Low Boolean degree *d* functions in Grassmann graphs

Finite Geometry
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(joint work with Jan De Beule, Jozefien D'haeseleer and Ferdinand Ihringer)

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Notations

Let $n \geq 2k \geq 2d \geq 1$ in $\mathbb{F}_q^n$. suppose that Π_i be the set of i-spaces in $\mathbb{F}_q^n$.

Gauss coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1)\cdots(q^{n-k+1} - 1)}{(q^k - 1)\cdots(q - 1)}$$

equals the number of k-spaces in $\mathbb{F}_q^n$.

d-space to k-space incidence matrix

Order the d- and k-spaces as the rows and columns of a matrix A and define

$$A_{ij} = \begin{cases} 1 & \text{if } D_i \subseteq K_j \\ 0 & \text{if } D_i \not\subseteq K_j \end{cases}$$

What are

Boolean degree *d* functions?

Boolean degree d functions

Boolean function and characteristic vectors

Definition

A Boolean function f on the k-spaces of $\mathbb{F}_q^n$, is a $\{0,1\}$-valued function on these subspaces.

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- For Boolean functions f:

$$\mathcal{L}_f := \{K \mid f(K) = 1\}.$$

  In particular the characteristic vector of this set χ_f are the values of f.

- Conversely for a characteristic vector $\chi_{\mathcal{L}}$, we can consider the Boolean function

$$\chi_{\mathcal{L}}^{+} : \ell \mapsto \chi_{\mathcal{L}}(\ell).$$

Boolean degree d functions

Example (d-space pencil)

- Fix d-space D and denote x_D as a Boolean function on the k-spaces such that

$$x_D(K) = \begin{cases} 1 & \text{if } D \subseteq K \\ 0 & \text{if } D \not\subseteq K \end{cases}$$

Boolean degree d functions

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Remark

For every d-space $D = \langle p_1, ..., p_d \rangle$, we have that

$$x_D(K) = x_{p_1}(K) \cdots x_{p_d}(K),$$

for every K.

Boolean degree *d* functions

Definition

A *Boolean degree d function f* in $\mathbb{F}_q^n$ is $\{0,1\}$-valued function on the lines such that

$$f = \sum_{D \in \Pi_D} c_D x_D,$$

where $c_D \in \mathbb{R}$ and x_D are Boolean functions for *d*-space pencils.

Remark: Clearly all unions, differences and complements of degree *d* functions are still degree *d*.

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Boolean degree d functions:

Trivial examples

Definition

For every d-space D and every $(n-d)$-space S, let us denote

- $x_{D,i}$ as the boolean degree d function that describes all k-spaces K with $\dim(K \cap D) = d - i$
- $x^{\perp}_{S,i}$ as the boolean degree d function that describes all k-spaces K with $\dim(K \cap S) = d - i$

for $i \in \{0, ..., d\}$.
Then all unions, differences, and complements of these are called *trivial* examples.

Connection with Association schemes

Association schemes

Definition

Definition

Let X be a finite set. A *d-class symmetrical association scheme* is a pair $(X, \mathcal{R})$, where $\mathcal{R} = \{\mathcal{R}_0, \mathcal{R}_1, ..., \mathcal{R}_d\}$ is a set of binary symmetrical relations such that:

1. $\{\mathcal{R}_0, \mathcal{R}_1, ..., \mathcal{R}_d\}$ is a partition of $X \times X$.
2. $\mathcal{R}_0$ is the identity relation.
3. There exist constants p_{ij}^l such that for $x, y \in X$, with $(x, y) \in \mathcal{R}_l$, there are exactly p_{ij}^l elements z with $(x, z) \in \mathcal{R}_i$ and $(z, y) \in \mathcal{R}_j$. These constants are called the *intersection numbers* of the association scheme.

Association schemes

Definition

We can define $d+1$ *adjacency matrices* $B_0, ..., B_d$ of dimension $n \times n$, such that

$$(B_k)_{ij} = \begin{cases} 1, \text{ if } (x_i, x_j) \in \mathcal{R}_k \\ 0, \text{ if } (x_i, x_j) \notin \mathcal{R}_k. \end{cases}$$

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Remark

We obtain pairwise orthogonal common (right) eigenspaces $V_0, ..., V_d$. In fact

$$\mathbb{R}^{|X|} = V_0 + V_1 + \cdots + V_d.$$

Association schemes

Grassmann scheme or q-Johnson scheme $J_q(n, k)$

Let $X = \Pi_k$ in $\mathbb{F}_q^n$, with $n \geq 3$. Consider now the following set of relations $\mathcal{R}' = \{\mathcal{R}_0, \mathcal{R}_1, \cdots, \mathcal{R}_k\}$, where two k-spaces K_1 and K_2 are in relation $\mathcal{R}_i$ if and only if

$$\dim(K_1 \cap K_2) = k - i.$$

Then $(\Pi_1, \mathcal{R}')$ is an k-class symmetrical association scheme.

Association schemes

Distribution of the eigenspaces

- Again we obtain pairwise orthogonal common (right) eigenspaces $V_0, ..., V_k$
- $\mathbb{R}^{|\Pi_k|} = V_0 + V_1 + V_2 + \cdots + V_k$

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Lemma

Let $n \geq 2k$ and consider $J_q(n,k)$, then we have for every $0 \leq d \leq k$ that $V_0 + \ldots + V_d = \langle x_D : \dim D = d \rangle$.

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Conclusion: Boolean degree d functions are those functions that arise from vectors in the first d eigenspaces of the association scheme.

