(Non-)embeddings of the Ree unitals in finite projective planes

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Outline

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Steiner systems; t-(v, k, λ) designs

Definition: Incidence structure

- We call the triple $(\mathcal{P}, \mathcal{B}, |)$ an incidence structure, provided $\mathcal{P}$, $\mathcal{B}$ are disjoint sets and $| \subseteq \mathcal{P} \times \mathcal{B}$.
- The incidence structure is called simple, if each block can be identified with the set of points with which it is incident.

Definition: Steiner system; t-designs

A t-(v, k, λ) design, or equivalently a Steiner system $S_\lambda(t, k, v)$, is a finite simple incidence structure consisting of v **points**

- such that **every block is incident with k points**
- and **every t-subset of points is incident with exactly λ blocks.**

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Examples of 2-designs: Affine planes, unitals

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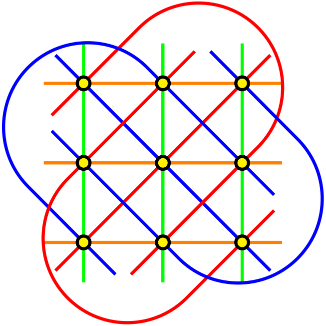
- A projective plane of order n is a 2-$(n^2+n+1, n+1, 1)$ design.
- An affine plane of order n is a 2-$(n^2, n, 1)$ design.
- An abstract unital of order n is a 2-$(n^3+1, n+1, 1)$ design.



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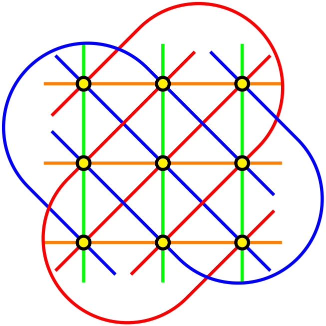

$\{1,2,3\}$ $\{1,5,9\}$
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Classical Hermitian unitals

- In PG$(2, q^2)$, **Hermitian curve** is given by the equation

$$\mathscr{H}_q : X^{q+1} + Y^{q+1} + Z^{q+1} = 0.$$

- The number of rational points is $|\mathscr{H}_q| = q^3 + 1$.
- There is a unique tangent at each point of $\mathscr{H}_q$.
- Any **non-tangent line** intersects $\mathscr{H}_q$ in $q + 1$ points.

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The classical Hermitian unital $\mathscr{H}_q$ of order q is given by the set of rational points of the Hermitian curves, and non-tangents lines in PG$(2, q^2)$.

- The group of projective linear transformations, fixing $\mathscr{H}_q$ is PGU$(3, q)$. It acts 2-**transitively** on $\mathscr{H}_q$.

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Embedding of unitals in projective planes

Definition: Embedding of unitals

Let $\mathscr{U} = (\mathcal{P}, \mathcal{B}, |)$ be an abstract unital and $\Pi = (\mathcal{P}', \mathcal{B}', |')$ a projective plane.

- The map $\varrho : \mathcal{P} \cup \mathcal{B} \to \mathcal{P}' \cup \mathcal{B}'$ is an embedding of $\mathcal{U}$, provided it is injective, $\varrho(\mathcal{P}) \subseteq \mathcal{P}'$, $\varrho(\mathcal{B}) \subseteq \mathcal{B}'$,
- and for all $P \in \mathcal{P}$, $B \in \mathcal{B}$

$$P \mid B \Leftrightarrow \varrho(P) \mid' \varrho(B).$$

- The embedding ϱ is admissible, if any automorphism α of $\mathscr{U}$ is induced by a collineation β of Π:

$$\varrho(P^{\alpha}) = \varrho(P)^{\beta}.$$

Remark. Special attention is paid to the embeddings of unitals of order n into planes of order n^2.

General embedding results

Results

1. [Buekenhout 1976] Unitals embedded in various translation planes. Buekenhout-Tits unitals of order $q = 2^{2n+1}$ in $\mathrm{PG}(2, q^2)$, $n \geq 1$.
2. [Buekenhout 1976, Metz 1979] BM-unitals embedded in $\mathrm{PG}(2, q^2)$ for all $q > 2$.
3. [Korchmáros, Siciliano, Szőnyi 2018], [Grundhöfer, Stroppel, Van Maldeghem 2019] The embedding of $\mathscr{H}_q$ in $\mathrm{PG}(2, q^2)$ is unique up to projective equivalence.

