Minimum size linear sets

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Linear sets: Definition

$$\Lambda = \mathrm{PG}(d, q^n) = \mathrm{PG}(W, \mathbb{F}_{q^n}) \quad W = V(d+1, q^n)$$

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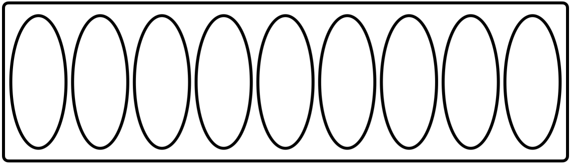
$\mathrm{PG}((d+1)n-1, q)$



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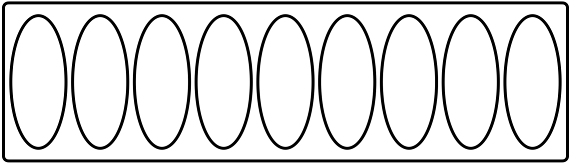
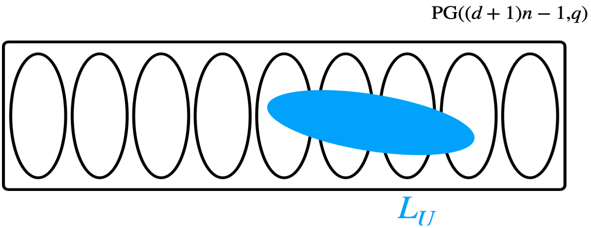


$\mathcal{S} = \{\langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} : \mathbf{u} \in W^*\}$ is a Desarguesian spread of $\mathrm{PG}(W, \mathbb{F}_q)$

Linear sets: Definition

U : $\mathbb{F}_q$-subspace of W



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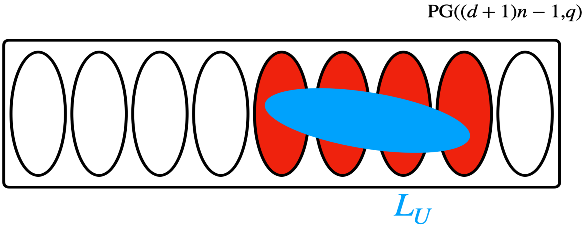

$$L_U = \{\langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} : \mathbf{u} \in U \setminus \{\mathbf{0}\}\}$$

L_U is said $\mathbb{F}_q$-**linear set** of $\Lambda = \mathrm{PG}(d, q^n)$

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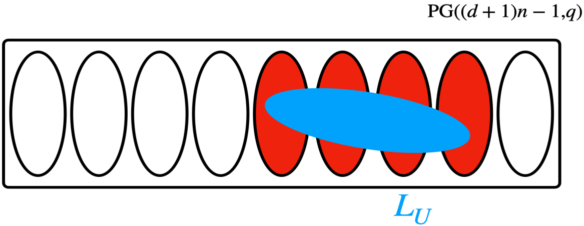

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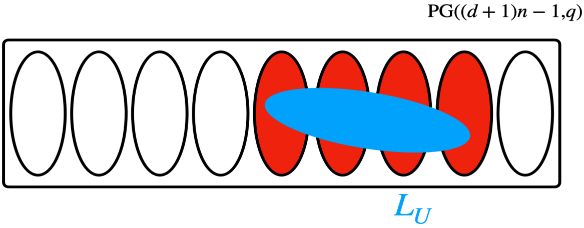

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$$k = \dim_{\mathbb{F}_q}(U) \leq dn$$

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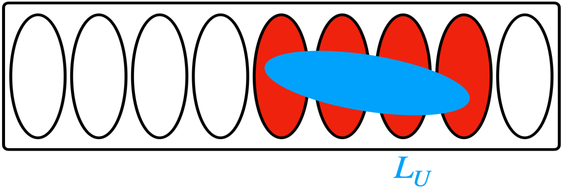


$$k = \dim_{\mathbb{F}_q}(U) \leq dn$$

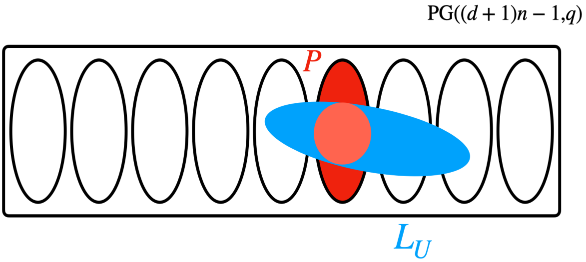
$$1 \leq |L_U| \leq \frac{q^k - 1}{q - 1}$$