Why study these objects?

Connecting with Blocks lemma

Lemma

Let G be a group acting on two finite sets X and X', with sizes n and m. Let $O_1, \ldots, O_s$, respectively $O'_1, \ldots, O'_t$ be the orbits of the action on X, respectively X'. Suppose that $R \subseteq X \times X'$ is a G-invariant relation and call $A = (a_{ij})$ the $n \times m$ matrix of this relation, i.e. $a_{ij} = 1$ if and only if $x_i R x'_j$ and $a_{ij} = 0$.

(i) *The vectors $A^T \chi_{O_i}$, $i = 1, \ldots, s$, are linear combinations of the vectors $\chi_{O'_j}$.*

(ii) *If A has full row rank, then $s \leq t$. If $s = t$, then all vectors $\chi_{O'_j}$ are linear combinations of the vectors $A^T \chi_{O_i}$, hence $\chi_{O'_j} \in \mathrm{Im}(A^T)$.*

Connecting with Blocks lemma

- Let X and X' be the d-spaces, respectively k-spaces of $\mathbb{F}_q^n$.
- Consider A to be the d-space-to-k-space incidence matrix.

Then for the characteristic $\chi_{O'_j}$ vector of the orbits of X' it holds that

$$\chi_{O'_j} \in \mathrm{Im}(A^T).$$

Connecting with Cameron-Liebler problems

Lemma

Let $n \geq 2k$. For f a real function on $J_q(n, k)$ the following are equivalent:

(a) *The function f has degree d.*

(b) *The function f lies in $V_0 + \cdots + V_d$.*

(c) *The function f is orthogonal to $V_{d+1} + \cdots + V_n$.*

(d) *The function f lies in the image of the d-space-to-k-space incidence matrix.*

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(d) *The function f lies in the image of the d-space-to-k-space incidence matrix.*

Remark: property (d) is equivalent with $\mathcal{L}_f$ being a Cameron-Liebler set of k-spaces (for $d = 1$).

How to study

Boolean degree d functions?

Common problems

A main problem for Boolean degree d functions

- Do there exist non-trivial examples?

Common problems

A main problem for Boolean degree *d* functions

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J. De Beule, J. D'haeseleer, J. Mannaert, and F. Ihringer
Degree 2 Boolean Functions on Grassmann Graphs
arXiv:2202.03940, submitted.

Common problems

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???

Non-existence conditions

Connecting Designs

Corollary

Let $n \geq 2k$. Consider a d-(n,k,λ) design $\mathcal{D}$ of $J_q(n,k)$. If $\mathcal{F}$ is a degree d subset of $J_q(n,k)$, then $|\mathcal{F} \cap \mathcal{D}| = |\mathcal{F}| \cdot |\mathcal{D}| / \begin{bmatrix} n \\ k \end{bmatrix}_q$.

Proof.

Use the fact that

$$\chi_D \in V_0 + V_{d+1} + \ldots + V_k$$

and,

$$\chi_F \in V_0 + V_1 + \ldots + V_d.$$

□

Non-existence conditions

Suzuki's construction and others

Lemma

Let $\mathcal{F}$ be a degree 2 family of 3-spaces in $\mathbb{F}_q^n$. Then $(q^3-1)|\mathcal{F}|$ is divisible by $q^{n-2}-1$.

Lemma

Let $m \geq 3$. Suppose that $\mathcal{F}$ is a set of 3-spaces in $\mathbb{F}_2^n$ of degree 2, then the following holds:

(a) *If $n = 8m$, then $C|\mathcal{F}|$ is divisible by $2^{8m-2}-1$, where $C \in \{42, 312\}$.*

(b) *If $n = 9m$, then $42 \cdot |\mathcal{F}|$ is divisible by $2^{9m-2}-1$.*

(c) *If $n = 10m$, then $210 \cdot |\mathcal{F}|$ is divisible by $2^{10m-2}-1$.*

(d) *If $n = 13m$, then $42 \cdot |\mathcal{F}|$ is divisible by $2^{13m-2}-1$.*

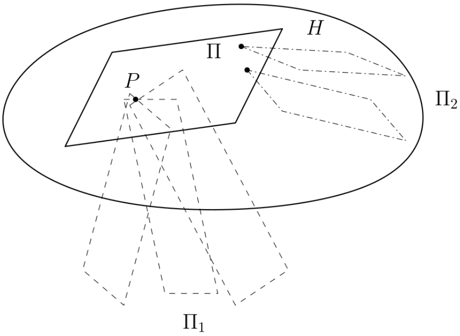
Boolean degree 2 functions

Non-trivial example

$(n, k) = (6, 3)$ of size $(q^2 + 1)q^3(q + 1)$

Pick a point P, a plane Π, and a hyperplane H such that $P \subseteq \Pi \subseteq H$.

- Let Π_1 be the set of all planes not in H which meet Π in a line through P.
- Let Π_2 the set of all planes in H whose meet with Π is a point different from P.



Non-trivial example

Other examples

(n, k, q)	size
$n = 8, k = 4$	$(q^4 + 1)(q^3 + 1)(q^2 + 1)\frac{q^5-1}{q-1}$
$n = 6, k = 3$	$(q + 1)(q^2 + 1)(q^3 + 1)$
$n = 6, k = 3$	$(q^2 + 1)q^2(q + 1)$
$n = 6, k = 3, q = 2$	$55, 75, 195$
$n = 6, k = 3, q = 2$	$80, 85, 177, 420$

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Thank you for your attention!

Are there any questions?

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