Partial permutation decoding of the binary code of the projective plane PG$(2, q)$, q even

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OUTLINE

Outline

1. Permutation decoding of linear codes
2. Linear codes from finite projective planes
3. Basis and 2-PD-set for code of PG$(2, 2^h)$

TRANSMISSION OF INFORMATION

In coding theory,

- **messages** encoded into codewords.
- **Linear** $[n, k, d]$**-code** C **over** $\mathbb{F}_q$ is:
  - k-dimensional subspace of $V(n, q)$,
  - *minimum (Hamming) distance* d = minimal number of positions in which two distinct codewords differ.
- If $d = 2t + 1$ or $d = 2t + 2$, then C is t**-error correcting**.

Transmission of information

- **Generator matrix of** $[n, k, d]$**-code** C**:**

$$G = (g_1 \cdots g_n)$$

  - $G = (k \times n)$ matrix of rank k,
  - rows of G form basis of C,
  - codeword of C = linear combination of rows of G.
- **Message** $(u_1, \ldots, u_k)$ becomes **codeword**

$$(u_1, \ldots, u_k) \cdot G = (c_1, \ldots, c_n).$$

Generator matrix in standard form

- Generator matrix of $[n, k, d]$-code C is in **standard form** when
$$G = (I_k\ A),$$
with A a $k \times (n-k)$ matrix.
- Message $(u_1, \ldots, u_k)$ becomes codeword
$$(u_1, \ldots, u_k) \cdot G = ((u_1, \ldots, u_k), (u_1, \ldots, u_k) \cdot A).$$
- First k positions are the **information positions** and last $n-k$ positions are the **check positions**.

PERMUTATION DECODING OF LINEAR CODES

(F.J. MacWilliams)

Permutation decoding of linear codes

- Let G be the group of the permutations on the positions which leave C invariant.
- An *s*-**PD-set** of permutations is set of elements of G such that for every error vector of weight s, there exists a permutation σ in G which moves the s errors out of the information positions.

Questions:

- Does a linear code have an s-PD set?
- If yes, construct a smallest possible s-PD set.
- How do we know that the s errors are out of the information positions?

Permutation decoding of linear codes

Theorem

Let C be a t-error correcting linear $[n, k, d]$-code, with generator matrix G in standard form $G = (I_k\ A)$ and parity check matrix $H = (-A^t\ I_{n-k})$.
Let c be a transmitted codeword of C and assume that the vector $y = c + e$ is received, where e is an error vector of weight at most t.
Then the errors are outside of the information positions if and only $wt(y \cdot H^t) < t$.

Outline

Codes from projective planes

- PG$(2, q)$, $q = p^h$, p prime, $h \geq 1$.
- Points P_j, $j = 1, \ldots, q^2 + q + 1$, and lines ℓ_i, $i = 1, \ldots, q^2 + q + 1$.
- **Incidence matrix**

$$G = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \leftarrow \text{lines } \ell_i$$

$$\uparrow$$
$$\text{points } P_j$$

with
$G_{ij} = 1$ iff $P_j \in \ell_i$,
$G_{ij} = 0$ iff $P_j \notin \ell_i$.

CODE DEFINED BY THE INCIDENCE MATRIX

- G = generator matrix of $[n, k, d]$-code $C = C(2, q)$ over $\mathbb{F}_p$, with
  - $n = q^2 + q + 1$,
  - $k = \binom{p+1}{2}^h + 1$,
  - $d = q + 1$.
- Similar code arises from AG$(2, q)$, $q = p^h$, p prime, $h \geq 1$.

MOORHOUSE BASIS FOR $C(AG(2,p))$, p PRIME

- Take one line M in PG$(2, q)$.
- Let $M = \{r_0, r_1, \ldots, r_p\}$.
- Take the p lines through r_0, different from M,
- Take $p-1$ lines through r_1, different from M,
- $\cdots$,
- Take $p-i$ lines through r_i, different from M,
- $\cdots$,
- Take one line through r_{p-1}, different from M.

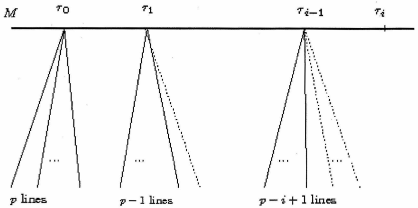


Fig. 1 The basis of Moorhouse

MOORHOUSE BASIS FOR $C(AG(2, p))$, p PRIME

- Line at infinity: $l_1 = [0, 0, 1]$.
- p lines $[1, 0, a]$, $0 \leq a \leq p - 1$, through point (0, 1, 0),
- $p - 1$ lines $[1, 1, a]$, $1 \leq a \leq p - 1$, through point $(1, -1, 0)$,
- $p - 2$ lines $[1, 2, a]$, $2 \leq a \leq p - 1$, through point $(1, -2^{-1}, 0)$,
- $p - i$ lines $[1, i, a]$, $i \leq a \leq p - 1$, through point $(1, -i^{-1}, 0)$
- $\ldots$,
- the line $[1, p - 1, p - 1]$ through the point $(1, 1, 0)$.

Moorhouse basis for $C(AG(2,p))$, p prime

Equivalent formulation: points as information set

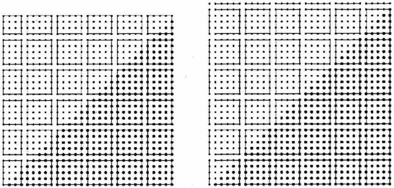
$$\begin{array}{ccccc}
(0,0) & (0,1) & (0,2) & \cdots & (0,p-1) \\
 & (1,1) & (1,2) & \cdots & (1,p-1) \\
 & & (2,2) & \cdots & (2,p-1) \\
 & & & \ddots & \\
 & & & & (p-1,p-1)
\end{array}$$

The information set I and check set H are equal to:

$$I = \{(i,j) : 0 \leq i \leq j \leq p-1\}$$

and

$$H = \{(i,j) : p-1 \geq i > j \geq 0\}.$$



A: $p = 29$.   B: $p = 31$.

