Designs in finite general linear groups

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(joint work with Alena Ernst)

Combinatorics: A big picture

sets

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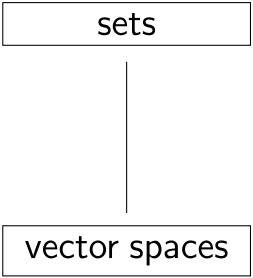

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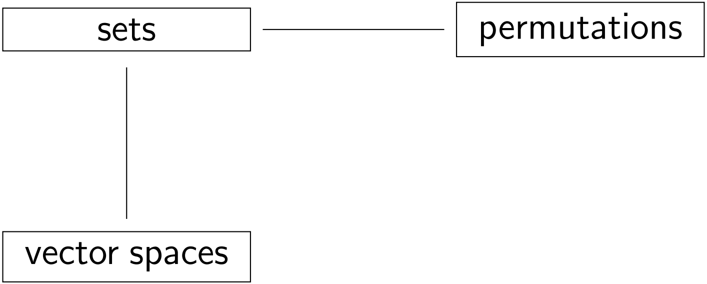

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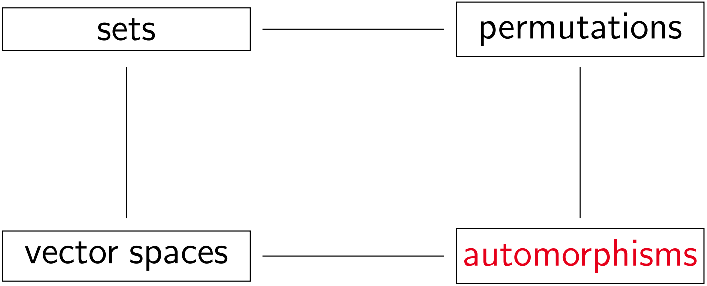

Coding and design theory

Coding Theory

Design Theory

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Coding Theory	Design Theory
Separation	Approximation

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Separation	Approximation
ordinary codes	orthogonal arrays
constant weight codes	combinatorial designs
subspace codes	subspace designs
permutation codes	transitive sets
⋮	⋮

Coding and design theory

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Common framework: association schemes

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Association scheme: A finite set X and a finite-dimensional subspace of $\mathbb{C}^{X \times X}$.

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D-code
$$A_i \neq 0 \Rightarrow i \in D$$

T-design
$$k \in T \Rightarrow A'_k = 0$$

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This motivates the present general definition [of T-designs], the "conjecture" being that T-designs will often have interesting properties.

— Delsarte's Thesis, 1973

Finite groups & association schemes

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The dual distribution of a subset Y of G then satisfies

$$A'_k = \chi_k(1) \sum_{x,y \in Y} \chi_k(x^{-1}y).$$

Designs in the symmetric group

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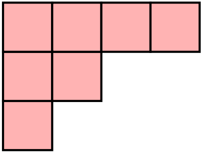

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σ-tabloid

A subset Y of S_n is transitive on σ-tabloids if the number of $\pi \in Y$ such that

a_1	a_2	a_3	a_4
a_5	a_6		
a_7			

$\xrightarrow{\pi}$

b_1	b_2	b_3	b_4
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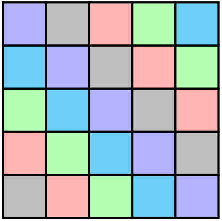
is independent of the two σ-tabloids.

Designs in S_5

transitive on (41)-tabloids



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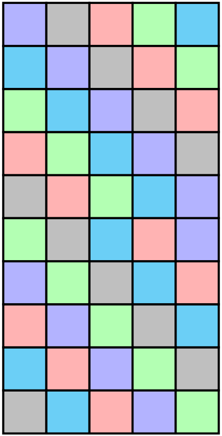

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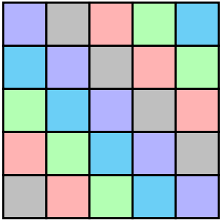

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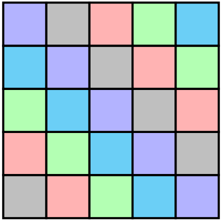

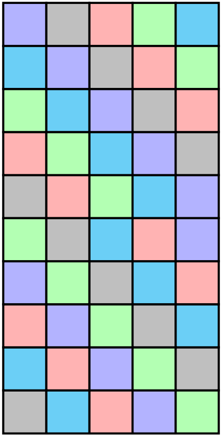

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Partitions and the symmetric group

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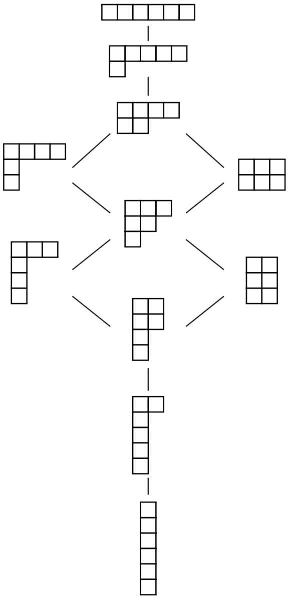
The partitions are partially ordered by dominance $\trianglelefteq$.



Characterisation of designs in S_n

Theorem (Martin-Sagan 2007) $Y \subseteq S_n$ is transitive on σ-tabloids if and only if $A'_\lambda = 0$ for all $\sigma \trianglelefteq \lambda \triangleleft (n)$.

