LINEAR CODES, ARCS, BLOCKING SETS AND THE MAIN PROBLEM IN CODING THEORY

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1. The Main Problem in Coding Theory

Given the positive integers k, d, and the prime power q, find the smallest value of n for which there exists a linear $[n, k, d]_q$-code. This value is denoted by $n_q(k, d)$.

The Griesmer bound:

$$n_q(k, d) \geq g_q(k, d) := \sum_{i=0}^{k-1} \lceil \frac{d}{q^i} \rceil$$

Griesmer code: an $[n, k, d]_q$-code with $n = g_q(k, d)$.

Theorem. Given the integer k and the prime power q, Griesmer $[g_q(k,d),k,d]_q$-codes exist for all sufficiently large d.

Define $\delta(k,q)$ by the following conditions:

- $n_q(k,d) = g_q(k,d)$ for all $d > \delta(k,q)$.
- $n_q(k,d) > g_q(k,d)$ for $d = \delta(k,q)$.

It is known that $\delta(k,q) \leq (k-2)q^{k-1}$.

It is conjectured that $\delta(k,q) = (k-2)q^{k-1} - (k-1)q^{k-2}$.

This is proved in some special cases.

2. The Equivalence of Linear codes and Minihypers

A **multiset** of points in $\mathrm{PG}(r,q)$ is a mapping $\mathcal{K} : \mathcal{P} \to \mathbb{N}_0$.

An (n,w)-**blocking set** (or, (n,w)-**minihyper**) in $\mathrm{PG}(r,q)$ is a multiset with

(i) $\mathcal{K}(\mathcal{P}) = n$,

(ii) $\mathcal{K}(H) \geq w$ for every hyperplane H in $\mathrm{PG}(r,q)$, and

(iii) $\mathcal{K}(H_0) = w$ for at least one hyperplane H_0.

Write the minimum distance d as

$$(*) \quad d = sq^{k-1} - a_{k-2}q^{k-2} - \ldots - a_1 q - a_0,$$

where $0 \le a_i < q$ and $s = \lceil \frac{d}{q^{k-1}} \rceil$.

Then existence of a Griesmer $[n, k, d]_q$-code is equivalent to that of a minihyper in $\mathrm{PG}(k-1, q)$ with parameters

$$(**) \quad (a_{k-2}v_{k-1} + \ldots + a_1 v_2 + a_0 v_1, a_{k-2}v_{k-2} + \ldots + a_1 v_1 + a_0 v_0)$$

and maximal point multiplicity s. Here, $v_i = (q^i - 1)/(q-1)$.

Minihypers with these parameters without restriction on s will be called Griesmer minihypers.

Example.

- Assume we want to construct a Griesmer $[139, 4, 104]_4$-code.

$$104 = 2 \cdot 4^3 - 1 \cdot 4^2 - 2 \cdot 4, \text{ i.e. } a_2 = 1, a_1 = 2.$$

  Equivalent to a $(v_3 + 2v_2, v_2 + 2v_1) = (31, 7)$-minihyper in $\mathrm{PG}(3, 4)$ with maximal point multiplicity $s = 2$.

- Assume we want to construct a Griesmer $[50, 4, 40]_4$-code.

$$40 = 1 \cdot 4^3 - 1 \cdot 4^2 - 2 \cdot 4, \text{ i.e. } a_2 = 1, a_1 = 2.$$

  Equivalent to a $(v_3 + 2v_2, v_2 + 2v_1) = (31, 7)$-minihyper in $\mathrm{PG}(3, 4)$ with maximal point multiplicity $s = 1$.

3. Standard Constructions

I. Minihypers in $\mathrm{PG}(k-1,q)$ with parameters

$$\left(\sum_{i=0}^{k-2} a_i v_{i+1}, \sum_{i=0}^{k-2} a_i v_i\right)$$

always can be constructed as the sum of a_{k-2} hyperplanes, a_{k-3} hyperlines and so on.

Minihypers obtained in this way are called **canonical**.

Remark. Although minihypers with the above parameters exist, the related Griesmer codes do not necessarily exist since the minihypers should also have a maximal point multiplicity $s = \lceil d/q^{k-1} \rceil$.

