A Note on Sperner's Theorem for Modules over Finite Chain Rings

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1. Preliminaries

Theorem. (E. Sperner, 1928) If $A_1, A_2, \ldots, A_m$ are subsets of $X = \{1, 2, \ldots, n\}$ such that A_i is not a subset of A_j if $i \neq j$, then $m \leq \binom{n}{\lfloor n/2 \rfloor}$.

Theorem. If $\mathcal{A}$ is an antichain in the partially ordered set of all subspaces of $\mathbb{F}_q^n$, then

$$|\mathcal{A}| \leq \begin{bmatrix} n \\ \lfloor n/2 \rfloor \end{bmatrix}_q$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1) \ldots (q^{n-k+1} - 1)}{(q^k - 1) \ldots (q - 1)}.$$

are the Gaussian coefficients.

2. Partially ordered sets

Let $\mathcal{P}$ be a partially ordered set with a partial order "$\preceq$".

- We say that the element y of a poset $\mathcal{P}$ covers the element $x \in \mathcal{P}$ if $x \prec y$ and $x \prec y' \preceq y$ implies $y = y'$. This is denoted by $x \prec\!\cdot\; y$.

- Ranked poset $\mathcal{P}$: there exists a function (rank function) $r : \mathcal{P} \to \mathbb{N}_0$ with $r(x) = 0$ for some minimal element and $r(y) = r(x) + 1$ for all x, y with $x \prec\!\cdot\; y$.

- Graded poset: a ranked poset in which all minimal elements have rank 0.

- $L_i(\mathcal{P})$ – the i-th level of $\mathcal{P}$: $L_i(\mathcal{P}) = \{x \in \mathcal{P} \mid r(x) = i\}$.

- the i-th Whitney number: $W_i(\mathcal{P}) = |L_i(\mathcal{P})|$

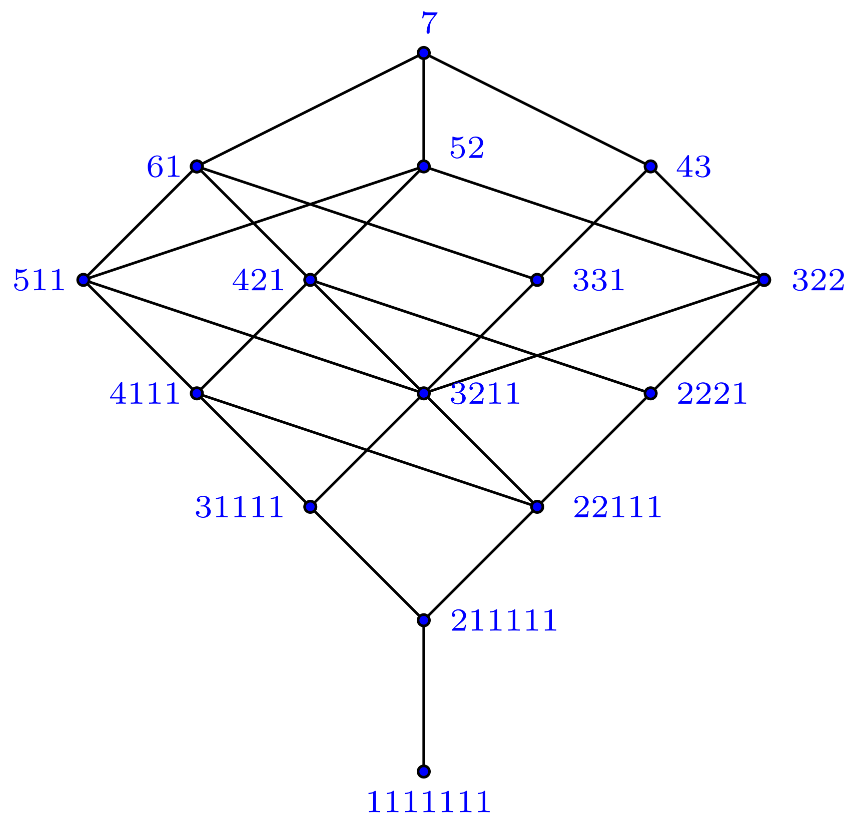
- A poset is said to have the Sperner property if the maximum cardinality of an antichain equals the largest Whitney number.

- The Hasse diagram of a partially ordered set is a directed graph $H(\mathcal{P}) = (\mathcal{P}, E(\mathcal{P}))$ where

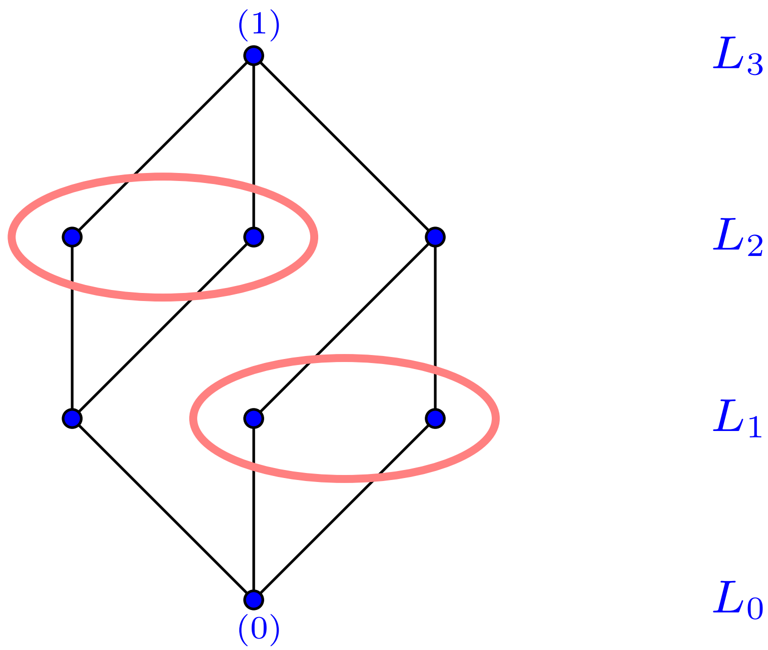
$$E(\mathcal{P}) = \{(x, y) \mid \text{ where } x \prec\!\cdot\; y\}.$$

The underlying nondirected graph is called the Hasse graph.

