Renitent lines

Peter Sziklai*, joint work with B. Csajbók and Zs. Weiner

*ELTE, Institute of Mathematics, Department of Computer Science
ELKH-ELTE Geometric and Algebraic Combinatorics Research Group

Irsee

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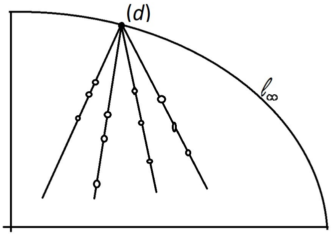
$\mathbb{F}_q$, $\mathrm{char} = p$, AG$(2, q)$, PG$(2, q)$

$\mathcal{T}$: a point set of AG$(2, q)$ might be multiset

geometric structure: intersection (numbers) with lines

regularity: when all the lines of a parallel class intersect in the same number of points (possibly mod p)



REGULARITY: this holds for many directions

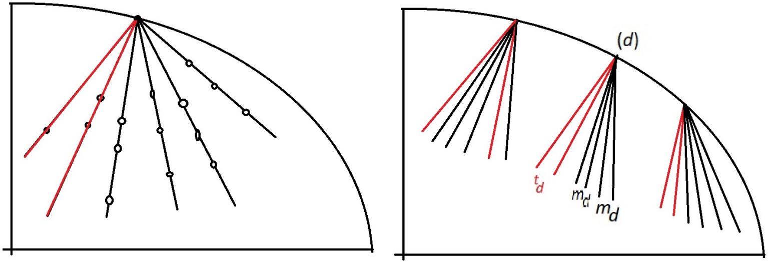
examples:

- graph of a function, from (∞)
- an additive pointset (subspace over a subfield)
- $q = 2^h$: a hyperoval in AG$(2, q)$, a maximal (k, n)-arc
- a KM-arc

almost regularity: almost all lines of a parallel class intersect in the same number of points (possibly mod p)

renitent lines: intersect not the typical way

ALMOST REGULARITY: this holds for many dir's (not necessarily the same way from every dir)



examples:

- $q = 2^h$: a $(q+1)$-arc in AG$(2, q)$: one renitent line from each direction
- ...
- ...

Question(s)

Is it true, that if $\mathcal{T}$ is almost regular, then it has a hidden structure, i.e. the non-regular intersections may be "corrected"?
Or at least they also possess some regularity themselves?
Is there a structure in the set of renitent lines?

This resembles Segre's theorem: let $\mathcal{K}$ be a $(q+2-t)$-arc in $\mathrm{PG}(2,q)$
"typical intersections": 0 or 2. Not typical lines: tangents
Segre: not typical lines are contained in a dual curve of low degree ($\leq 2t$)
(just motivation, not the same situation as there are two types of "typical intersections" if q is odd)

structure of a set of lines: contained in a dual curve of low degree
(algebraic envelope, class)

Lemma (Lemma of renitent lines (Csajbók, Weiner))

Let $\mathcal{T}$ be a point set of $\mathrm{AG}(2,q)$*. A line ℓ with slope d is called* renitent *if there exists an integer m_d such that $|\ell \cap \mathcal{T}| \not\equiv m_d \pmod{p}$ but every other line with slope d meets $\mathcal{T}$ in m_d modulo p points. Then the renitent lines are concurrent.*

Lemma (Lemma of renitent lines (Csajbók, Weiner))

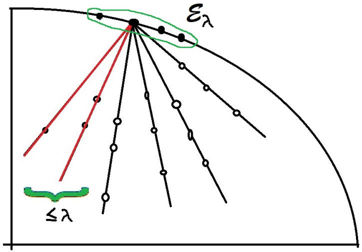
Let $\mathcal{T}$ be a point set of $\mathrm{AG}(2,q)$*. A line ℓ with slope d is called* renitent *if there exists an integer m_d such that $|\ell \cap \mathcal{T}| \not\equiv m_d \pmod{p}$ but every other line with slope d meets $\mathcal{T}$ in m_d modulo p points. Then the renitent lines are concurrent.*

Now we define **renitent lines** in the following, more general setting and then prove various generalizations of the lemma above.

Definition

Let $\mathcal{M}$ be a multiset of AG$(2,q)$. For some $\lambda \leq (q-1)/2$, a direction (d) is *$(q-\lambda)$-uniform* if there are at least $(q-\lambda)$ affine lines with slope d meeting $\mathcal{M}$ in the same number of points mod p. This number will be called the *typical intersection number* at (d). The rest of the lines with direction (d) will be called *renitent*.

