ERDŐS-KO-RADO FOR FLAGS IN SPHERICAL BUILDINGS

joint work with Jan De Beule and Klaus Metsch

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Sixth Irsee Conference
September 2, 2022

VRIJE UNIVERSITEIT BRUSSEL

Context

Theorem (Erdős-Ko-Rado 1938)

Let C be a family in $\binom{[n]}{d}$ such that any two elements in C intersect non-trivially. If $n \geq 2d$ then

$$|C| \leq \binom{n-1}{d-1}.$$

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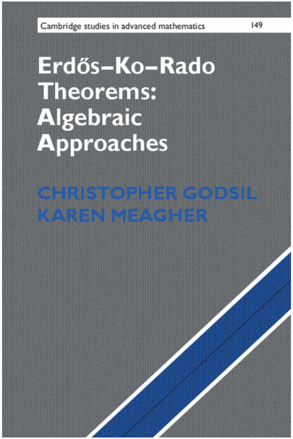
General EKR-problem

Describe in different contexts the largest sets of objects such that no two are "far away".

A successful strategy

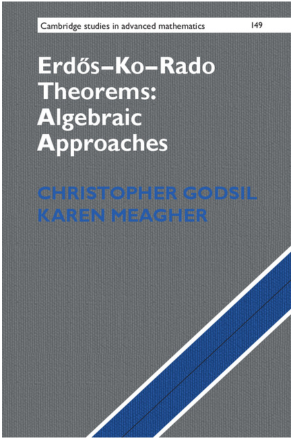


A successful strategy

1. Rephrase the problem

Construct a graph:

- vertex set = finite set Ω
- adjacency = “far-awayness”



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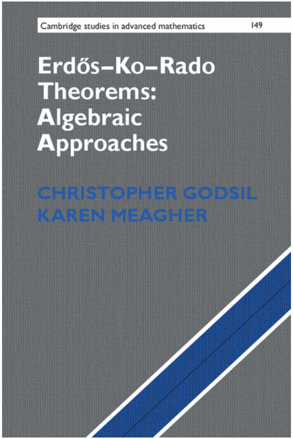
$\longrightarrow$ find the largest cocliques.



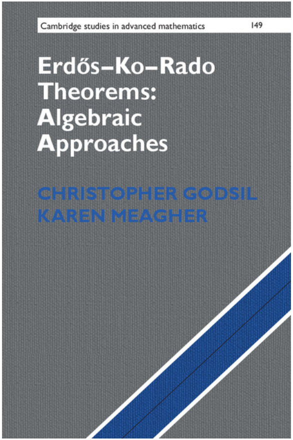
A successful strategy

1. Rephrase the problem

2. The ratio bound

Let Γ be a k-regular graph on n vertices whose adjacency matrix $A(\Gamma)$ has smallest eigenvalue λ. Then if C is a coclique we have

$$|C| \leq \frac{n}{1 - \frac{k}{\lambda}}.$$



Finding the eigenvalues

Objective

Find a ‘nice’ matrix algebra $\mathcal{A}$ containing $A(\Gamma)$.

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Commutative is nice

There exists a unitary matrix U such that

$$U^* \mathcal{A} U = \{U^* A U \mid A \in \mathcal{A}\}$$

is an algebra of diagonal matrices.

Erdős-Ko-Rado results in finite geometry

Theorem (Hsieh 1975)

Let C be a set of d-dimensional subspaces of $\mathbb{F}_q^n$ such that any two elements in C intersect non-trivially. If $n \geq 2d$ then

$$|C| \leq \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q .$$

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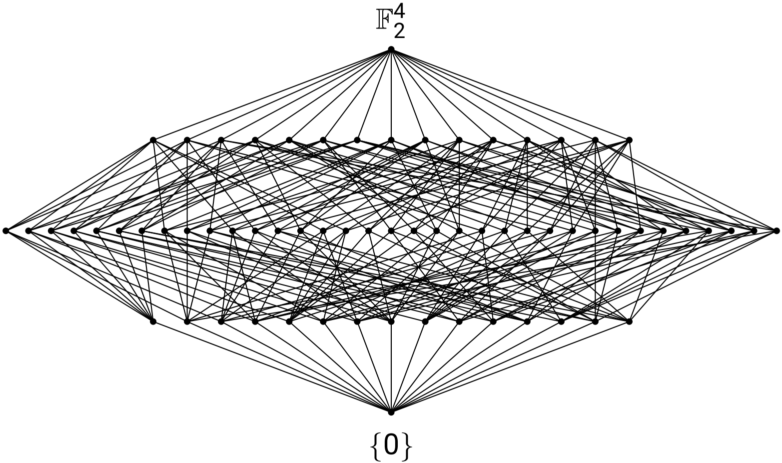
Theorem (Pepe-Storme-Vanhove 2011)