The poset of the partitions of $n = 7$



A poset without the Sperner proerty



3. Modules over Finite Chain Rings

Theorem. Let R be a finite chain ring of length m and with residue field $\mathbb{F}_q$. For any finite module ${}_RM$ there exists a uniquely determined partition

$$\lambda = (\lambda_1, \ldots, \lambda_k) \vdash \log_q |M|,$$

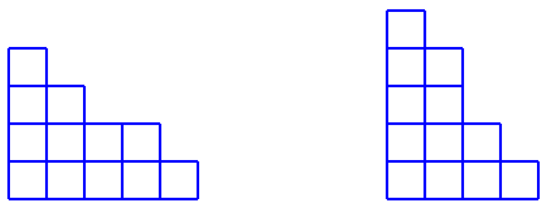
$m \geq \lambda_1 \geq \ldots \geq \lambda_k > 0$, such that

$${}_RM \cong R/(\mathrm{rad}\, R)^{\lambda_1} \oplus \ldots \oplus R/(\mathrm{rad}\, R)^{\lambda_k}.$$

- The partition λ is called the **shape of** ${}_RM$.
- The conjugate partition λ' to λ is called the **conjugate shape** of ${}_RM$.

  The **conjugate partition** $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ is defined by:

  $\lambda'_i =$ number of parts in λ that are greater or equal to i



$\lambda = (4, 3, 2, 2, 1)$ $\quad$ $\lambda' = (5, 4, 2, 1)$

- The number k is called the **rank** of ${}_RM$.

4. Counting Formulas

Theorem. Let ${}_RM$ be a module of shape $\lambda = (\lambda_1, \ldots, \lambda_n)$. For every sequence $\mu = (\mu_1, \ldots, \mu_n)$, $\mu_1 \geq \ldots \geq \mu_n \geq 0$, satisfying $\mu \leq \lambda$ the module ${}_RM$ has exactly

$$\begin{bmatrix} \lambda \\ \mu \end{bmatrix}_{q^m} := \prod_{i=1}^{m} q^{\mu'_{i+1}(\lambda'_i - \mu'_i)} \cdot \begin{bmatrix} \lambda'_i - \mu'_{i+1} \\ \mu'_i - \mu'_{i+1} \end{bmatrix}_q$$

submodules of shape μ.

If $\lambda = (\underbrace{m, \ldots m}_{k_m}, \underbrace{(m-1), \ldots, (m-1)}_{k_{m-1}}, \ldots, \underbrace{1, \ldots, 1}_{k_1})$

then we shall write $m^{k_m}(m-1)^{k_{m-1}} \ldots 1^{k_1}$.

• The family of all submodules of a finitely generated left R-module ${}_RM$ ordered by inclusion is a graded poset. If ${}_RM = {}_RR^n$ we denote this poset by $\mathcal{P}_n$.

• Rank function: $r(L) = \sum_{i=1}^n \lambda_i = \log_q |L|$, where ${}_RL < {}_RM$ and L has shape $(\lambda_1, \ldots, \lambda_n)$.