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This implies the Livingstone-Wagner Theorem since

$$(n-t, t) \trianglelefteq (n-t+1, t-1) \quad \text{for } 1 \leq t \leq n/2.$$

Compositions and flags

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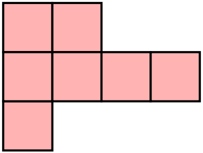

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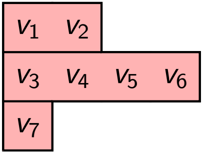

σ-flag

$$V_1 = \langle v_1, v_2 \rangle$$
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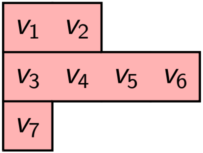

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Our interest: subsets of $\mathrm{GL}(n, q)$ that are transitive on σ-flags.

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The conjugacy classes and the (complex) irreducible characters of $\mathrm{GL}(n, q)$ are naturally indexed by mappings

$$\underline{\lambda} : \{\text{monic irr. polynomials in } \mathbb{F}_q[X]\} \setminus \{X\} \to \text{Partitions}$$

of finite support such that

$$\sum_f \deg(f)\, |\underline{\lambda}(f)| = n.$$

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Signed flags

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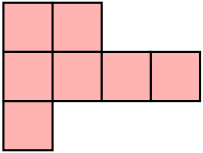

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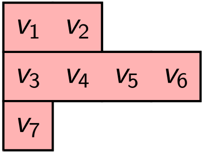

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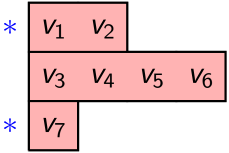

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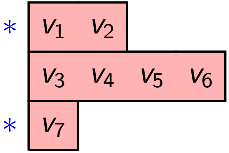

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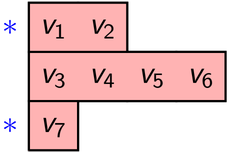

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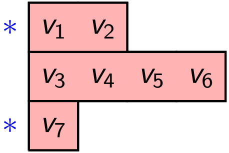	$V_1 = \langle v_1, v_2 \rangle$	(v_1, v_2)
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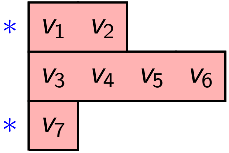

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Two special cases:

$(t, n-t)$-flag with signature $(\circ\circ)$: t-space of V.
$(t, n-t)$-flag with signature $(*\circ)$: basis of a t-space of V.

The general characterisation

For $q = 3$, let $\underline{\lambda}$ be given by

$$\begin{array}{cccc} X - 1 & X + 1 & X^2 + 1 & X^2 + X - 1 \\ (31) & (33) & (2) & (21) \end{array}$$

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Theorem (Ernst-S. 2022) A subset $Y \subseteq \mathrm{GL}(n, q)$ is transitive on signed flags of type (σ, τ) with fixed signature if and only if $A'_{\underline{\lambda}} = 0$ for all

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This has similar consequences as the result for unsigned flags.

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If G is a subgroup of $GL(n, q)$ with stronger transitivity and does not contain $SL(n, q)$, then one of the following holds:

(n, q)	group	(σ, τ)
$(2, 3)$	$G \cong \Gamma L(1, 3^2)$	$((1^2), \emptyset)$
$(2, 5)$	$\lvert G \rvert = 96$	$((1^2), \emptyset)$
$(3, 2)$	$G \cong \Gamma L(1, 2^3)$	$((1^3), \emptyset)$
$(4, 2)$	$G \cong A_7$	$((31), \emptyset)$
$(5, 2)$	$G \cong \Gamma L(1, 2^5)$	$(\emptyset, (32))$

Transitive subgroups of $\mathrm{GL}(n, q)$

A Singer cycle is sharply transitive nonzero points of $\mathbb{F}_q^n$ and it contains a subgroup that is sharply transitive on 1-spaces.

If G is a subgroup of $\mathrm{GL}(n, q)$ with stronger transitivity and does not contain $\mathrm{SL}(n, q)$, then one of the following holds:

(n, q)	group	(σ, τ)
$(2, 3)$	$G \cong \Gamma\mathrm{L}(1, 3^2)$	$((1^2), \emptyset)$
$(2, 5)$	$\lvert G \rvert = 96$	$((1^2), \emptyset)$
$(3, 2)$	$G \cong \Gamma\mathrm{L}(1, 2^3)$	$((1^3), \emptyset)$
$(4, 2)$	$G \cong A_7$	$((31), \emptyset)$
$(5, 2)$	$G \cong \Gamma\mathrm{L}(1, 2^5)$	$(\emptyset, (32))$

$\mathrm{SL}(n, q)$ is transitive on bases of $(n - 1)$-spaces.

A recursive construction

t-design in $\mathrm{GL}(V)$: Transitive on bases of t-spaces of $V = \mathbb{F}_q^n$.

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Theorem (Ernst-S. 2022) Write $V = U \oplus W$ with $\dim U = k$.

If there is a (1) a t-design Y in $\mathrm{GL}(U)$,
(2) a t-design Z in $\mathrm{GL}(W)$,
(3) a t-design D in $J_q(n, k)$,

then there is a t-design in $\mathrm{GL}(V)$ of size $|Y| \cdot |Z| \cdot |D| \cdot q^{k(n-k)}$.

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Example. Taking $Y = Z \cong \mathrm{GL}(3,2)$ and D a 2-design in $J_2(6,3)$ of size 279 gives a 2-design in $\mathrm{GL}(6,2)$ of size $\frac{1}{5}|\mathrm{GL}(6,2)|$.

An existence result

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Through our algebraic characterisation this also gives similar existence results for subsets of $\mathrm{GL}(n, q)$ that are transitive on signed flags.

Designs in finite general linear groups

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