

A pattern for scattered polynomials

Università degli Studi di Padova

Definition (Longobardi-Z. 202x).

Let $f(x)=\sum_i a_i x^{q^i} \in \mathbb{F}_{q^n}[x]$ be scattered and

$$\Delta_f = \{(i-j) \mod n \colon a_i a_j \neq 0\} \cup \{n\}.$$

Let $r = r_f = \gcd \Delta_f$. If $r>1$, then $f(x)$ is in *standard form.*

☞ If $f(x) = F(x^{q^s})$ is in standard form, then $f(x) = \sum_k b_k x^{q^{s+kr}}$ where $r>1$, $(s,r)=1$.

The *stabilizer* of $U_f = \{(y, f(y))\colon y \in \mathbb{F}_{q^n}\}$ is

$$G_f = \mathrm{GL}(2,q^n)_{\{U_f\}} \text{ (i.e. setwise stabilizer).}$$

$|G_f| \geq q-1$ since $\mathrm{diag}(\alpha,\alpha) \in G_f$ for all $\alpha \in \mathbb{F}_q^*$.
If $|G_f| = q-1$, then G_f is trivial.

☞ If $f(x) = \sum_k b_k x^{q^{s+kr}} = F(x^{q^s})$ is in standard form, then

$$\{\mathrm{diag}(\alpha, \alpha^{q^s})\colon \alpha \in \mathbb{F}_{q^r}^*\} \subseteq G_f.$$

Indeed, $f(\alpha y) = \alpha^{q^s} F(y^{q^s}) = \alpha^{q^s} f(y)\ \forall y \in \mathbb{F}_{q^n}$.

A pattern for scattered polynomials

A pattern for scattered polynomials

Università degli Studi di Padova

Definition (Sheekey 2016).

The *rank distance code associated with* $f(x)$ is the following subspace of $\mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_{q^n})$:

$$\mathcal{C}_f = \langle x, f(x) \rangle_{\mathbb{F}_{q^n}}$$

A pattern for scattered polynomials

Definition (Sheekey 2016).

The *rank distance code associated with* $f(x)$ is the following subspace of $\mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_{q^n})$:

$$\mathcal{C}_f = \langle x, f(x) \rangle_{\mathbb{F}_{q^n}}$$

Definition.

The *right idealizer* of $\mathcal{C}_f$ is

$$I_R(\mathcal{C}_f) = \{\varphi \in \mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_{q^n}) \colon g \circ \varphi \in \mathcal{C}_f,\ \forall g \in \mathcal{C}_f\}.$$

A pattern for scattered polynomials

Università degli Studi di Padova

Definition (Sheekey 2016).

The *rank distance code associated with* $f(x)$ is the following subspace of $\mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_{q^n})$:

$$\mathcal{C}_f = \langle x, f(x)\rangle_{\mathbb{F}_{q^n}}$$

Definition.

The *right idealizer* of $\mathcal{C}_f$ is

$$I_R(\mathcal{C}_f) = \{\varphi \in \mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_{q^n})\colon g \circ \varphi \in \mathcal{C}_f,\ \forall g \in \mathcal{C}_f\}.$$

Theorem.

$I_R(\mathcal{C}_f)$ is a field isomorphic to $\mathbb{F}_{q^t}$, $t \mid n$, and $t = n$ if and only if $f(x)$ is of pseudoregulus type.

Università degli Studi di Padova

Definition (Sheekey 2016).

The *rank distance code associated with* $f(x)$ is the following subspace of $\mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_{q^n})$:

$$\mathcal{C}_f = \langle x, f(x)\rangle_{\mathbb{F}_{q^n}}$$

Definition.

The *right idealizer* of $\mathcal{C}_f$ is

$$I_R(\mathcal{C}_f) = \{\varphi \in \mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_{q^n}) \colon g \circ \varphi \in \mathcal{C}_f,\ \forall g \in \mathcal{C}_f\}.$$

Theorem.

$I_R(\mathcal{C}_f)$ is a field isomorphic to $\mathbb{F}_{q^t}$, $t \mid n$, and $t = n$ if and only if $f(x)$ is of pseudoregulus type.

Proof. As regards the case $t = n$, Csajbók-Marino-Polverino-Zhou (2020) prove that if an MRD code $\mathcal{C}$ has both right and left idealizers isomorphic to $\mathbb{F}_{q^n}$, then it is equivalent to

$$\langle x^{q^{t_i}} \,:\, i = 0, 1, \ldots, k-1\rangle_{\mathbb{F}_{q^n}}$$

and if $\mathcal{C} = \mathcal{C}_f$ this is equivalent to $\langle x, x^{q^s}\rangle_{\mathbb{F}_{q^n}}$.