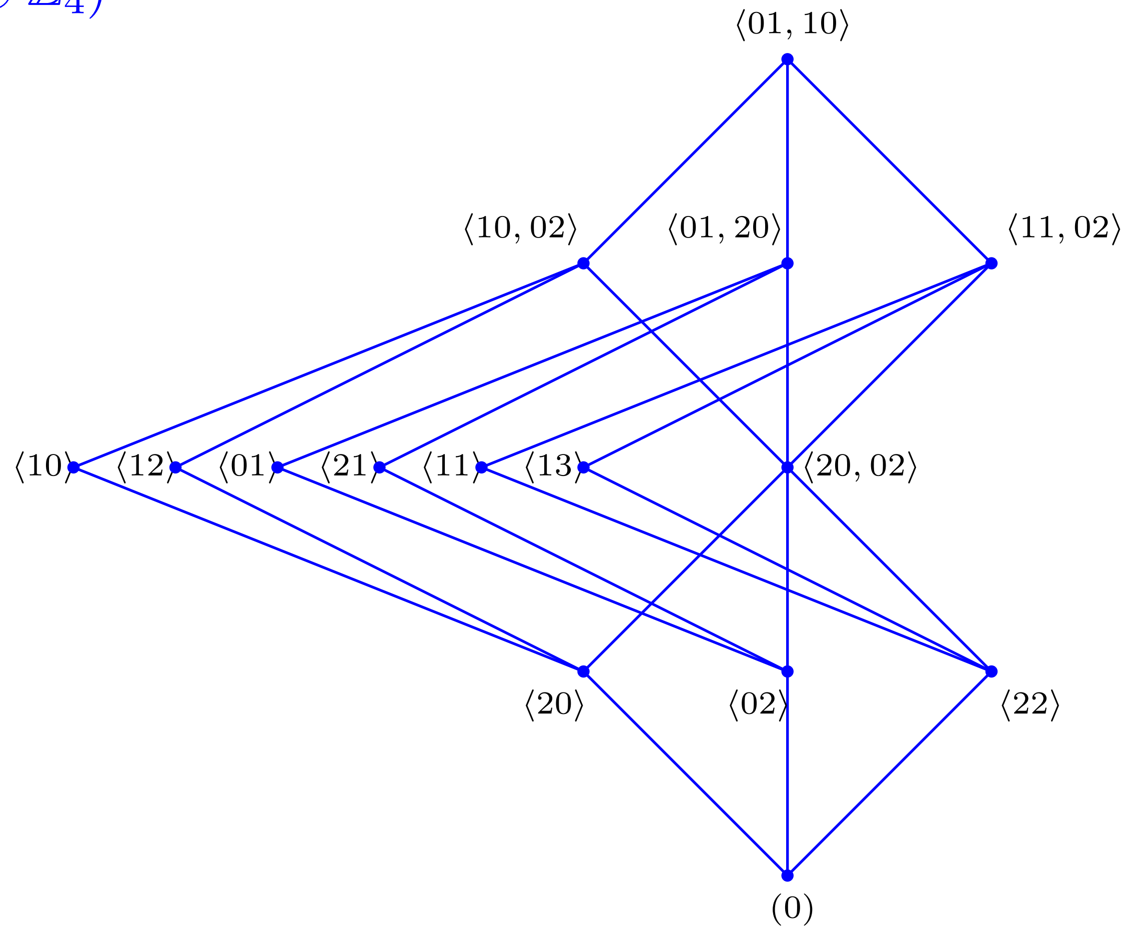
• We have $r(\mathcal{P}_n) = mn$, where m is the length of R.

• The k-th Whitney number:

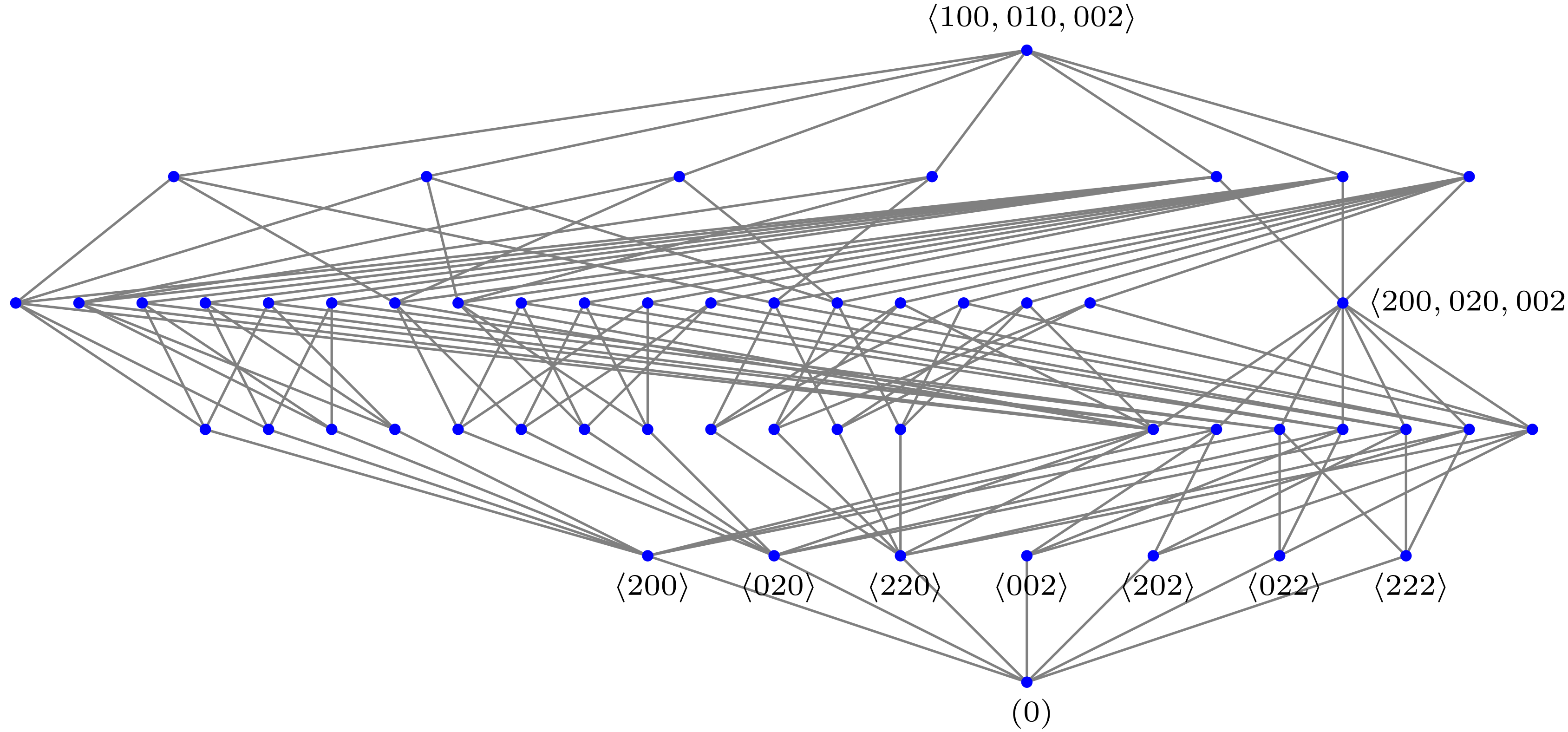
$$W_k(\mathcal{P}_n) = \sum_{\boldsymbol{\mu}} \begin{bmatrix} \boldsymbol{m}_n \\ \boldsymbol{\mu} \end{bmatrix}_{q^m},$$

where the sum is over all shapes $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ with $\sum_i \mu_i = k$.