Linear sets: weight of a point

$$P = \langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} \in \Lambda = \mathrm{PG}(d, q^n)$$

The **weight of P in L_U** is

$$w_{L_U}(P) := dim_{\mathbb{F}_q}(U \cap \langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}}) \leq k$$



Weight spectrum and Weight distribution

weight spectrum of L_U linear set in $\mathrm{PG}(d, q^n)$ of rank $k \leq n$

$$(w_1, \ldots, w_t)$$

$1 \leq w_1 < w_2 < \ldots < w_t \leq k$ and for every $P \in L_U$

$$w_{L_U}(P) \in \{w_1, \ldots, w_t\}$$

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weight distribution of L_U

$$(N_{w_1}, \ldots, N_{w_t})$$

N_{w_i} the number of points of L_U having weight w_i.

Weight spectrum and Weight distribution

weight spectrum of L_U linear set in $\mathrm{PG}(1,q^n)$ of rank k

$$(w_1,\ldots,w_t)$$

weight distribution of L_U

$$(N_{w_1},\ldots,N_{w_t})$$

$$|L_U| = N_{w_1} + \ldots + N_{w_t},$$

$$\sum_{i=1}^{t} N_{w_i}(q^{w_i}-1) = q^k - 1.$$

Minimum size linear sets on the projective line

$$\ell = \mathrm{PG}(1, q^n)$$

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Theorem (De Beule and Van de Voorde)

If L_U is an $\mathbb{F}_q$-linear set of rank k in $\mathrm{PG}(1, q^n)$ with at least one point of weight one $\Rightarrow$

$$|L_U| \geq q^{k-1} + 1.$$

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If $|L_U| = q^{k-1} + 1 \Rightarrow L_U$ of **minimum size**

Trace example

$$U = \{(x, \mathrm{Tr}_{q^n/q}(x))\colon x \in \mathbb{F}_{q^n}\},$$
$$\mathrm{Tr}_{q^n/q}(x) = x + x^q + \ldots + x^{q^{n-1}}$$
$$|L_U| = q^{n-1} + 1$$

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weight spectrum $(1, n-1)$
weight distribution $(q^{n-1}, 1)$

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If $L_{U'}$ has rank n and there exists P s.t. $w_{L_{U'}}(P) = n-1 \Rightarrow L_{U'}$ is equivalent to L_U

Minimum size linear sets on the projective line

Theorem (Lunardon and Polverino)

$$\ell = \mathrm{PG}(1, q^n), \quad \mathbb{F}_q(\lambda) = \mathbb{F}_{q^n}, \quad n \geq 3$$

$$U = \langle 1, \lambda \rangle_{\mathbb{F}_q} \times \langle 1, \lambda, \ldots, \lambda^{n-3} \rangle_{\mathbb{F}_q} \subset \mathbb{F}_{q^n} \times \mathbb{F}_{q^n}$$

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$n > 4$

weight spectrum $(1, 2, n-2)$

weight distribution $(q^{n-1} - q^{n-3}, q^{n-3}, 1)$

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$n = 4$

weight spectrum $(1, 2)$

weight distribution $(q^3 - q, q + 1)$

Minimum size linear sets on the projective line

Definition

$$b \in \mathbb{F}_{q^n}^*, \quad \mathbb{F}_q(\lambda) = \mathbb{F}_{q^s} \leq \mathbb{F}_{q^n}$$

$$S = b\langle 1, \lambda, \ldots, \lambda^{t-1} \rangle_{\mathbb{F}_q} \quad t \leq s$$

S of *polynomial type* w.r.t. the element λ

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Example

$$U = \langle 1, \lambda \rangle_{\mathbb{F}_q} \times \langle 1, \lambda, \ldots, \lambda^{n-3} \rangle_{\mathbb{F}_q} \subset \mathbb{F}_{q^n} \times \mathbb{F}_{q^n}$$

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Theorem (Jena and Van de Voorde)

$$\ell = \mathrm{PG}(1, q^n), \quad \mathbb{F}_q(\lambda) = \mathbb{F}_{q^s}, \quad 1 < s \leq n$$
$$1 \leq t_1 \leq t_2, \quad t_1 + t_2 \leq s + 1$$