A *sharply* $(q-\lambda)$-uniform direction (d) is a direction incident with *exactly* $(q-\lambda)$ affine lines meeting $\mathcal{M}$ in the same number of points modulo p.



Different directions might have different typical intersection numbers. Uniquely determined for each $(q-\lambda)$-uniform direction because $\lambda \leq (q-1)/2$.

The basic case

Theorem (Cs-Sz-W)

Take a multiset $\mathcal{T}$ of $\mathrm{AG}(2,q)$ and let $\mathcal{E}_\lambda$ denote a set of at most q directions which are $(q-\lambda)$-uniform, with typical intersection numbers $m_d \pmod p$ for each $(d) \in \mathcal{E}_\lambda$, such that the following hold:

- (i) $0 < \lambda \le \min\{q-2, p-1\}$,
- (ii) *for each $(d) \in \mathcal{E}_\lambda$ the renitent lines meet $\mathcal{T}$ in the same number, say t_d, of points modulo p,*
- (iii) *for each $(d) \in \mathcal{E}_\lambda$ $\quad t_d - m_d$ mod p does not depend on the choice of (d).*

Then the renitent lines with direction in $\mathcal{E}_\lambda$ are contained in a dual curve of degree λ.

Note: $\min\{|\mathcal{E}_\lambda|, \lambda\} \le \min \deg \le \lceil \sqrt{2 \cdot |\mathcal{E}_\lambda| \cdot \lambda} \rceil - 1$

Proof

First we show that the number of renitent lines is the same at each direction (d) of $\mathcal{E}_\lambda$. Let (d) and (e) denote two directions in $\mathcal{E}_\lambda$ which are sharply $(q-\lambda_d)$-uniform and sharply $(q-\lambda_e)$-uniform, respectively. Then

$$(q-\lambda_d)m_d + \lambda_d t_d \equiv |\mathcal{T}| \equiv (q-\lambda_e)m_e + \lambda_e t_e \pmod{p},$$

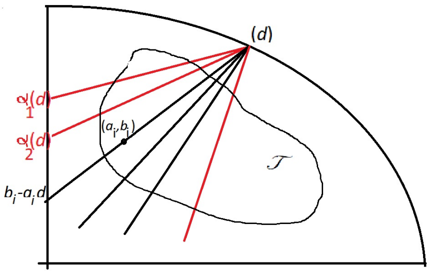
hence

$$\lambda_d(t_d - m_d) \equiv \lambda_e(t_e - m_e) \pmod{p}.$$

By *(iii)*, $t_d - m_d \equiv t_e - m_e \pmod{p}$ and $t_d \not\equiv m_d \pmod{p}$, thus $\lambda_d \equiv \lambda_e \pmod{p}$. Then $\lambda_d = \lambda_e$ as $0 \leq \lambda_e, \lambda_d \leq \lambda \leq p-1$.

$\Rightarrow$ directions in $\mathcal{E}_\lambda$ are sharply $(q-\lambda)$-uniform

$|\mathcal{E}_\lambda| \leq q$, wlog $(0:1:0) \notin \mathcal{E}_\lambda$.
For each $(1:d:0) \in \mathcal{E}_\lambda$ put $(0:\alpha_1(d):1), (0:\alpha_2(d):1), \ldots, (0:\alpha_\lambda(d):1)$ for the points of the Y-axis on the renitent lines with slope d.



$s := |\mathcal{T}|$ and $\mathcal{T} = \{(a_i : b_i : 1)\}_{i=1}^{s}$.
The line joining $(1:d:0)$ and $(a_i : b_i : 1)$ meets the Y-axis at the point $(0 : b_i - a_i d : 1)$, hence for each $(1:d:0) \in \mathcal{E}_\lambda$ the multiset

$$M_d := \{(b_i - a_i d)\}_{i=1}^{s} = (m_d \bmod p) \cdot \mathbb{F}_q \cup (c = t_d - m_d \bmod p) \cdot \{\alpha_i\}_{i=1}^{\lambda}$$

where $c \in \{1, \ldots, p-1\}$.

Proof

Define the polynomials

$$\pi_k(V) := \sum_{i=1}^{s} (b_i - a_i V)^k \in \mathbb{F}_q[V]$$

of degree at most k.