$\mathcal{P}(\mathbb{Z}_4 \oplus \mathbb{Z}_4)$



$\mathcal{P}(\mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus 2\mathbb{Z}_4)$



Problem. Let R be a finite chain ring and let ${}_RM$ be a (left) module over R.

What is the size of the largest antichain in the poset $\mathcal{P}(M)$ of all submodules of ${}_RM$?

5. A Sperner-type Theorem for Free Modules

Let $\mathcal{P}$ be a graded poset.

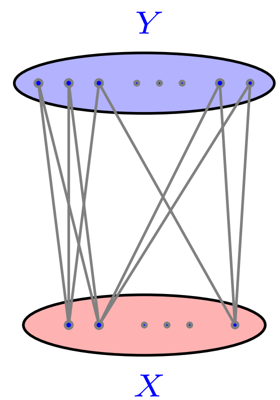
It is said that level L_i can be matched into level L_j, where $j = i - 1$ or $i + 1$, if there is a matching of size W_i in the subgraph of the Hasse graph of $\mathcal{P}$ defined on the vertices from $L_i \cup L_j$.

Theorem. Let $\mathcal{P}$ be a graded poset. If there exist indices g and h such that L_i can be matched into L_{i+1} for all $i = 0, 1, \ldots, g - 1$, and L_j can be matched into L_{j-1} for all $j = h + 1, \ldots, n$ then there exists a largest antichain which is contained in levels $L_g, L_{g+1}, \ldots, L_h$.

Let R be a finite chain ring with nilpotency index 2 and residue field $\mathbb{F}_q$ and let ${}_RM = {}_RR^n$.

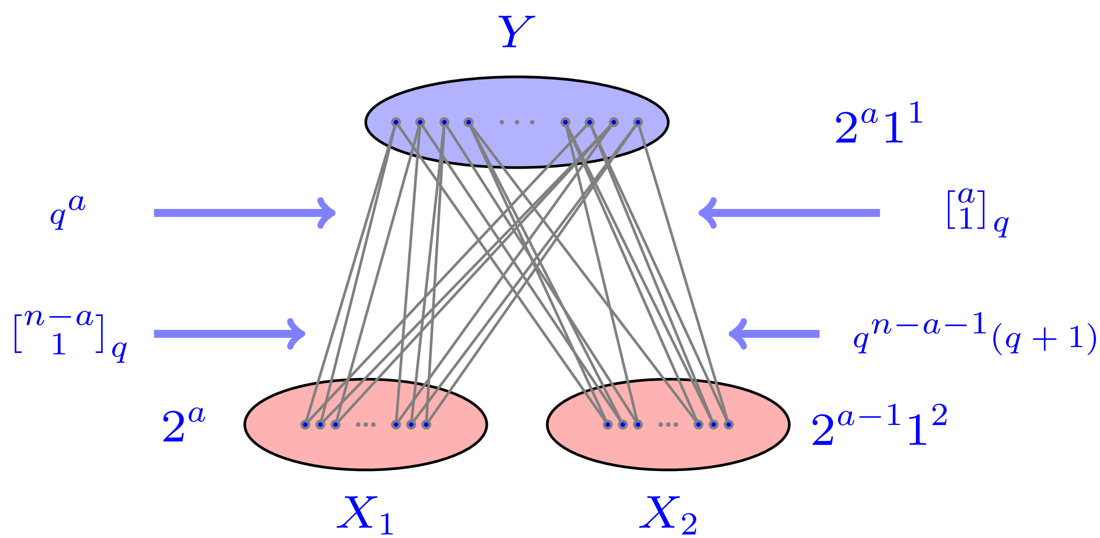
Lemma A. Let a, b be non-negative integers with $2a + b \leq n$. Denote by X be the set of all submodules of ${}_RR^n$ of shape $2^{a-1}1^{b+1}$, and by Y – the set of all submodules of ${}_RR^n$ of shape 2^a1^b. Let $G = (X \cup Y, E)$ be the bipartite graph with edges given by set-theoretical inclusion. The X can be matched into Y.

$q^{(a-1)(n-a-b)}\begin{bmatrix} n-a+1 \\ b+1 \end{bmatrix}_q \begin{bmatrix} n \\ a-1 \end{bmatrix}_q$ $\quad$ 2^a1^b



$q^{a(n-a-b)}\begin{bmatrix} n-a \\ b \end{bmatrix}_q \begin{bmatrix} n \\ a \end{bmatrix}_q$ $\quad$ $2^{a-1}1^{b+1}$

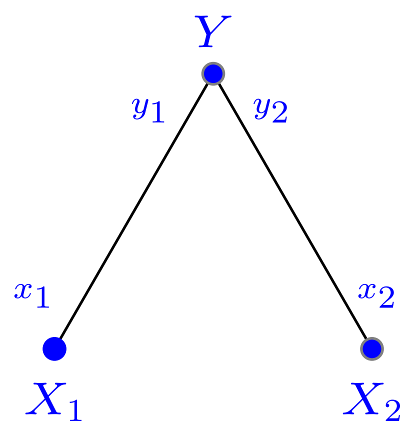
Lemma B. Let a be a non-negative integer with $2a+1 \le n$. Denote by X be the set of all submodules of ${}_R R^n$ of shape 2^a, and $2^{a-1}1^2$, and by Y – the set of all submodules of ${}_R R^n$ of shape $2^a 1^1$. Let $G = (X \cup Y, E)$ be the bipartite graph with edges given by set-theoretical inclusion. The X can be matched into Y.