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$$dim_q U = k = t_1 + t_2$$
$$|L_U| = q^{k-1} + 1 = q^{t_1 + t_2 - 1} + 1$$

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- $s = n$, $t_1 = 1, t_2 = n - 1 \Rightarrow$ Trace example

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- $s = n, \ t_1 = 1, t_2 = n - 1 \Rightarrow$ Trace example
- $s = n, \ t_1 = 2, t_2 = n - 2 \Rightarrow$ Lunardon-Polverino construction

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If $t_1 < t_2$
weight spectrum $(1, 2, .., i, .., t_1, t_2)$
weight distribution
$(q^{k-1} - q^{k-3}, ..., .., q^{k-2i+1} - q^{k-2i-1}, ..., q^{t_2 - t_1 + 1}, 1)$
If $t_1 = t_2$
weight spectrum $(1, 2, .., i, .., t_1)$
weight distribution
$(q^{k-1} - q^{kn-3}, q^{k-3} - q^{k-5}, .., q^{k-2i+1} - q^{k-2i-1}, .., q + 1)$

V. Napolitano, O. Polverino, PS and F. Zullo: Classifications and constructions of minimum size linear sets. ArXiv.

A classification result: n prime

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If there exist $P, Q \in L_U$ with $P \neq Q$ such that

$$w_{L_U}(P) + w_{L_U}(Q) = k$$

L_U is a linear set with two points of complementary weights

A classification result: n prime

Theorem (Napolitano, Polverino, PS and Zullo)

$L_U \subset \ell = \mathrm{PG}(1, q^n)$, n prime $rank(L_U) = k \leq n$

L_U with points of complem. weights t_1 and t_2, $(t_1 \leq t_2)$.

$$U = S \times T$$

$$dim_{\mathbb{F}_q} S = t_1, \quad dim_{\mathbb{F}_q} T = t_2$$

If $N_{t_1} \geq q^{t_2 - t_1} + 2$ then
$|L_U| = q^{k-1} + 1$ and S and T are of polynomial type with respect to the same element λ.

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Main Tools:

- q-analog of Vosper's theorem (Bachoc, Serra and Zémor)
- q-analog of Cauchy-Davenport inequality (Bachoc, Serra and Zémor)

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n prime $\Longrightarrow$ L_U is of polynomial type

A lifting construction

$$\textbf{n=gt}, \quad g, t \geq 2$$

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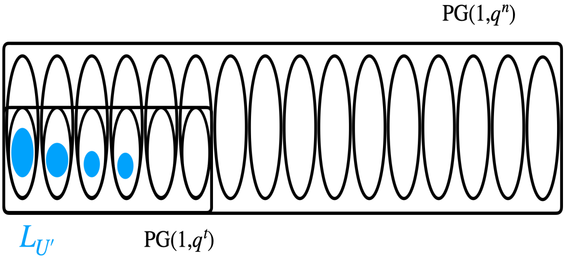

A lifting construction

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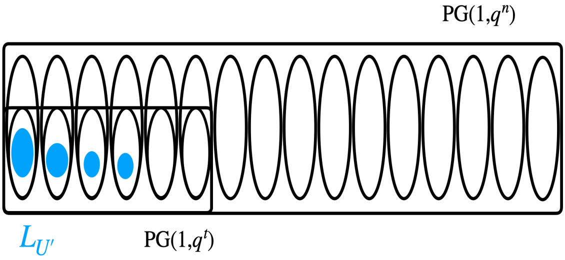
$$\mathbb{F}_{q^t} \leq \mathbb{F}_{q^n}, \quad \overline{S} \leq_{\mathbb{F}_{q^t}} \mathbb{F}_{q^n}, \quad dim_{\mathbb{F}_{q^t}} \overline{S} = h < n/t$$
$$b \in \mathbb{F}_{q^n}^* \quad : \quad \overline{S} \cap b\mathbb{F}_{q^t} = \{0\}.$$



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$$\mathbf{U} = \{(\mathbf{s} + \mathbf{bu_1}, \mathbf{u_2}) \colon \mathbf{s} \in \overline{\mathbf{S}}, (\mathbf{u_1}, \mathbf{u_2}) \in \mathbf{U}'\} \subset \mathbb{F}_{q^n} \times \mathbb{F}_{q^n}$$