A pattern for scattered polynomials

Definition (Sheekey 2016).

The *rank distance code associated with* $f(x)$ is the following subspace of $\mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_{q^n})$:

$$\mathcal{C}_f = \langle x, f(x) \rangle_{\mathbb{F}_{q^n}}$$

Definition.

The *right idealizer* of $\mathcal{C}_f$ is

$$I_R(\mathcal{C}_f) = \{\varphi \in \mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_{q^n}) \colon g \circ \varphi \in \mathcal{C}_f,\ \forall g \in \mathcal{C}_f\}.$$

Theorem.

$I_R(\mathcal{C}_f)$ is a field isomorphic to $\mathbb{F}_{q^t}$, $t \mid n$, and $t = n$ if and only if $f(x)$ is of pseudoregulus type.

Proof. As regards the case $t = n$, Csajbók-Marino-Polverino-Zhou (2020) prove that if an MRD code $\mathcal{C}$ has both right and left idealizers isomorphic to $\mathbb{F}_{q^n}$, then it is equivalent to

$$\langle x^{q^{t_i}} : i = 0, 1, \dots, k-1 \rangle_{\mathbb{F}_{q^n}}$$

and if $\mathcal{C} = \mathcal{C}_f$ this is equivalent to $\langle x, x^{q^s} \rangle_{\mathbb{F}_{q^n}}$.

Theorem (Longobardi-Marino-Trombetti-Zhou 202x).

$I_R(\mathcal{C}_f)$ and $G_f \cup \{0\}$ are isomorphic fields. (R.: G_f, setwise stabilizer of U_f in $\mathrm{GL}(2, q^n)$.)

Simultaneous diagonalization

Theorem (Longobardi-Z. 202x).

All elements of G_f are simultaneously diagonalizable.

Simultaneous diagonalization

Theorem (Longobardi-Z. 202x).

All elements of G_f are simultaneously diagonalizable.

Proof. Use the complete description by Beard (1972) and Willett (1973) of all subrings of $\mathbb{F}_q^{N\times N}$ which are fields.

Simultaneous diagonalization

Theorem (Longobardi-Z. 202x).

All elements of G_f are simultaneously diagonalizable.

Proof. Use the complete description by Beard (1972) and Willett (1973) of all subrings of $\mathbb{F}_q^{N\times N}$ which are fields.

Corollary.

If $f(x) = \sum_k b_k x^{q^{s+kr}}$ is in standard form, then

$$G_f = \{\mathrm{diag}(\alpha, \alpha^{q^s}) : \alpha \in \mathbb{F}_{q^r}^*\}.$$

Simultaneous diagonalization

Theorem (Longobardi-Z. 202x).

All elements of G_f are simultaneously diagonalizable.

Proof. Use the complete description by Beard (1972) and Willett (1973) of all subrings of $\mathbb{F}_q^{N\times N}$ which are fields.

Corollary.

If $f(x) = \sum_k b_k x^{q^{s+kr}}$ is in standard form, then

$$G_f = \{\mathrm{diag}(\alpha, \alpha^{q^s}) \colon \alpha \in \mathbb{F}_{q^r}^*\}.$$

If $f(x)$ is in standard form, then G_f only depends on s and r, i.e. on the shape of $f(x)$.

Simultaneous diagonalization

The elements of G_f are simultaneously diagonalizable $\rightsquigarrow$ $\mathbb{F}_{q^n}$-basis v_1, v_2 of $\mathbb{F}_{q_n}^2$, eigenvectors of any $\varphi \in G_f$

Simultaneous diagonalization

The elements of G_f are simultaneously diagonalizable $\rightsquigarrow$ $\mathbb{F}_{q^n}$-basis v_1, v_2 of $\mathbb{F}_{q_n}^2$, eigenvectors of any $\varphi \in G_f$ $\rightsquigarrow$ for $|G_f| > q-1$, two "transversal points" $V_1 = \langle v_1 \rangle_{\mathbb{F}_{q^n}}$, $V_2 = \langle v_2 \rangle_{\mathbb{F}_{q^n}}$ are projectively defined by f.

Simultaneous diagonalization

The elements of G_f are simultaneously diagonalizable $\rightsquigarrow$ $\mathbb{F}_{q^n}$-basis v_1, v_2 of $\mathbb{F}_{q_n}^2$, eigenvectors of any $\varphi \in G_f$ $\rightsquigarrow$ for $|G_f| > q-1$, two "transversal points" $V_1 = \langle v_1 \rangle_{\mathbb{F}_{q^n}}$, $V_2 = \langle v_2 \rangle_{\mathbb{F}_{q^n}}$ are projectively defined by f.