Lemma C. Let $G = (X \cup Y, E)$ be a bipartite graph with $X = X_1 \cup X_2$ and $|X| \leq |Y|$. Each vetex from X_i is adjacent to x_i vertices of Y, and each vertex of Y is adjacent to y_i vertices of X_i, $i = 1, 2$. If

$$y_1 + y_2 \leq \min(x_1, x_2),$$

then G has a matching of $|X|$ edges.



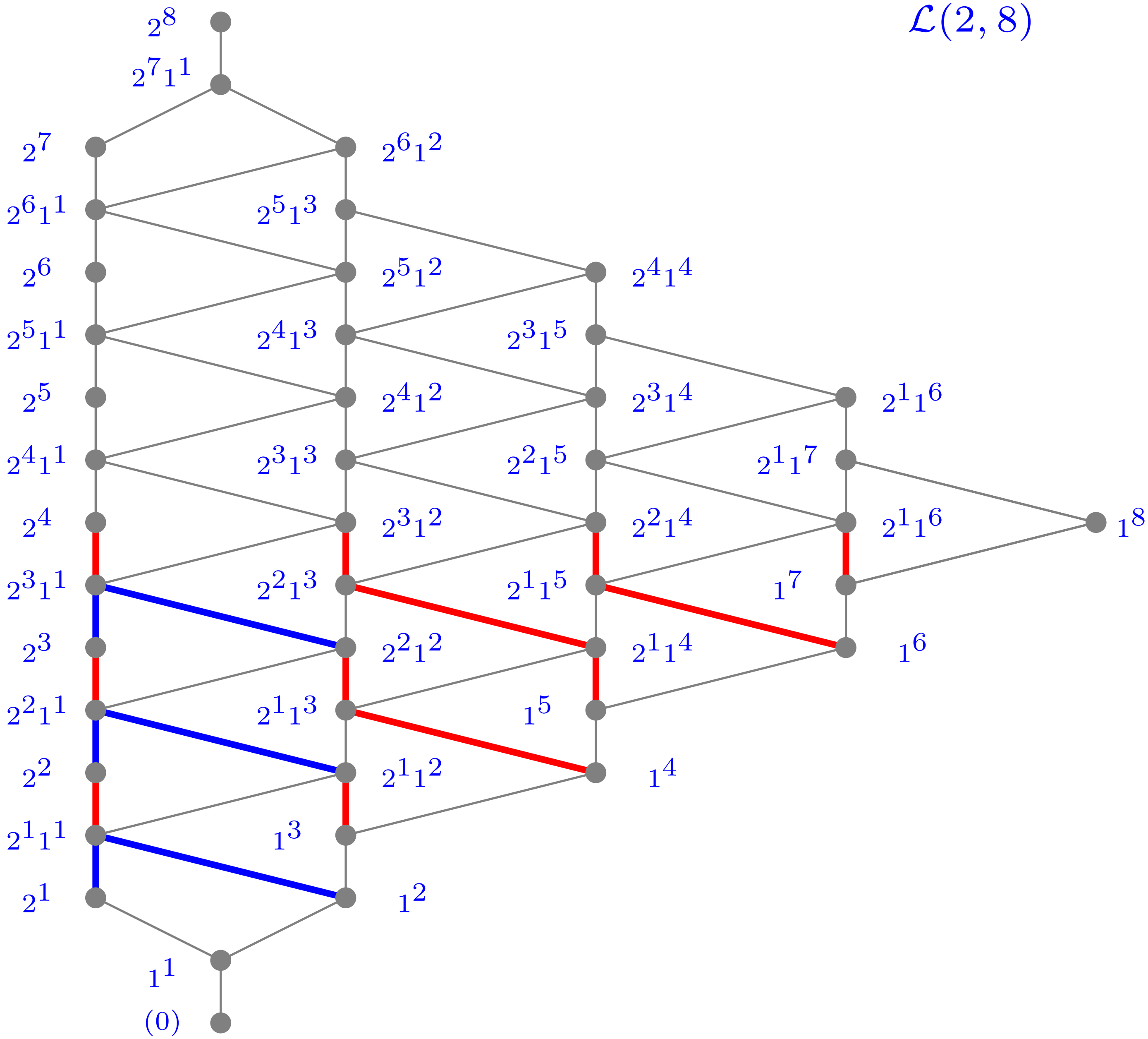
• $\mathcal{L}(m,n)$: the poset of all n-tuples $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $m \geq \lambda_1 \geq \ldots \geq \lambda_n \geq 0$ and $\sum \lambda_i \leq mn$ with partial order defined by

$$\lambda \preceq \mu \Longleftrightarrow \lambda_1 \leq \mu_1, \ldots \lambda_n \leq \mu_n.$$

• $\mathcal{L}(m,n)$ can be graded by the rank function $r(\lambda) = \sum_{i=1}^{n} \lambda_i$.