Consider the translations $\tau_{a,b} : (x, y) \mapsto (x, y) + (a, b)$.

THEOREM (KEY, MACDONOUGH, MAVRON)

Let C_A be the p-ary code from the affine plane $AG(2, p)$, $p \geq 5$ prime. Let $n = \lfloor \frac{p+1}{6} \rfloor$ and let $Y = \{\tau_{un,-vn} : 0 \leq u, v \leq 5\}$. For the predefined information set I, Y is a 2-PD-set of size 36 for the code of $AG(2, p)$ when $p \equiv -1 \pmod 6$ and $Y \cup \{\tau_{1,1}\}$ is a 2-PD-set of size 37 for the code of $AG(2, p)$ when $p \equiv 1 \pmod 6$.

Outline

NOTATIONS

- Let $q = 2^h$ and let α be a primitive element of $\mathbb{F}_{2^h}$.
- Let

$$\beta = a_{h-1}\alpha^{h-1} + a_{h-2}\alpha^{h-2} + \cdots + a_1\alpha + a_0, \beta \neq 0,$$

  where all $a_i \in \mathbb{F}_2$.
- $|\beta| = |\{i : a_i \neq 0\}|$.
- **Leading position** of β: $lp(\beta) = \max\{i : a_i \neq 0\} + 1$.
- **Leading position** of point $b = (0, 1, \beta)$ is $lp(\beta)$.
- **Leading position** of $(0, 1, 0)$ is 0 and **leading position** of $(0, 0, 1)$ is $+\infty$.

Basis of P. Vandendriessche

Theorem

The line $X_0 = 0$ and the set of lines

$$\{\langle (0, 1, \beta), (1, 0, \gamma)\rangle : |\gamma| + lp(\beta) \leq h\}$$

together form a basis for code of PG$(2, 2^h)$, with $5 \leq h \leq 9$.

The line $X_0 = 0$ has homogeneous coordinates $[1, 0, 0]$. The set of lines from the previous theorem consists of lines with homogeneous coordinates $[\gamma, \beta, 1]$, where $|\gamma| + lp(\beta) \leq h$.
Question: Is this also basis for $h > 9$?

2-PD-set for code of PG$(2, 2^h)$, $5 \leq h \leq 9$.

- $$\hat{\tau}_{u,v}([\gamma, \beta, 1]) = [\gamma + u, \beta + v, 1],$$
  $$\hat{\tau}_{u,v}([1, 0, 0]) = [1, 0, 0],\ \hat{\tau}_{u,v}([\gamma, 1, 0]) = [\gamma, 1, 0].$$
- $\sigma_1 : [u, v, w] \mapsto [v, u, w]$,
- $\sigma_2 : [u, v, w] \mapsto [w, v, u]$.

2-PD-SET FOR CODE OF PG$(2, 2^h)$, $5 \leq h \leq 9$.

THEOREM (CRNKOVIĆ, MOSTARAC, RODRIGUES, STORME)

Let $\Pi = \mathrm{PG}(2, 2^h)$*,* $5 \leq h \leq 9$*, and let*
$C : [2^{2h} + 2^h + 1, 3^h + 1, 2^h + 1]_2$ *be its binary code.*
Let

$$a = (1, 0, ..., 0), a' = (0, 1, 0, ..., 0),$$

$$b = (1, ..., 1, 0), c = (1, ..., 1).$$

Then following set S *is 2-PD-set of size 16 for* C*, for the information set* I*:*

$$S = \{\hat{\tau}_{0,0}, \hat{\tau}_{a,a}, \hat{\tau}_{a,b}, \hat{\tau}_{a,c}, \hat{\tau}_{a',b}, \hat{\tau}_{b,a}, \hat{\tau}_{b,b}, \hat{\tau}_{b,c}, \hat{\tau}_{c,a}, \hat{\tau}_{c,b}, \hat{\tau}_{c,c}, \sigma_1,$$

$$\hat{\tau}_{a,b}\sigma_1, \hat{\tau}_{a,c}\sigma_1, \hat{\tau}_{b,c}\sigma_1, \hat{\tau}_{a,c}\sigma_2\}.$$

Search for these permutations

Example:

- Assume two errors in the positions $[\gamma_1, \beta_1, 1]$, with $|\gamma_1| + lp(\beta_1) \leq h$, and $[\gamma_2, \beta_2, 1]$, with $|\gamma_2| + lp(\beta_2) \leq h$.
- Find translations $\tau_{u,v}$ such that

$$(\gamma_i, \beta_i) + (u, v) = (\gamma_i + u, \beta_i + v),$$

with

$$|\gamma_i + u| + lp(\beta_i + v) > h, i = 1, 2.$$

3-PD-SET FOR CODE OF PG$(2, 2^9)$

THEOREM (CRNKOVIĆ, MOSTARAC, RODRIGUES, STORME)

Let $\Pi = \mathrm{PG}(2, q)$, $q = 2^h$, and let G be its automorphism group. Furthermore, let $C_{\mathrm{gen}} = [q^2 + q + 1, 3^h + 1, q + 1]_2$ be the binary code of Π. If $h = 9$, a 3-PD-set for C_{gen} consisting of 75 elements can be found in G, for the information set I.

Thank you very much for your attention!