Simultaneous diagonalization

The elements of G_f are simultaneously diagonalizable $\rightsquigarrow$ $\mathbb{F}_{q^n}$-basis v_1, v_2 of $\mathbb{F}_{q_n}^2$, eigenvectors of any $\varphi \in G_f$ $\rightsquigarrow$ for $|G_f| > q-1$, two "transversal points" $V_1 = \langle v_1 \rangle_{\mathbb{F}_{q^n}}$, $V_2 = \langle v_2 \rangle_{\mathbb{F}_{q^n}}$ are projectively defined by f.

Proposition.

$$V_1, V_2 \notin L_f$$

Proof. The orbits in $L_f \setminus \{V_1, V_2\}$ under G_f are of size $|G_f|/(q-1)$, obtain a contradiction.

Simultaneous diagonalization

Università degli Studi di Padova

The elements of G_f are simultaneously diagonalizable $\rightsquigarrow$ $\mathbb{F}_{q^n}$-basis v_1, v_2 of $\mathbb{F}_{q_n}^2$, eigenvectors of any $\varphi \in G_f$ $\rightsquigarrow$ for $|G_f| > q - 1$, two "transversal points" $V_1 = \langle v_1 \rangle_{\mathbb{F}_{q^n}}$, $V_2 = \langle v_2 \rangle_{\mathbb{F}_{q^n}}$ are projectively defined by f.

Proposition.

$$V_1, V_2 \notin L_f$$

Proof. The orbits in $L_f \setminus \{V_1, V_2\}$ under G_f are of size $|G_f|/(q-1)$, obtain a contradiction.

Proposition.

If $f(x)$ is in standard form, then $f(x)$ is nonsingular.

Simultaneous diagonalization

The elements of G_f are simultaneously diagonalizable $\rightsquigarrow$ $\mathbb{F}_{q^n}$-basis v_1, v_2 of $\mathbb{F}_{q^n}^2$, eigenvectors of any $\varphi \in G_f$ $\rightsquigarrow$ for $|G_f| > q-1$, two "transversal points" $V_1 = \langle v_1 \rangle_{\mathbb{F}_{q^n}}$, $V_2 = \langle v_2 \rangle_{\mathbb{F}_{q^n}}$ are projectively defined by f.

Proposition.

$$V_1, V_2 \notin L_f$$

Proof. The orbits in $L_f \setminus \{V_1, V_2\}$ under G_f are of size $|G_f|/(q-1)$, obtain a contradiction.

Proposition.

If $f(x)$ is in standard form, then $f(x)$ is nonsingular.

Open problem.

Do V_1 and V_2 depend only on the linear set L_f?

Simultaneous diagonalization

Simultaneous diagonalization

Theorem (Longobardi-Z. 202x).

If $|G_f| > q-1$, then $f(x)$ is GL-equivalent to a scattered polynomial $g(x)$ in standard form. Such $g(x)$ is essentially unique, i.e. the only polynomials in standard form which are GL-equivalent to $f(x)$ are

$$ag(bx) \text{ and } ag^{-1}(bx) \text{ for } a, b \in \mathbb{F}_{q^n}^*.$$

Simultaneous diagonalization

Theorem (Longobardi-Z. 202x).

If $|G_f| > q-1$, then $f(x)$ is GL-equivalent to a scattered polynomial $g(x)$ in standard form. Such $g(x)$ is essentially unique, i.e. the only polynomials in standard form which are GL-equivalent to $f(x)$ are

$$ag(bx) \text{ and } ag^{-1}(bx) \text{ for } a, b \in \mathbb{F}_{q^n}^*.$$

Example

For $q \equiv 1 \pmod 4$, $f(x) = x^q + x^{q^2} - x^{q^4} + x^{q^5} \in \mathbb{F}_{q^6}[x]$ is scattered and GL-equivalent to

$$g(x) = (1-\rho)x^q - x^{q^3} + (1+\rho)x^{q^5}$$

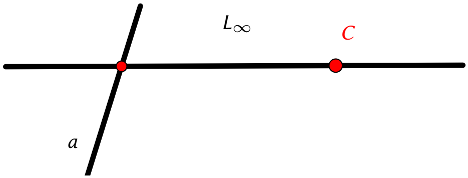
where $\rho^2 = -1$.

