Small complete caps in $\mathrm{PG}(4n+1, q)$

Francesco Pavese

Joint work with A. Cossidente, B. Csajbók and G. Marino

Polytechnic University of Bari – Italy

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parity check matrix of $\mathcal{C}$: generator matrix of $\mathcal{C}^\perp$

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- 11*-complete cap of* $\mathrm{PG}(4,3) \to [11,6]_3$ *ternary Golay code*

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trivial lower bound: $t_2(r-1, q) \geq \sqrt{2}q^{\frac{r-2}{2}}$

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Kim & Vu 2003

$$\sqrt{q} \log^c q$$

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$$\mathrm{PG}(r-1, q),\ q \textit{ even},$$

Pambianco & Storme 1996

$$3\left(q^{\frac{r-2}{2}} + \cdots + q\right) + 2,\ r \textit{ even}$$

Giulietti 2006

$$\frac{5}{2}q^{\frac{r-2}{2}} + 3\left(q^{\frac{r-4}{2}} + \cdots + q\right) + 2,\ r \textit{ even}$$

Bartoli & Giulietti & Marino & Polverino 2018

$$3q^{\frac{r-2}{2}} + 4q^{\frac{r-3}{2}} + 3\frac{q^{(r-3)/2}-1}{q-1},\ r \textit{ odd},\ q \textit{ square}$$

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- Giulietti, The geometry of covering codes: small complete caps and saturating sets in Galois spaces, *Surveys in combinatorics* 2013.

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Small caps in $\mathrm{PG}(4n+1, q)$, $q > 2$

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Özbudak, *On maximal curves and linearized permutation polynomials over finite fields*, 2001.

A geometric description

$(2n+1) \times (2n+1)$ symmetric matrices over $\mathbb{F}_{q^{2n+1}}$

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a_0 & \dots & a_{n-1} & a_n & a_n^{q^{n+1}} & a_{n-1}^{q^{n+2}} & \dots & a_1^{q^{2n}} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
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vector space of dimension $(n+1)(2n+1)$ over $\mathbb{F}_q$

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\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_1^{q^{2n}} & \dots & a_n^{q^{2n}} & a_n^{q^n} & a_{n-1}^{q^{n+1}} & a_{n-2}^{q^{n+2}} & \dots & a_0^{q^{2n}}
\end{pmatrix}$$

$$W = \{M(a_0, \dots, a_n) \colon a_i \in \mathbb{F}_{q^{2n+1}}\}$$

vector space of dimension $(n+1)(2n+1)$ over $\mathbb{F}_q$

$$\mathrm{PG}(W) \simeq \mathrm{PG}(n(2n+3), q)$$

A geometric description

$(2n+1)\times(2n+1)$ symmetric matrices over $\mathbb{F}_{q^{2n+1}}$

$$M(a_0,\dots,a_n) =$$

$$\begin{pmatrix}
a_0 & \dots & a_{n-1} & a_n & a_n^{q^{n+1}} & a_{n-1}^{q^{n+2}} & \dots & a_1^{q^{2n}} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n-1} & \dots & a_0^{q^{n-1}} & a_1^{q^{n-1}} & a_2^{q^{n-1}} & a_3^{q^{n-1}} & \dots & a_n^{q^{2n}} \\
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\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
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$$\tilde{\Pi}_i = \{M(0,\dots,0,a_i,0,\dots,0)\colon a_i \in \mathbb{F}_{q^{2n+1}} \setminus \{0\}\} \simeq \mathrm{PG}(2n, q)$$

A geometric description

Veronese variety of $\mathrm{PG}(W)$: locus of the zeros of all determinants of 2×2 submatrices of $M(a_0, \ldots, a_n)$

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$\alpha_i \in \mathbb{F}_{q^{2n+1}} \setminus \{0\}$, $1 \le i \le n$

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$\tilde{\mathcal{V}}_\omega$ *projection of* $\mathcal{V}_{\omega,\alpha_2,\ldots,\alpha_n}$ *from* $\langle \tilde{\Pi}_2, \ldots, \tilde{\Pi}_n \rangle$ *onto* $\langle \tilde{\Pi}_0, \tilde{\Pi}_1 \rangle$

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$$\tilde{\mathcal{V}}_\omega \simeq \mathcal{V}_\omega$$

THANK YOU