II. Given a minihyper $\mathcal{F}$ in $\mathrm{PG}(k-1,q)$ with parameters

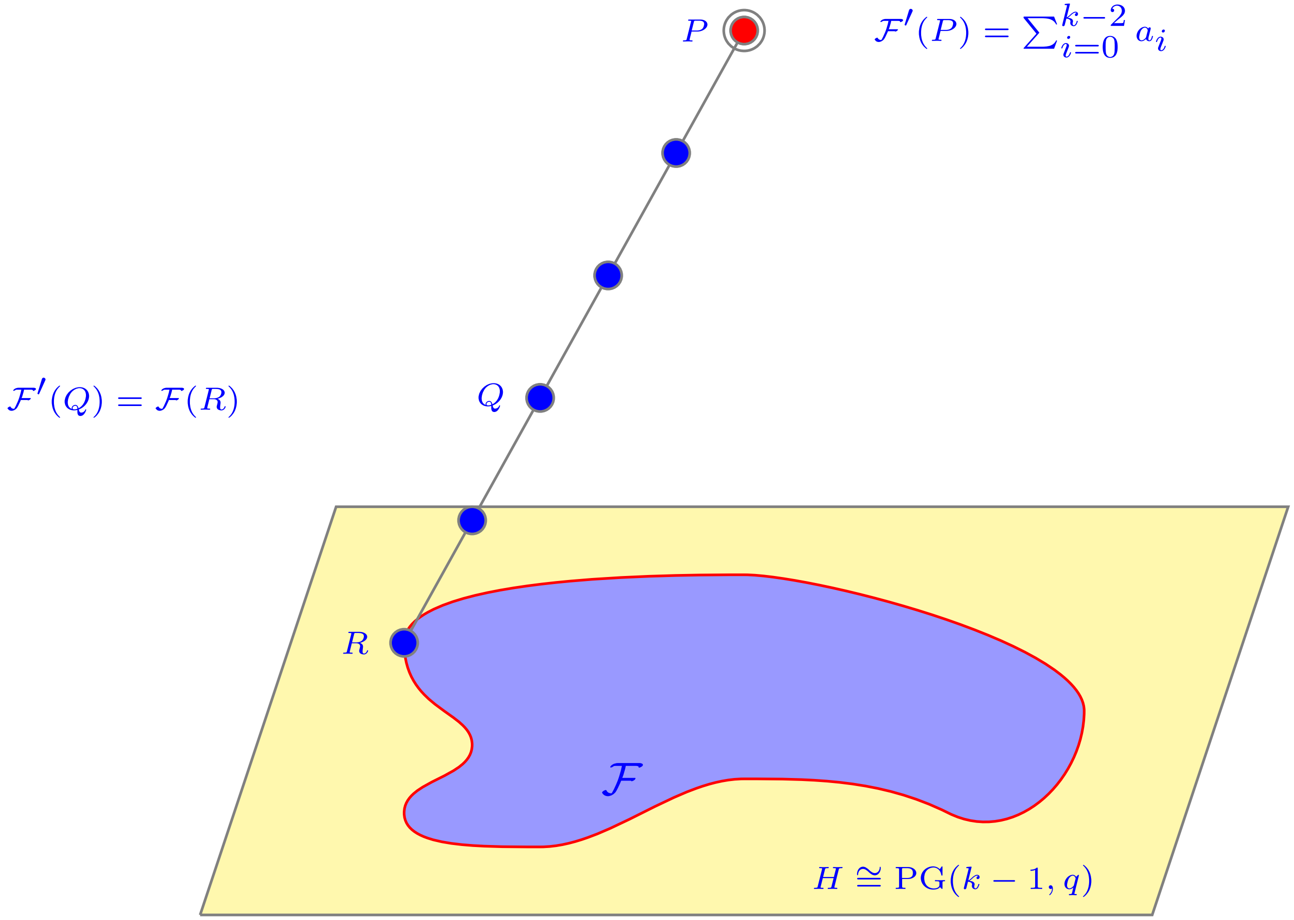
$$(\sum_{i=0}^{k-2} a_i v_{i+1}, \sum_{i=0}^{k-2} a_i v_i)$$

we can construct a munihyper $\mathcal{F}'$ in $\mathrm{PG}(k,q)$ in the following way:

- Fix a hyperplane H in $\mathrm{PG}(k,q)$ and take a copy of $\mathcal{F}$ in H.
- Let $P \in \mathrm{PG}(k,q) \setminus H$ and set $\mathcal{F}'(P) = \sum a_i$.
- For every point $Q \neq P$ set

$$\mathcal{F}'(Q) = \mathcal{F}(R),$$

where $R = H \cap \langle P, Q \rangle$.