Consequences on translation planes

Consequences on translation planes

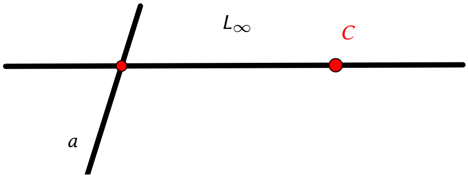
An *affine central collineation* of an affine plane $\mathcal{A}$ fixes an affine line a (the *axis*) pointwise, as well as all lines through a point C (the *center*) at infinity.

An *affine central collineation* of an affine plane $\mathcal{A}$ fixes an affine line a (the *axis*) pointwise, as well as all lines through a point C (the *center*) at infinity.



Consequences on translation planes

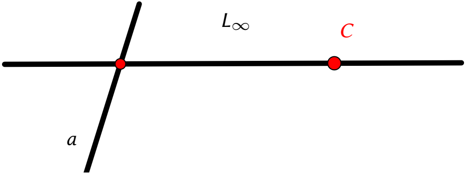
An *affine central collineation* of an affine plane $\mathcal{A}$ fixes an affine line a (the *axis*) pointwise, as well as all lines through a point C (the *center*) at infinity.



$a \cap L_\infty$ is the *co-center* of the central collineation.

Consequences on translation planes

An *affine central collineation* of an affine plane $\mathcal{A}$ fixes an affine line a (the *axis*) pointwise, as well as all lines through a point C (the *center*) at infinity.

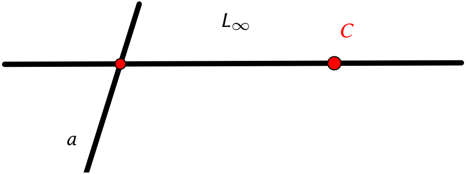


$a \cap L_\infty$ is the *co-center* of the central collineation.

If $\mathcal{A} = \mathcal{A}_f$, $f(x)$ scattered, then any affine central collineation κ is of type

$$\kappa : x \mapsto v + d\varphi(x),\ v \in \mathbb{F}_{q^n}^2,\ d \in \mathbb{F}_{q^n}^*,\ \varphi \in G_f.$$

An *affine central collineation* of an affine plane $\mathcal{A}$ fixes an affine line a (the *axis*) pointwise, as well as all lines through a point C (the *center*) at infinity.



$a \cap L_\infty$ is the *co-center* of the central collineation.

If $\mathcal{A} = \mathcal{A}_f$, $f(x)$ scattered, then any affine central collineation κ is of type

$$\kappa : x \mapsto v + d\varphi(x),\ v \in \mathbb{F}_{q^n}^2,\ d \in \mathbb{F}_{q^n}^*,\ \varphi \in G_f.$$

☞ If $|G_f| > q - 1$, the center and the co-center are the transversal points of f.

Consequences on translation planes

Consequences on translation planes

Theorem (Jha-Johnson 2008 for the Lunardon-Polverino polynomials; Longobardi-Z. 202x for the general case).

(i) If $|G_f| = q - 1$, then $\mathcal{A}_f$ admits no nontrivial central collineation.

Consequences on translation planes

Theorem (Jha-Johnson 2008 for the Lunardon-Polverino polynomials; Longobardi-Z. 202x for the general case).

(i) If $|G_f| = q - 1$, then $\mathcal{A}_f$ admits no nontrivial central collineation.

(ii) If $|G_f| > q - 1$, then the central collineations fixing O are in two cyclic groups of homologies of order $(q^r - 1)/(q - 1)$. In this case the intersection of the full collineation group with $\mathsf{GL}(2, q^n)$ is the direct product of one of those homology groups for the *kernel homology group* of the associated Desarguesian plane (maps of type $(x, y) \mapsto (dx, dy)$, $d \in \mathbb{F}_{q^n}^*$).

Consequences on translation planes

Theorem (Jha-Johnson 2008 for the Lunardon-Polverino polynomials; Longobardi-Z. 202x for the general case).

(i) If $|G_f| = q-1$, then $\mathcal{A}_f$ admits no nontrivial central collineation.

(ii) If $|G_f| > q-1$, then the central collineations fixing O are in two cyclic groups of homologies of order $(q^r-1)/(q-1)$. In this case the intersection of the full collineation group with $\mathrm{GL}(2,q^n)$ is the direct product of one of those homology groups for the *kernel homology group* of the associated Desarguesian plane (maps of type $(x,y) \mapsto (dx,dy)$, $d \in \mathbb{F}_{q^n}^*$).

Theorem (Jha-Johnson 2008 for the Lunardon-Polverino polynomials; Longobardi-Z. 202x for the general case).

If $f(x)$ is a scattered polynomial not GL-equivalent to a polynomial of pseudoregulus type, then $\mathcal{A}_f$ is neither an André plane nor a generalized André plane.

Thank you!