The linear programming bounds for classical association schemes

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(joint work with Kai-Uwe Schmidt)

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Basic coding-theoretic problem

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This problem goes back to the 1940s.

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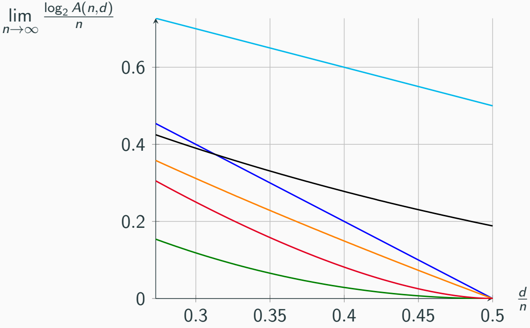
The LP optimum can be computed numerically with an LP solver.

Asymptotic bounds for binary codes



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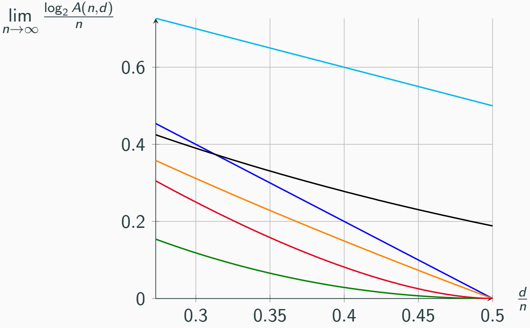

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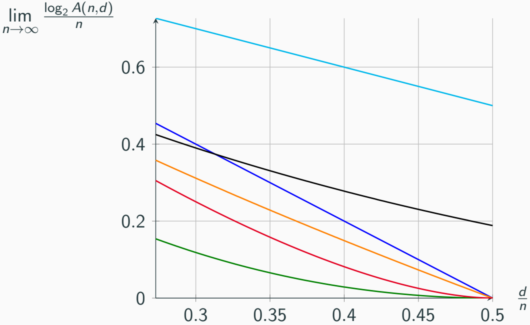

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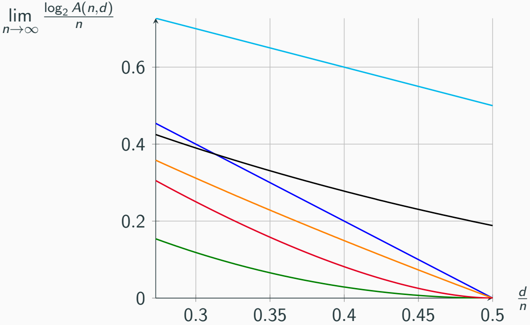

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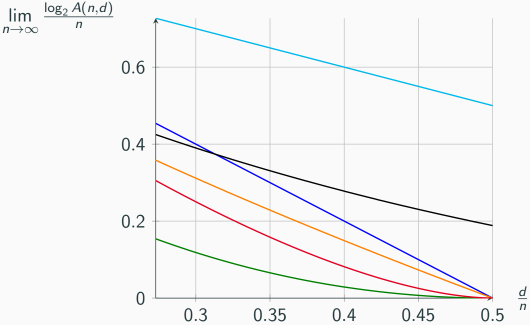

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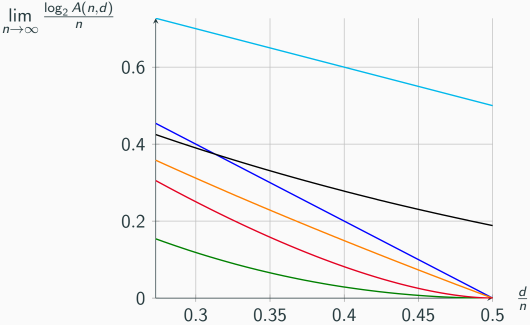

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Here: Codes in different classical association schemes.

Association schemes

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(related to the eigenspace V_k and the relation R_i)

Codes in association schemes

Inner distribution of a subset Y of X: $a = (a_0, a_1, \ldots, a_n)^T$ with

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All entries of Qa are nonnegative and $\sum_{i=0}^{n} a_i = |Y|$.

Linear program (Delsarte, 1973)

Find $x_0, x_1, \ldots, x_n$ that maximize $x_0 + x_1 + \cdots + x_n$ subject to

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Especially, $|Y| \leq \mathrm{LP}(d)$ for all d-codes Y.

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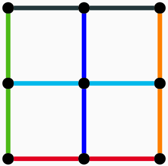

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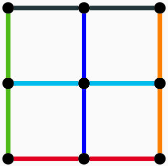

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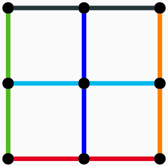

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Polar space schemes: X is the set of n-spaces in a polar space of rank n and $\rho(x, y) = n - \dim(x \cap y)$.