P
$\mathcal{F}'(P) = \sum_{i=0}^{k-2} a_i$
$\mathcal{F}'(Q) = \mathcal{F}(R)$
Q
R
$\mathcal{F}$
$H \cong \mathrm{PG}(k-1,q)$

Theorem. The minihyper $\mathcal{F}'$ in $\mathrm{PG}(k,q)$ constructed above has parameters

$$(\sum_{i=0}^{k-2} a_i v_{i+2}, \sum_{i=0}^{k-2} a_i v_{i+1})$$

Minihypers obtained by this construction are called **lifted** minihypers.

Propositon. A lifted minihyper of a canonical minihyper is also canonical.

Example. (cont.)

Consider $(v_3 + 2v_2, v_2 + 2v_1) = (31, 7)$-minihypers in $\mathrm{PG}(3, 4)$.

Construction I gives the sum of a plane and two lines.

Construction II uses a $(v_2 + 2v_1, v_1 + 2v_0) = (7, 1)$-blocking set in $\mathrm{PG}(2, 4)$.

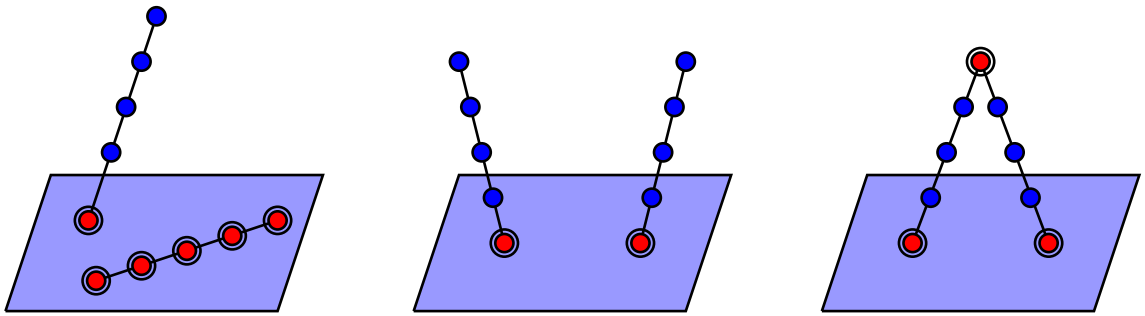
- If the blocking set is a line plus two points we get a canonical minihyper.

- If the blocking set is a Baer subplane we get a lifted non-cannonical minihyper.

In both subcases Construction II gives a point of multiplicity 3.

It can be proved that these are the only $(31, 7)$-minihypers in $\mathrm{PG}(3, 4)$.

$s = 2$



- A Griesmer code with parameters $[50, 4, 40]_4$ does not exist. $(s = 1)$

- There exist three (non-isomorphic) Griesmer $[139, 4, 104]_4$-codes. $(s = 2)$

- There exists nine (non-isomorphic) Griesmer $[224, 4, 168]_4$-codes. $(s = 3)$

Problem. Find conditions on the numbers a_i and on the maximal point multiplicity s that force a minihyper with the parameters

$$(**) \quad (a_{k-2}v_{k-1} + \ldots + a_1v_2 + a_0v_1, a_{k-2}v_{k-2} + \ldots + a_1v_1 + a_0v_0)$$

to be canonical or lifted.

4. Related results

- N. Hamada, T. Helleseth, M. Deza – various characterization results for minihypers mainly for the projective case
  - Graphs and Combinatorics (1989)
    $(v_{\mu+1} + 2v_\mu, v_\mu + 2v_{\mu-1})$ in $\mathrm{PG}(t, q)$, $t \geq 3$, $q \geq 5$, $2\mu \leq t$.
  - Discrete Mathematics (1989)
    $(v_{\alpha+1} + v_{\beta+1}, v_\alpha + v_\beta)$ in $\mathrm{PG}(t, q)$, $t \geq \alpha + \beta + 1$, $q \geq 3$.
    $(v_{\alpha+1} + v_{\beta+1} + v_{\gamma+1}, v_\alpha + v_\beta + v_\gamma)$ in $\mathrm{PG}(t, q)$, $\alpha < \beta < \gamma$,$t \geq \beta + \gamma + 1$, $q \geq 3$.
    $(2v_{\alpha+1} + 2v_{\beta+1}, 2v_\alpha + 2v_\beta)$ in $\mathrm{PG}(t, q)$, $\alpha < \beta$, $t \geq 2\beta + 1$, $q \geq 5$.
  - Discrete Mathematics (1992)
    $(v_{\gamma+1} + 2v_{\alpha+1}, v_\gamma + 2v_\alpha)$ in $\mathrm{PG}(k-1, q)$, $0 \leq \alpha < \gamma < k-1$, $k \geq 3$.