Let C be a set of generators in a rank d polar space such that any two elements in C intersect non-trivially. Then in most cases

$$|C| \leq |star|.$$

The lattice of subspaces



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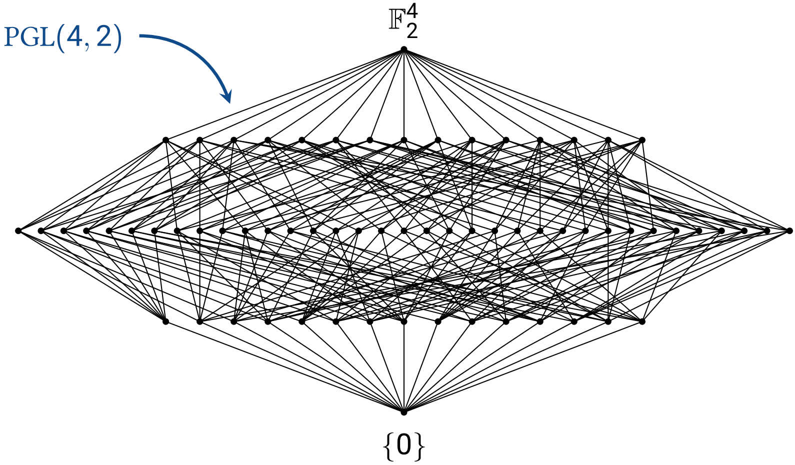

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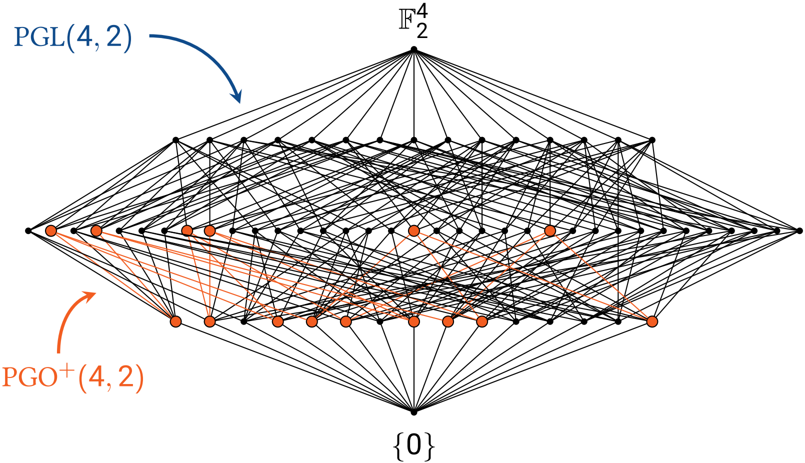

Use the lattice structure

We will focus on vector spaces.

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Definition

A **flag** in $\mathbb{F}_q^n$ is a set of subspaces $\{V_0, \ldots, V_k\}$, such that $V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_k$.
It is **maximal** if it cannot be extended.

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Definition

Two maximal flags $\{V_0, \ldots, V_n\}$ and $\{W_0, \ldots, W_n\}$ are **opposite** if $V_i \cap W_{n-i} = \{0\}$.

Buildings for dummies

Buildings for dummies explained by a dummy

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Fact

For any two maximal flags $\mathcal{F}$ and $\mathcal{F}'$ there exists a basis $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{F}_q^4$ such that

$$\mathcal{F} = (\{0\}, \langle e_1 \rangle, \langle e_1, e_2 \rangle, \langle e_1, e_2, e_3 \rangle, \langle e_1, e_2, e_3, e_4 \rangle)$$

and

$$\mathcal{F}' = (\{0\}, \langle e_i \rangle, \langle e_i, e_j \rangle, \langle e_i, e_j, e_k \rangle, \langle e_i, e_j, e_k, e_l \rangle),$$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

A matrix algebra?

Corollary

Every relation corresponds to an element in $\mathrm{Sym}(4)$.

For example: (12) $\leftrightarrow$ “change the one-dimensional space”

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A matrix algebra!

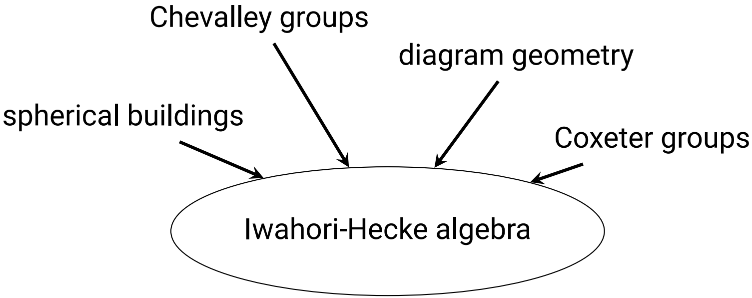
Chevalley groups

diagram geometry

spherical buildings

Coxeter groups

A matrix algebra!