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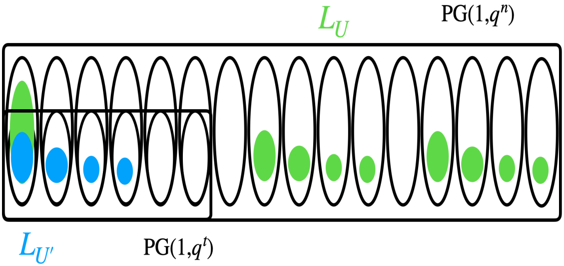

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$L_{U'} \subseteq \mathrm{PG}(1, q^t)$ of rank m

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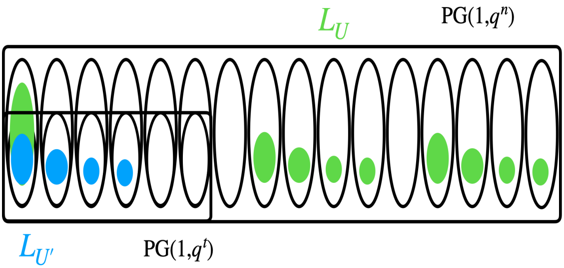
$L_U \subseteq \mathrm{PG}(1, q^n)$ has rank $k = ht + m$.



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- $\forall P \in L_U \setminus \{\langle(1,0)\rangle_{\mathbb{F}_{q^n}}\}$, we have $w_{L_U}(P) \in \{w'_1, \ldots, w'_c\}$;

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$L_U \subseteq \mathrm{PG}(1, q^n)$ has rank $k = ht + m$.

Theorem (Napolitano, Polverino, PS and Zullo)

- $w_{L_U}(\langle (1, 0)\rangle_{\mathbb{F}_{q^n}}) = ht + w_{L_{U'}}(\langle (1, 0)\rangle_{\mathbb{F}_{q^t}})$;
- $\forall P \in L_U \setminus \{\langle (1, 0)\rangle_{\mathbb{F}_{q^n}}\}$, we have $w_{L_U}(P) \in \{w'_1, \ldots, w'_c\}$;
- $N_{w'_i} = q^{ht} N'_{w'_i}$, for any $i \in \{1, \ldots, c\}$.

A lifting construction

$L_{U'} \subseteq \mathrm{PG}(1, q^t)$ of rank m
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- $N_{w'_i} = q^{ht} N'_{w'_i}$, for any $i \in \{1, \ldots, c\}$.
- $|L_U| = q^{ht}(|L_{U'}| - \varepsilon) + 1$, where $\varepsilon \in \{0, 1\}$;

A lifting construction

Theorem (Napolitano, Polverino, PS and Zullo)

$L_{U'} \subseteq \mathrm{PG}(1, q^t)$ *of rank* $m = t_1 + t_2$ *of minimum size*

$$U' = \langle 1, \mu, \ldots, \mu^{t_1-1} \rangle_{\mathbb{F}_q} \times \langle 1, \mu, \ldots, \mu^{t_2-1} \rangle_{\mathbb{F}_q}$$

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1. $|L_U| = q^{ht+m-1} + 1 \quad \Longrightarrow \quad L_U$ *of minimum size;*
2. $t_2 \leq t_1 + 1$

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$L_U \subseteq \mathrm{PG}(1, q^n)$ *of rank* $k = \ell t + m$ *obtained lifting* $L_{U'}$

1. $|L_U| = q^{ht+m-1} + 1 \implies$ L_U *of minimum size;*
2. $t_2 \leq t_1 + 1 \implies$ *w. spec.* $(1, 2, .., i, ..t_2, k - t_2)$
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$$A \in \mathrm{GL}(d+2, q^t) \quad \phi_A : \mathbb{F}_{q^t}^2 \to \mathbb{F}_{q^t}^2,$$

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$|L_U| = q^{ht+m-1} + 1 \quad \Longrightarrow \quad$ *L_U of minimum size;*

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- *"new" (with respect to the action of $\Gamma\mathrm{L}(2, q^n)$);*
- *"new" weight spectrum;*
- *"new" minimum size linear set not admitting points of complementary weights.*

A classification result

Theorem(Napolitano, Polverino, PS and Zullo)

$$L_U \subset \ell = \mathrm{PG}(1, q^n), \quad rank(L_U) = k \leq n$$

L_U of minimum size with a point of weight $k-2$

$$\swarrow \qquad \searrow$$

polynomial type ?