- Discrete Mathematics (1996)
  $(3v_{\mu+1}, 3v_\mu)$ in $\mathrm{PG}(k-1, q)$, $k \geq 3$, $q \geq 5$.
- J. Stat. Plann. and Inference (1996)
  uniqueness of $[87, 5, 57]_3$:
  $(2v_3 + 2v_2, 2v_2 + 2v_1) = (34, 10)$ in $\mathrm{PG}(4, 3)$.
  existence of $[258, 6, 171]_3$:
  $(2v_4 + 2v_3, 2v_3 + 2v_2) = (106, 34)$ in $\mathrm{PG}(5, 3)$.

- S. Ball, R. Hill, I. Landjev, H. N. Ward (2001)

  Characterization of certain $(q^2+q+2, q+2)$-arcs in $\mathrm{PG}(2, q)$. i.e. (qv_2, qv_1)-minihypers with maximal point multiplicity $s = 2$.

- T. Maruta (2004)

  Characterization of $(4v_3, 4v_2)$-minihypers in $\mathrm{PG}(3, 4)$.

- R. Hill, H. N. Ward (2007)

  $(x(q+1), x)$-minihyper in $\mathrm{PG}(2, q)$

- L. Storme, I. Landjev (2010)

  $(x(q+1), x)$-minihyper in $\mathrm{PG}(2, q)$

- I. Landjev, P. Vandendriesche (2012)

  For every $x \leq q - q/p$, an (xv_t, xv_{t-1})-minihyper in $\mathrm{PG}(t, q)$, $q = p^h$, is canonical.

5. General Results

Theorem A. Assume that every minihyper with parameters

$$\left(\sum_{i=0}^{k-2} a_i v_{i+1}, \sum_{i=0}^{k-2} a_i v_i\right)$$

in $\mathrm{PG}(k-1, q)$ is canonical. Then for any $r \geq k-1$ every minihyper with the same parameters in $\mathrm{PG}(r, q)$ is also canonical.

Theorem B. Assume that every minihyper with parameters

$$\left(\sum_{i=0}^{k-2} a_i v_{i+1}, \sum_{i=0}^{k-2} a_i v_i\right)$$

in $\mathrm{PG}(k-1,q)$ is canonical. Then every minihyper with parameters

$$\left(\sum_{i=0}^{k-2} b_i v_{i+1}, \sum_{i=0}^{k-2} b_i v_i\right)$$

in $\mathrm{PG}(k-1,q)$, where $b_i \leq a_i$, is also canonical.

Theorem C. Assume that every minihyper with parameters

$$\left(\sum_{i=0}^{k-2} a_i v_{i+1}, \sum_{i=0}^{k-2} a_i v_i\right)$$

in $\mathrm{PG}(k-1, q)$ is canonical. Then every minihyper with parameters

$$\left(\sum_{i=0}^{k-2} a_i v_{i+2}, \sum_{i=0}^{k-2} a_i v_{i+1}\right)$$

in $\mathrm{PG}(k, q)$ is also canonical.

6. Minihypers with parameters $(v_3 + (q-2)v_2, v_2 + (q-2)v_1)$ in $\mathrm{PG}(3, q)$

- $q = 3$: $(v_3 + v_2, v_2 + v_1) = (17, 5)$

  the sum of a plane and a line

- $q = 4$: $(v_3 + 2v_2, v_2 + 2v_1) = (31, 7)$

  the sum of a plane and two lines, or lifted from a Baer subplane

- $q = 5$: $(v_3 + 3v_2, v_2 + 3v_1) = (49, 9)$

  the sum of a plane and three lines, or lifted from a the projective triangle

- $q = 7$: $(v_3 + 5v_2, v_2 + 5v_1) = (97, 13)$:

  the sum of a plane and five lines, or lifted from a $(13, 1)$ blocking set in $\mathrm{PG}(2, 7)$.

  Are there other constructions?

We consider only geometries over prime fields $\mathbb{F}_p$.