• $\mathcal{L}(m,n)$ is self dual: $(\lambda_1, \ldots, \lambda_n) \to (m - \lambda_n, \ldots, m - \lambda_1)$.

$\mathcal{L}(2,8)$



2^8
$2^7 1^1$
2^7
$2^6 1^2$
$2^6 1^1$
$2^5 1^3$
2^6
$2^5 1^2$
$2^4 1^4$
$2^5 1^1$
$2^4 1^3$
$2^3 1^5$
2^5
$2^4 1^2$
$2^3 1^4$
$2^1 1^6$
$2^4 1^1$
$2^3 1^3$
$2^2 1^5$
$2^1 1^7$
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$2^3 1^2$
$2^2 1^4$
$2^1 1^6$
1^8
$2^3 1^1$
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1^6
$2^2 1^1$
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1^5
2^2
$2^1 1^2$
1^4
$2^1 1^1$
1^3
2^1
1^2
1^1
(0)

Theorem. Let R be a finite chain ring with nilpotency index 2 and residue field of order q. Let $\mathcal{P} = \mathcal{P}(R^n)$ be the partially ordered set of all submodules of ${}_RR^n$ with partial order given by inclusion. Then $\mathcal{P}$ has the Sperner proerty and the size of a maximal antichain in $\mathcal{P}$ is equal to

$$\sum_{\mu \prec \mathbf{2}_n} \begin{bmatrix} \mathbf{2}_n \\ \mu \end{bmatrix}_{q^m},$$

where the sum is over all sequences $\mu = (\mu_1, \ldots, \mu_n) \prec \mathbf{2}_n$ with

$$\sum_{i=1}^{n} \mu_i = n.$$

Theorem. Let R be a finite chain ring with nilpotency index m and residue field of order q. Let $\mathcal{P}_n = \mathcal{P}_n(R)$ be the partially ordered set of all submodules of ${}_R R^n$ with partial order given by inclusion. Then $\mathcal{P}$ has the Sperner proerty and the size of a maximal antichain in $\mathcal{P}$ is equal to

$$\sum_{\mu \prec \boldsymbol{m}_n} \begin{bmatrix} \boldsymbol{m}_n \\ \mu \end{bmatrix}_{q^m},$$

where the sum is over all partitions $\mu = (\mu_1, \ldots, \mu_n) \prec \boldsymbol{m}_n$ with

$$\sum_{i=1}^{n} \mu_i = \lfloor \frac{mn}{2} \rfloor.$$

6. Partial Results for Non-free Modules

Let R be a finite chain ring of nilpotency index 2 and with residue field $\mathbb{F}_q$.

Set $\Gamma = \{\gamma_0 = 0, \gamma_1 = 1, \gamma_2, \dots, \gamma_{q-1}\}$ and $\operatorname{rad} R = R\theta$.

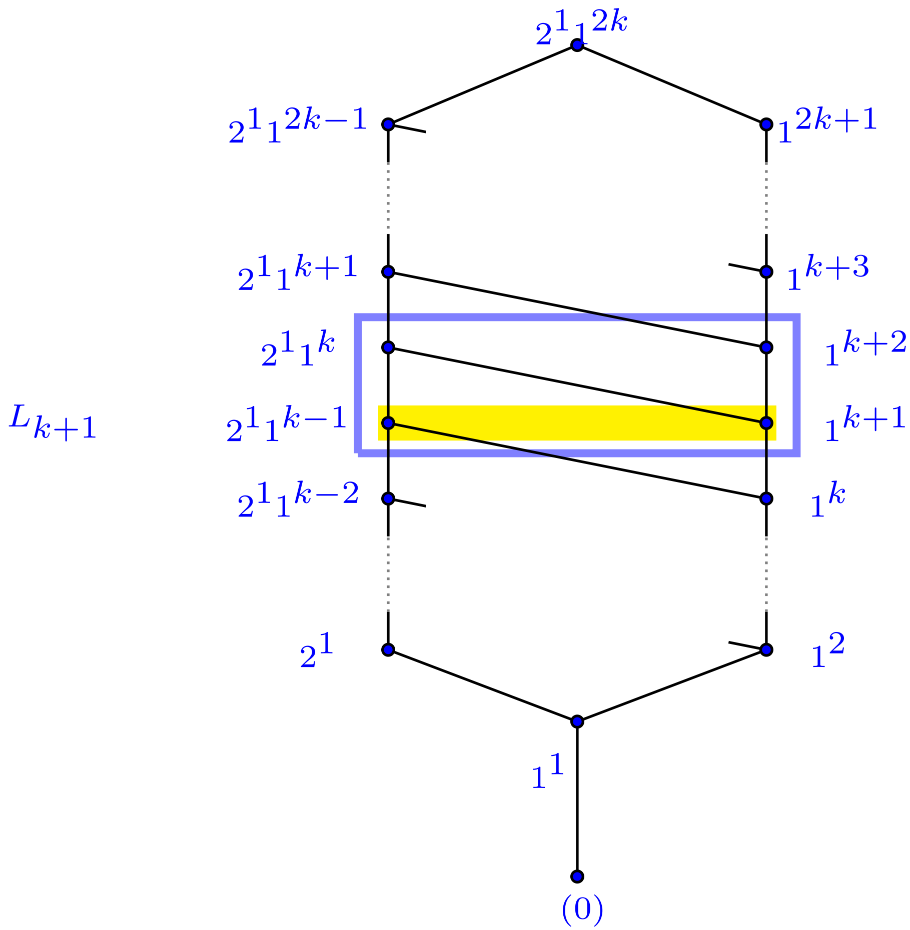
Let ${}_RM$ be a module of shape $2^1 1^n$, e.g. the module generated by the rows of

$$A = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \theta & 0 & \dots & 0 \\ 0 & 0 & \theta & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \theta \end{pmatrix}.$$

Consider the poset $\mathcal{P}(M)$ of all submodules of ${}_RM$.