As $\sum_{\gamma \in \mathbb{F}_q} \gamma^k = 0$ for $0 \le k \le q-2$ and since $\pi_k(d)$ is the k-th power sum of M_d,

$$\pi_k(d) = c \sum_{i=1}^{\lambda} \alpha_i(d)^k \text{ for } (1 : d : 0) \in \mathcal{E}_\lambda. \tag{1}$$

$\sigma_i(X_1, \dots, X_\lambda)$: the i-th elementary symmetric polynomial in the variables $X_1, \dots, X_\lambda$.

Also, for $d \in \mathcal{E}_\lambda$ put $\sigma_i(d) = \sigma_i(\alpha_1(d), \dots, \alpha_\lambda(d))$.

For $p - 1 \ge j \ge 1$ define the following polynomial of degree at most j:

$$S_j(V) := (-1)^j \sum_{\substack{n_1+2n_2+\ldots+jn_j=j \\ n_1,n_2,\ldots,n_j>0}} \prod_{i=1}^{j} \frac{(-\pi_i(V)/c)^{n_i}}{n_i! \, i^{n_i}} \in \mathbb{F}_q[V].$$

Proof

from the Newton-Girard identities it follows that $S_j(d) = \sigma_j(d)$ for each $(1 : d : 0) \in \mathcal{E}_\lambda$.

affine curve of degree λ defined by

$$f(U,V) := U^\lambda - S_1(V)U^{\lambda-1} + S_2(V)U^{\lambda-2} - \ldots + (-1)^{\lambda-1}S_{\lambda-1}(V)U + (-1)^\lambda S_\lambda(V).$$

make it projective: $g(U,V,W) := W^\lambda f(U/W, V/W)$
it contains the point $(\alpha_i(d) : d : 1)$ for each $d \in \mathcal{E}_\lambda$ and $1 \leq i \leq \lambda$. Indeed:

$$g(U,d,1) = U^\lambda - \sigma_1(d)U^{\lambda-1} + \sigma_2(d)U^{\lambda-2} - \ldots + (-1)^{\lambda-1}\sigma_{\lambda-1}(d)U + (-1)^\lambda \sigma_\lambda(d) =$$

$$\prod_{i=1}^{\lambda}(U - \alpha_i(d)).$$

So the renitent lines $[d : -1 : \alpha_i(d)]$ are contained in an algebraic envelope of class λ.

Pushing it further

Theorem (Cs-Sz-W)

Take a multiset $\mathcal{T}$ *of* $\mathrm{AG}(2,q)$ *and let* $\mathcal{F}_\lambda$ *denote a set of* $(q-\lambda)$*-uniform directions. For each* $(d) \in \mathcal{F}_\lambda$ *denote the typical intersection number by* $m_d \pmod p$ *and denote the intersection numbers of the renitent lines by* $t_{d,1}, t_{d,2}, \ldots, t_{d,\lambda_d}$*, for some* $0 < \lambda_d \leq \lambda$*. For* $c \in \mathbb{F}_p \setminus \{0\}$ *define the integers* $\lambda_{d,i}(c) \in \{1, \ldots, p-1\}$ *such that*

$$c\lambda_{d,i}(c) \equiv t_{d,i} - m_d \pmod p$$

and assume that

$$\Lambda_d(c) := \sum_{i=1}^{\lambda_d} \lambda_{d,i}(c) \leq \min\{q-2, p-1\}$$

holds for each $(d) \in \mathcal{F}_\lambda$ *(the sum is taken over natural numbers). Then for a fixed* c*,* $\Lambda(c) := \Lambda_d(c)$ *does not depend on* d *and the renitent lines with direction in* $\mathcal{F}_\lambda$ *are contained in an algebraic envelope of class* $\Lambda(c)$*.*
Moreover, the intersection multiplicity of this envelope with the pencil centered at (d) *at a renitent line incident with* (d) *and with intersection number* $t_{d,i}$ *is* $\lambda_{d,i}(c)$*.*

Remark

Here if we assume $t_{d,1} = t_{d,2} = \ldots = t_{d,\lambda_d} =: t_d$ *for each* $(d) \in \mathcal{F}_\lambda$*, we also assume that* $t_d - m_d$ *does not depend on the choice of* d*, and further assume* $0 < \lambda \leq \min\{q-2, p-1\}$*, then with the choice* $c \equiv t_d - m_d \pmod{p}$ *we obtain the previous Theorem without the restriction* $|\mathcal{E}_\lambda| \leq q$*.*