We denote by $\mathcal{F}$ a minihyper in $\mathrm{PG}(3,p)$ with parameters

$(v_3 + (p-2)v_2, v_2 + (p-2)v_1) = (2p^2 - 1, 2p - 1)$

(1) Plane multiplicities (H. N. Ward): $\equiv -1 \pmod{p}$

$2p-1, 3p-1, \ldots, p^2-1, p^2+p-1, p^2+2p-1, \ldots, 2p^2-1$.

(2) If there exists a plane of multiplicity $\geq p^2+2p-1$ then $\mathcal{F}$ is the sum of a plane and $p-2$ lines.

(3) There is no plane of multiplicity p^2+p-1.

(4) Dualization: for $j = 0, 1, \ldots, p-1$

$$((j+2)p-1)\text{-planes} \longrightarrow j\text{-points}$$

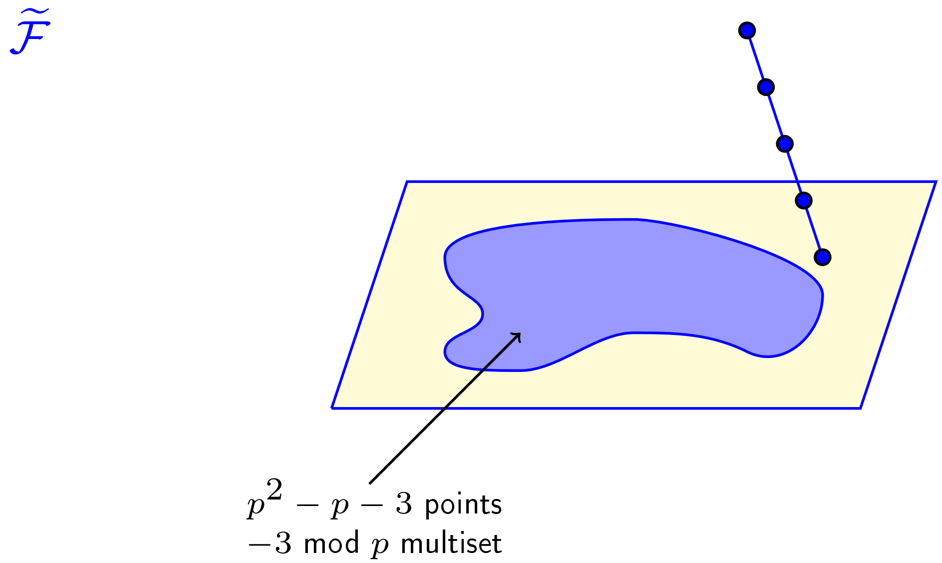
$$j\text{-points} \longrightarrow ((j+1)p-2)\text{-planes}$$

Denote the dual multiset by $\widetilde{\mathcal{F}}$: hence $|\widetilde{\mathcal{F}}| = p^2 - 2$

plane multiplicities: $p-2, 2p-2, \ldots, p^2-p-2, p^2-2$

(5) If $\widetilde{\mathcal{F}}$ has a (p^2-2)-plane then $\mathcal{F}$ is lifted.

(6) If $\widetilde{\mathcal{F}}$ has a (p^2-p-2)-plane then it has the following structure:



In $\mathrm{PG}(2,7)$ the plane multiset is the dual of a $(12,1)$-blocking set.

This gives new $(97,13)$-minihypers with a 5-point.

There are no other $(97,13)$-minihypers in $\mathrm{PG}(3,7)$.

Theorem D. Every $(2p^2-1, 2p-1)$-minihyper in $\mathrm{PG}(3,p)$ with maximal point multiplicity at most $\dfrac{p+1}{2}$ is canonical.

Corollary. Every $(v_t+(p-2)v_{t-1}, v_{t-1}+(p-2)v_{t-2}), t \geq 3$-minihyper in $\mathrm{PG}(t,p)$ with maximal point multilicity at most $\dfrac{p+1}{2}$ is canonical.

Open problem. For what values of a, $1 \leq a \leq p-2$ is a minihyper in $\mathrm{PG}(3,p)$ with parameters

$$((p-1-a)v_3 + av_2, (p-1-a)v_2 + av_1)$$

canonical?

For $a = 1$ we have the following theorem:

Theorem E. Every (p^3-p^2-1, p^2-p-1)-minihyper in $\mathrm{PG}(3,p)$ is canonical.

Corollary. Every $((p-2)v_t+v_{t-1}, (p-2)v_{t-1}+v_{t-2})$-minihyper in $\mathrm{PG}(t,p)$ is canonical.

THANK YOU FOR YOUR ATTENTION!