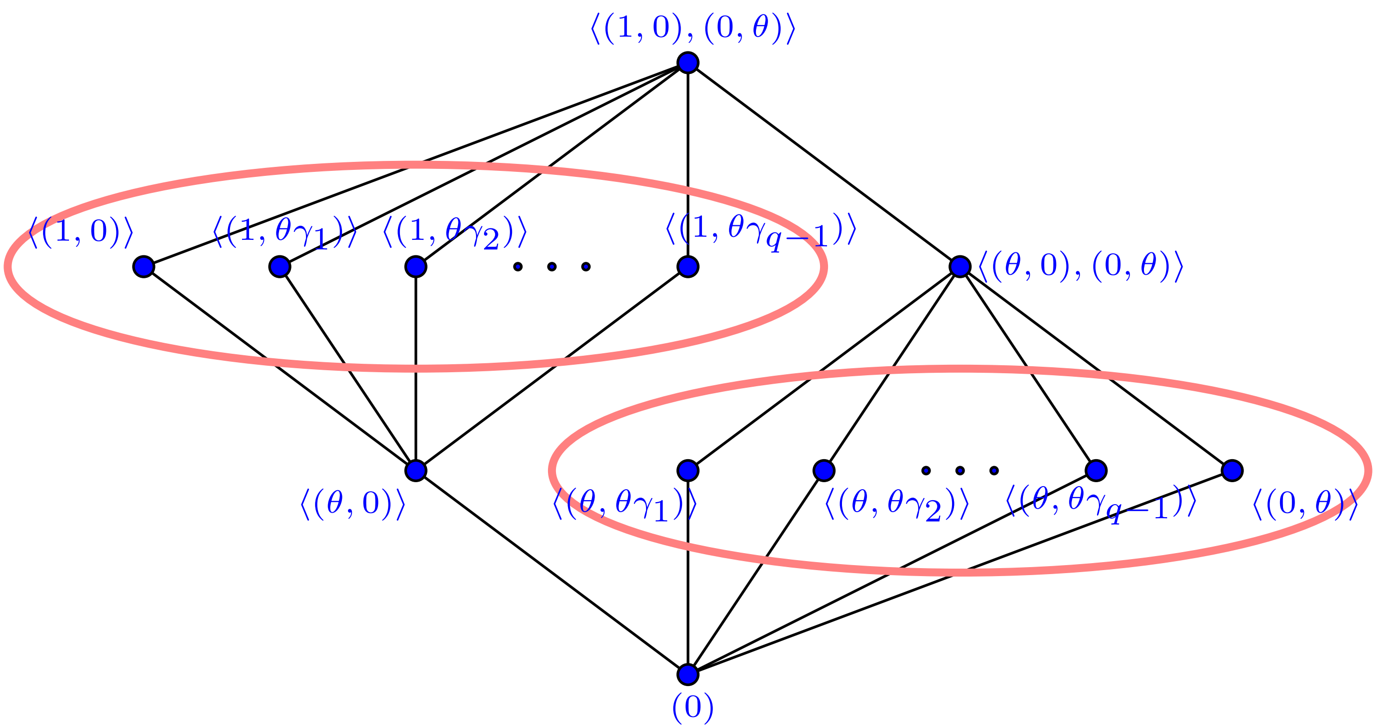
Modules of shape $2^1 1^{2k}$

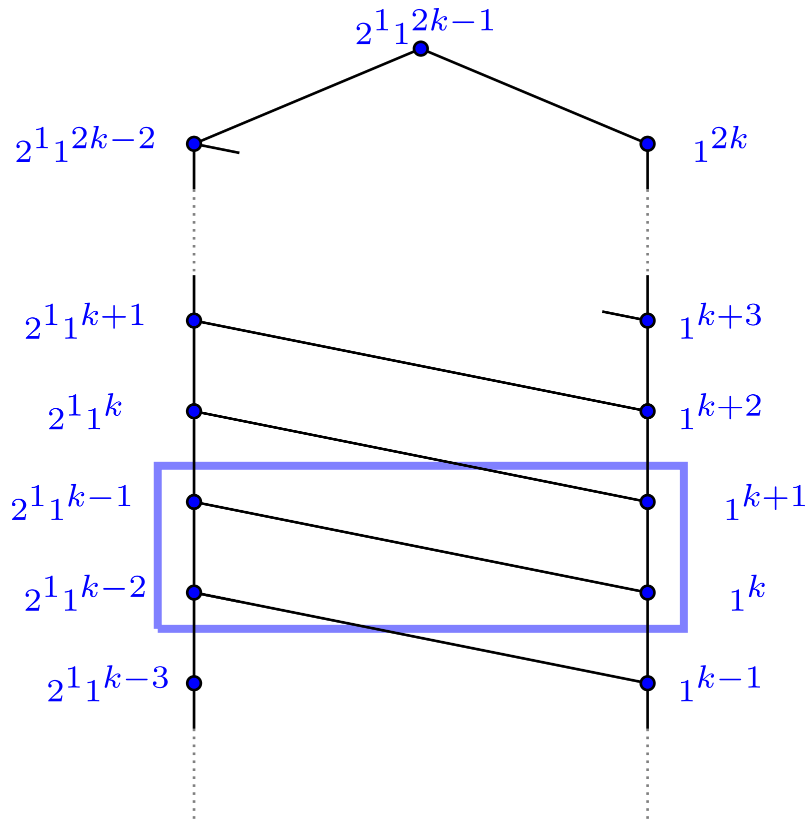
Theorem. Let M be a module of shape $2^1 1^{2k}$ over the finite chain ring R of nilpotency index 2. Then $\mathcal{P}(M)$ has the Sperner property and the maximal antichain has size $W_{k+1}(\mathcal{P})$.