Six polar spaces

There are six polar spaces of rank n (up to isomorphism).

form	**name**	**type**
Hermitian	Hermitian	${}^2A_{2n-1}$
Hermitian	Hermitian	${}^2A_{2n}$
alternating	symplectic	C_n
quadratic	hyperbolic	D_n
quadratic	parabolic	B_n
quadratic	elliptic	${}^2D_{n+1}$

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$J_q(n, m+n)$	q-Hahn polynomials
Polar space schemes	q-Krawtchouk polynomials

Hermitian polar space ${}^2A_{2n-1}$

The association scheme of ${}^2A_{2n-1}$ has two orderings

$$1^{\text{st}} \text{ ordering:} \quad V_0 \;\; V_1 \;\; \cdots \;\; V_n$$

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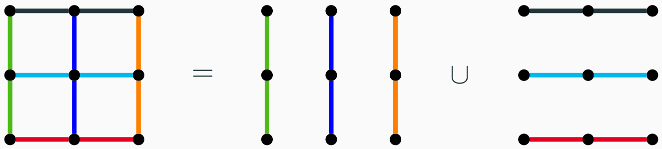
2^{nd} ordering: $Q_k(x)$ becomes a q-Hahn polynomial instead of a q-Krawtchouk polynomial.

Bipartite halves $\frac{1}{2}D_m$

Hyperbolic polar space D_m



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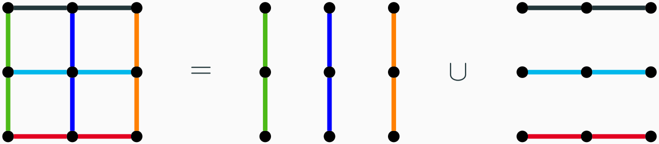

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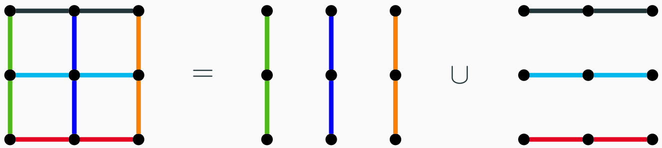

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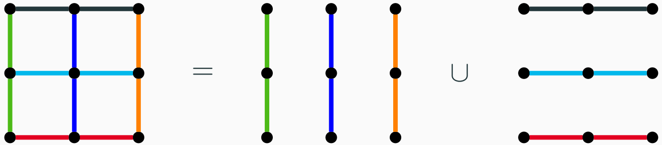

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Here, $Q_k(x)$ is also a q-Hahn polynomial.

Connection between the cubic and triangular type

$\mathrm{Bil}_q(m,n)$, $\mathrm{Alt}_q(m)$, $\mathrm{Her}_q(n)$	affine q-Krawtchouk polynomial
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Triangular Type: Anticode bound $|Y| \leq \frac{\begin{bmatrix} m+n \\ n-d+1 \end{bmatrix}_q}{\begin{bmatrix} n \\ n-d+1 \end{bmatrix}_q}$ for $J_q(n, m+n)$.

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Triangular Type: Anticode bound $|Y| \leq \frac{\left[{m+n \atop n-d+1}\right]_q}{\left[{n \atop n-d+1}\right]_q}$ for $J_q(n, m+n)$.

This bound is sharp up to a constant factor (Silva, Kschischang, Kötter, 2008).

The LP optimum

Theorem (Schmidt, W., 2022)

Triangular types $J_q(n, m+n)$, $\frac{1}{2}D_m$, ${}^2A_{2n-1}$ (with odd d for ${}^2A_{2n-1}$)

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Steiner systems

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Existence? Nontrivial examples only known for $(t, n, v) = (2, 3, 13)$ and $q = 2$ (Braun, Etzion, Östergård, Vardy, Wassermann, 2016).

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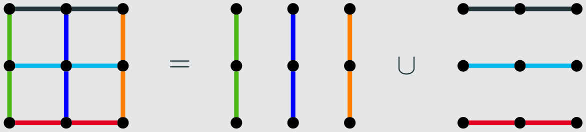

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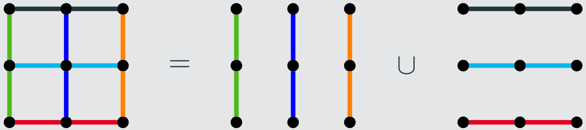

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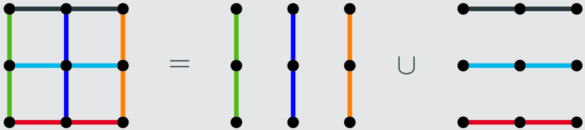

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- 1-Steiner systems are spreads in polar spaces.

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There are no t-Steiner systems in polar spaces of rank n if $(n, t) = (4, 2)$ or $(n, t) = (5, 3)$ (Cossidente, Marino, Pavese, Smaldore, 2022).

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Except for $\frac{1}{2}D_n$ and spreads in some polar spaces, no other nontrivial Steiner systems are known.

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Conjecture

$\frac{1}{2}D_n$ are the only nontrivial t-Steiner systems with $t > 1$.