Outline

Ree groups (Ree 1960)

- Let $\mathrm{Ree}(q) = {}^2G_2(q)$ be the **Ree group** of order

$$(q^3+1)q^3(q-1), \qquad q = 3^{2n+1}.$$

- $\mathrm{Ree}(q)$ has a unique conjugacy class of involutions.
- The Sylow 2-subgroups are elementary abelian of order 8.
- For $q > 3$, $\mathrm{Ree}(q)$ is **simple.**
- $\mathrm{Ree}(3)$ is isomorphic to

$$\mathrm{P\Gamma O}(3,8) \cong \mathrm{P\Gamma L}(2,8) \cong \mathrm{PSL}(2,8) \rtimes C_3$$

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Ree unitals (Lüneburg 1966)

- Then Ree(q) has a **2-transitive action** on $q^3 + 1$ points.
- Ree(q) has a unique conjugacy class of involutions.
- Any involution t fixes exactly $q + 1$ points.

Definition: Ree unital of order $q = 3^{2n+1}$

The blocks of the Ree unital $\mathcal{R}(q)$ are the sets of **fixed points of the involutions** of Ree(q).

- $\mathcal{R}(q)$ admits the Ree(q) as a **2-transitive automorphism group;** the full automorphism group is larger, for $n \geq 1$.
- The properties of the smallest Ree unital $\mathcal{R}(3)$ differ from the general case.

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Embedding results of the Ree unital

1. [Lüneburg 1966] The Ree unital $\mathcal{R}(q)$ has no admissible embeddings in projective planes of order q^2 (desarguesian or not).
2. [Grüning 1986] proved that $\mathcal{R}(3)$ has no embedding in a projective plane of order 9.
3. [Grüning 1986] constructed an **admissible embedding** of $\mathcal{R}(3)$ in $PG(2,8)$. (He attributes the idea to Piper.)
4. [Montinaro 2008] extended these results by showing that for $q \neq 3$ and $n \leq q^4$, $\mathcal{R}(q)$ has no admissible embedding in a projective plane of order n.
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The dual setting of the $\mathcal{R}(3) \to \mathrm{PG}(2,8)$ embedding

The unital:

- Let $\mathcal{K}$ be a non-singular conic in $\mathrm{PG}(2,8)$, with nucleus N. The set $\mathcal{O} = \mathcal{K} \cup \{N\}$ is a hyperoval.
- There are 63 external points, 28 external lines, and each external point is incident with 4 external lines.
- The external points and the external lines form a **dual unital** $\mathcal{U}$ of order 3.

The group:

- $G = \mathrm{P\Gamma O}(3,8)$ is the group of projective semilinear transformations of $\mathrm{PG}(2,8)$, preserving $\mathcal{O}$.
- G acts 2-**transitively** on the set of external lines.
- $\mathcal{U}$ has a 2-**transitive automorphism** group and $\mathcal{R}(3) \cong \mathcal{U}$.

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Extending Montinaro's result

Theorem 1 (GN)

Let F be a field and $\varphi : \mathcal{R}(3) \to \mathrm{PG}(2, F)$ an embedding. Then the following hold:

1. $\mathbb{F}_8$ is a **subfield** of F, and the image of φ is contained in a **subplane of order** 8**.**
2. The embedding is **unique** up to $\mathtt{Aut}(\mathcal{R}(3))$ and $\mathrm{PGL}(3, F)$.
3. The embedding is **admissible.**

Proof.

Elementary computations with External Pentagons of the conic (set of **five points**, determining **ten external lines**,

and Super O'Nan Configurations of $\mathcal{R}(3)$ (set of **five pairwise intersecting blocks**, in general position) (Brouwer 1981). □

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Sylow 2-subgroups of the Ree group

Corollary

Let F be a field and $\varphi : \mathcal{R}(3) \to \mathrm{PG}(2, F)$ an embedding. Let $S = \{1, a_1, \ldots, a_7\}$ be a Sylow 2-subgroup of Ree(3). Then the lines $\varphi(a_1), \ldots, \varphi(a_7)$ are **concurrent.**

Main result

Theorem 2 (GN)

Let n be a positive integer, and $q = 3^{2n+1}$. Suppose that Π is a projective plane such that for each embedding $\varphi : \mathcal{R}(3) \to \Pi$, the image $\varphi(\mathcal{R}(3))$ is **contained in a pappian subplane.**

- Then the Ree unital $\mathcal{R}(q)$ has **no embedding** in Π.
- In particular, $\mathcal{R}(q)$ has no embedding in a projective plane **over a field.**

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THANK YOU FOR YOUR ATTENTION!