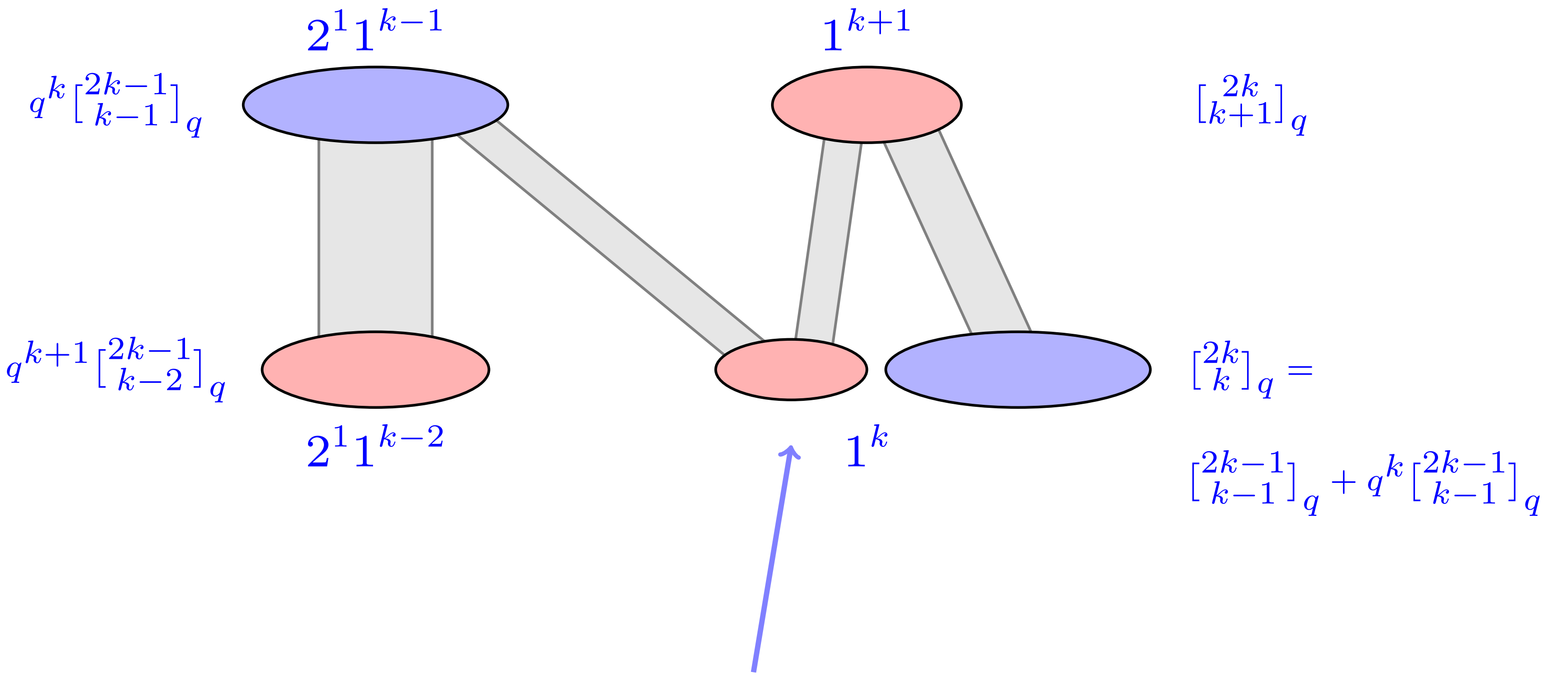

$2^1 1^{2k}$
$2^1 1^{2k-1}$
1^{2k+1}
$2^1 1^{k+1}$
1^{k+3}
$2^1 1^k$
1^{k+2}
L_{k+1}
$2^1 1^{k-1}$
1^{k+1}
$2^1 1^{k-2}$
1^k
2^1
1^2
1^1
(0)

Modules of shape $2^1 1^{2k-1}$

$\mathcal{P}(M) = \mathcal{P}(R \oplus \operatorname{rad} R)$




$2^1 1^{2k-1}$
$2^1 1^{2k-2}$
1^{2k}
$2^1 1^{k+1}$
1^{k+3}
$2^1 1^{k}$
1^{k+2}
$2^1 1^{k-1}$
1^{k+1}
$2^1 1^{k-2}$
1^{k}
$2^1 1^{k-3}$
1^{k-1}



$$\begin{pmatrix} \theta & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{pmatrix}$$

Theorem. Let M be a module of shape $2^1 1^{2k-1}$ over the finite chain ring R of nilpotency index 2. Then $\mathcal{P}(M)$ does not have the Sperner property and the maximal antichain has size $2q^k \begin{bmatrix} 2k-1 \\ k-1 \end{bmatrix}_q$.