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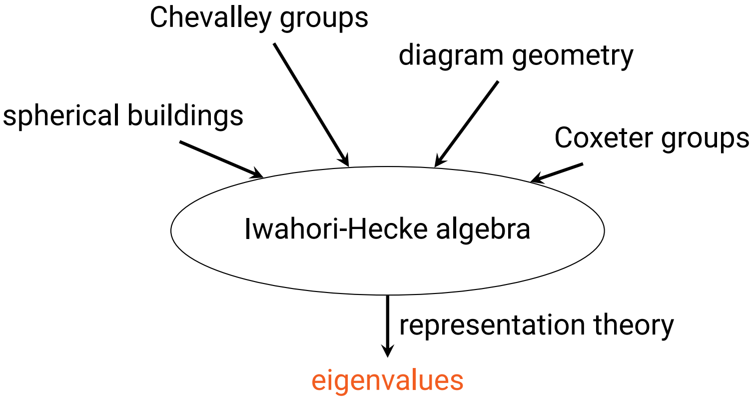

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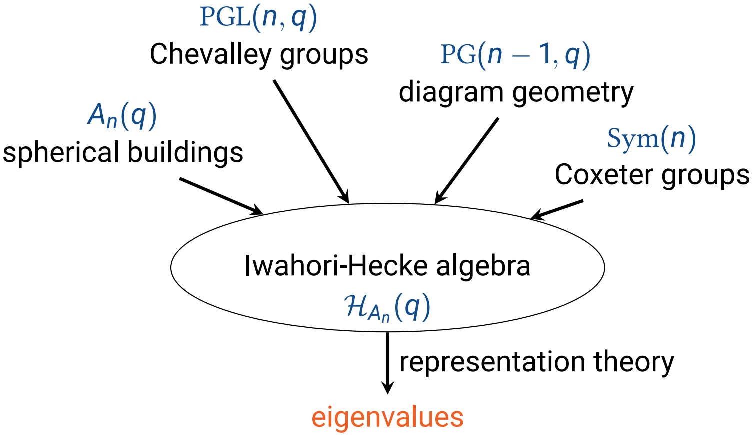

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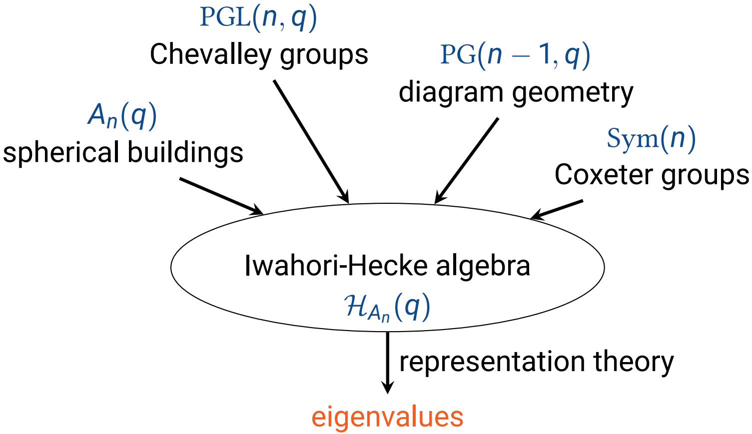

$$\boxed{\mathcal{H}_{A_n}(1) = \mathbb{C}[\mathrm{Sym}(n)]}$$

Towards eigenvalues

Each partition $\mu \vdash n$ gives rise to an eigenvalue q^{e_μ}.

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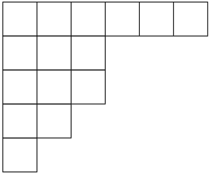
Eigenvalue for $(6, 3, 3, 2, 1) \vdash 15$



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Eigenvalue for $(6, 3, 3, 2, 1) \vdash 15$

$$e_\mu = \binom{15}{2} + \begin{array}{llllll} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & & & \\ 0 & 1 & 2 & & & \\ 0 & 1 & & & & \\ 0 & & & & & \end{array} - \begin{array}{llllll} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & & & \\ 2 & 2 & 2 & & & \\ 3 & 3 & & & & \\ 4 & & & & & \end{array} = 109.$$

Summary

Theorem (De Beule-M.-Metsch 2022)

We can compute the eigenvalues of opposition of

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Corollary

We can derive EKR bounds for all these cases.

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Conclusion

1. A computer can do the work!
2. We did not achieve our initial goal.
3. Life without commutativity is fine.
4. Not the end of the story: eigenvalues are everywhere!

Thank you for your attention!

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Finite Geometry and Friends

18-22 September 2023

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