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polynomial type lifting construction

S. Adriaensen and PS. Minimum size linear sets in projective spaces.
Ongoing project

$$\Lambda = \mathrm{PG}(d, q^n)$$

Theorem (De Beule and Van de Voorde)

If L_U spans the entire space and there is at least one hyperplane π of $\mathrm{PG}(d, q^n)$ meeting L_U in a canonical subgeometry of π

$$\Downarrow$$

$$|L_U| \geq q^{k-1} + q^{k-2} + \ldots + q^{k-d} + 1.$$

Theorem (Jena and Van de Voorde)

$$\Lambda = \mathrm{PG}(d, q^n), \quad \mathbb{F}_q(\lambda) = \mathbb{F}_{q^s}, \quad 1 < s \leq n$$
$$1 \leq t_1, \ldots, t_{d+1} \quad t_i + t_j \leq s + 1$$
$$U = \langle 1, \lambda, \ldots, \lambda^{t_1 - 1} \rangle_{\mathbb{F}_q} \times \ldots \times \langle 1, \lambda, \ldots, \lambda^{t_{d+1} - 1} \rangle_{\mathbb{F}_q}$$
$$dim_q U = k = t_1 + \ldots + t_{d+1}$$

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$$dim_q U = k = t_1 + \ldots + t_{d+1}$$

$$\Downarrow$$

$$|L_U| = q^{k-1} + \ldots + q^{k-d} + 1$$

If $k \leq s + d \Rightarrow$ there is at least one hyperplane π of $\mathrm{PG}(d, q^n)$ meeting L_U in a canonical subgeometry of π

Example (Adriaensen and PS)

$$\Lambda = \mathrm{PG}(3, q^8), \quad n = 8, \quad \mathbb{F}_{q^2}(\lambda) = \mathbb{F}_{q^8}$$

$$U_1 = \langle 1, \lambda \rangle_{\mathbb{F}_{q^2}} \times \langle 1, \lambda \rangle_{\mathbb{F}_{q^2}} \subseteq \mathbb{F}_{q^8} \times \mathbb{F}_{q^8}$$

$$U = U_1 \times \mathbb{F}_q^2 \subseteq \mathbb{F}_{q^8}^4 \qquad L_U \subseteq \mathrm{PG}(3, q^8)$$

$$dim_q U = k = 10$$

$$|L_U| = q^{k-1} + q^{k-2} + q^{k-4} + 1 = q^9 + q^8 + q^6 + 1$$
$$< q^{k-1} + q^{k-2} + q^{k-3} + 1 = q^9 + q^8 + q^7 + 1$$

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Theorem (Adriaensen and PS)

$\Lambda = \mathrm{PG}(d, q^n)$ $\quad L_U \subseteq \Lambda$ of rank k

If there exists an $(r-1)$-dimensional space η of $\mathrm{PG}(d, q^n)$ meeting L_U in a canonical subgeometry of η

$$\Downarrow$$

$$|L_U| \geq q^{k-1} + q^{k-2} + \ldots + q^{k-r} + l_\eta,$$

where l_η is the number of full secant r-spaces through η.

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where l_P is the number of secant lines to L_U through P.

$$P \in L_U \quad w_{L_U}(P) = 1 \quad \Rightarrow \quad |L_U| \geq q^{k-1} + I_P,$$

where I_P is the number of secant lines to L_U through P.

L_U $\mathbb{F}_q$-linear blocking set in $\mathrm{PG}(2, q^n)$ $\quad P \in L_U \quad w_{L_U}(P) = 1$

n prime

$$\Downarrow$$

$$|I_P| \geq q^{n-1} + 1 \quad |L_U| \geq q^n + q^{n-1} + 1$$

Theorem (Adriaensen and PS)

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$$U = U_1 \times \mathbb{F}_q^2 \subseteq \mathbb{F}_{q^8}^4 \qquad L_U \subseteq \mathrm{PG}(3, q^8)$$

$$dim_q U = k = 10$$

$$(x_1, x_2, x_3, x_4) \quad \ell : x_1 = x_2 = 0$$

$$\Downarrow$$

$$r = 2 \qquad l_\ell = q^6 + 1$$

$$|L_U| = q^{k-1} + q^{k-2} + q^{k-4} + 1 = q^9 + q^8 + q^6 + 1$$
$$= q^{k-1} + q^{k-2} + l_\ell = q^9 + q^8 + l_\ell$$

Thank you for your attention!