Remark

If $|\mathcal{T}| \equiv 0 \pmod{p}$ *then this Theorem cannot be applied. Indeed, in that case for each* $(d) \in \mathcal{F}_\lambda$*,*

$$\sum_{i=1}^{\lambda_d}(t_{d,i} - m_d) \equiv 0 \pmod{p}$$

and hence $\Lambda_d = \sum_{i=1}^{\lambda_d} \lambda_{d,i} \equiv 0 \pmod{p}$*, which is not possible if* $\Lambda_d \leq p-1$ *(by definition* $\lambda_{d,i} > 0$ *for each* i*).*

examples

Example

$\lambda = 3, p \neq 2$, $m_d = 1$, assume that the renitent lines meet $\mathcal{T}$ modulo p in the multiset $\{3, 3, 5\}$ for each $(d) \in \mathcal{F}_\lambda$.
With the choice $c = 2$ it follows that the renitent lines are contained in a curve of degree $\Lambda(c) = (3-1)/2 + (3-1)/2 + (5-1)/2 = 4$ whenever $p \geq 5, q > 5$.

One might think that to obtain a curve of the lowest degree, the best option is to chose c as the greatest common divisor of the values $t_{d,i} - m_d$. not true:

Example

Put $\lambda = 2$, $p = 13$, and assume that $m_d = 1$ and the renitent lines meet $\mathcal{T}$ modulo 13 in the multiset $\{2, 8\}$ for each $(d) \in \mathcal{F}_\lambda$.
choosing $c = 1$: renitent lines are in a curve of deg $\Lambda(c) = (2-1) + (8-1) = 8$.
choosing $c = 7$: renitent lines are in a curve of deg $\Lambda(c) = 1/7 + 7/7 = 2 + 1 = 3$.

The general case

Theorem (Cs-Sz-W)

Take any $\mathcal{T} \subseteq \mathrm{AG}(2,q)$ and an integer $0 < \lambda \leq (q-1)/2$. Let $\mathcal{E}_\lambda$ denote a set of $(q-\lambda)$-uniform directions of size at most q. Then the renitent lines with slope in $\mathcal{E}_\lambda$ are contained in an algebraic envelope of class λ^2. Furthermore, if a direction of $\mathcal{E}_\lambda$ is not sharply $(q-\lambda)$-uniform, then the line pencil centered at that direction is fully contained in the envelope.

Proof: using a recursion technique (with Vandermonde-matrices and linear algebra) connecting elementary symmetric polynomials to weighted power sums

Corollary

Suppose that the assumptions of the previous theorem holds and also that there exists a sharply $(q-\lambda)$*-uniform direction. Then there are at most* $\lambda^2-\lambda$ *directions incident with less than* λ *renitent lines. More precisely, if* λ_d *denotes the number of renitent lines with slope* d *for every* $(d)\in\mathcal{E}_\lambda$*, then*

$$\sum_{(d)\in\mathcal{E}_\lambda}(\lambda-\lambda_d)\le\lambda^2-\lambda.$$

With the resultant method

Theorem (Cs-Sz-W)

Take a multiset $\mathcal{M}$ of AG$(2,q)$, *$q>2$, and fix an integer $\lambda>0$. Let $\mathcal{F}_\lambda$ denote the set of $(q-\lambda)$-uniform directions. If $|\mathcal{F}_\lambda|>\lambda^2+\lambda$ then for each point R of the plane it holds that R is incident with at most λ or with at least $|\mathcal{F}_\lambda|+1-\lambda$ renitent lines.*

If $\lambda=1$ it implies that

With the resultant method

Theorem (Cs-Sz-W)

Take a multiset $\mathcal{M}$ of $\mathrm{AG}(2, q)$, $q > 2$, and fix an integer $\lambda > 0$. Let $\mathcal{F}_\lambda$ denote the set of $(q - \lambda)$-uniform directions. If $|\mathcal{F}_\lambda| > \lambda^2 + \lambda$ then for each point R of the plane it holds that R is incident with at most λ or with at least $|\mathcal{F}_\lambda| + 1 - \lambda$ renitent lines.

If $\lambda = 1$ it implies that the renitent lines are concurrent.

Thank you for your attention!