The direct sum of q-matroids

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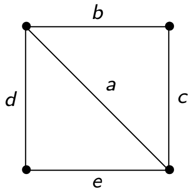
Matroid: a pair (E, r) with

- E finite set;
- $r : 2^E \to \mathbb{N}_0$ a function, the *rank function*, with for all $A, B \in E$:
  (r1) $0 \leq r(A) \leq |A|$
  (r2) If $A \subseteq B$ then $r(A) \leq r(B)$.
  (r3) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ (semimodular)

Example

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Example



But: most matroids don't come from a matrix or graph.

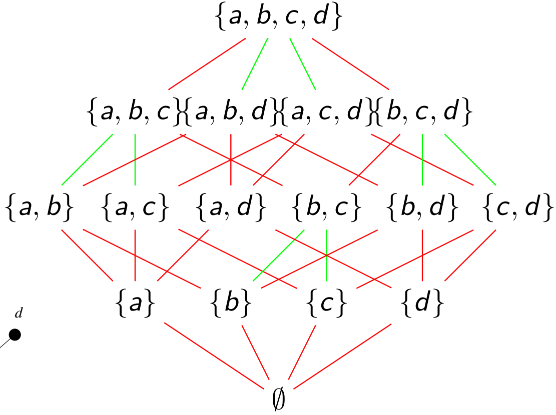
Independent set: subset with rank equal to cardinality

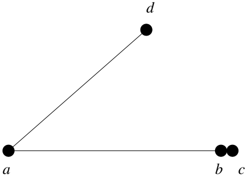
Loop: singleton of rank 0

Restriction $M|_X$: only consider elements in $X \subseteq E$

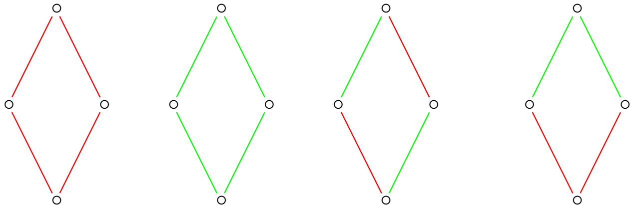
Contraction M/X: only consider elements containing $X \subseteq E$ and remove X

Example





Matroid $\Longleftrightarrow$ only the following diamonds:



one zero mixed prime

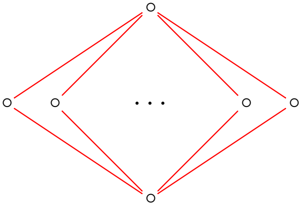
red means rank $+1$, green means rank $+0$

q-Analogues

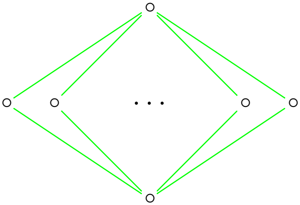
lattice	Boolean	subspace lattice of $\mathbb{F}_q^n$
atom	element	1-dim subspace
height	size	dimension
# atoms	n	$[n]_q := \frac{q^n-1}{q-1}$
meet $\wedge$	intersection	intersection
join $\vee$	union	sum

From q-analogue to 'normal': let $q \to 1$.

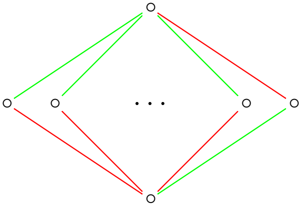
q-Matroid $\iff$ only the following "diamonds":



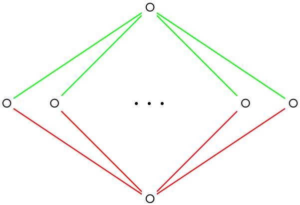
one



zero

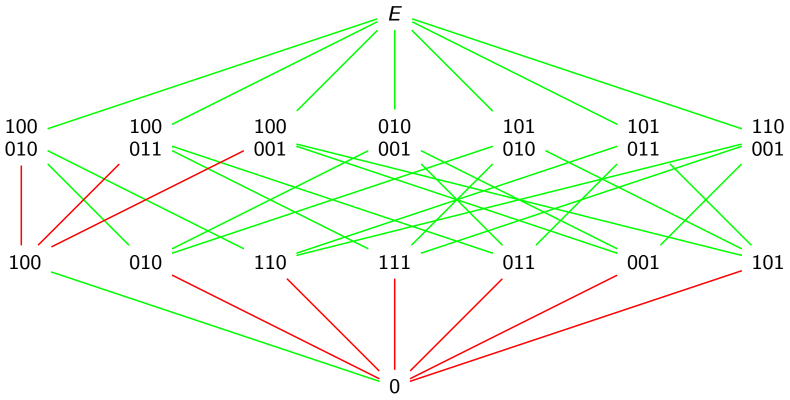


mixed



prime

Example



E
100
010
100
011
100
001
010
001
101
010
101
011
110
001
100
010
110
111
011
001
101
0

Definition

The direct sum of the matroids $M_1 = (E_1, r_1)$ and $M_2 = (E_2, r_2)$ is the matroid M on ground set $E = E_1 \sqcup E_2$ with for all $A \subseteq E$,

$$r(A) = r_1(A \cap E_1) + r_2(A \cap E_2).$$

Its independent sets are union of independent set in M_1 and M_2.

Definition

The direct sum of the matroids $M_1 = (E_1, r_1)$ and $M_2 = (E_2, r_2)$ is the matroid M on ground set $E = E_1 \sqcup E_2$ with

$$M|_{E_1} = M/E_2 = M_1 \text{ and } M|_{E_2} = M/E_1 = M_2.$$

Sets: let $E = E_1 \sqcup E_2$.

For all $A \subseteq E$ we have $A = A_1 \sqcup A_2$ with $A_1 \subseteq E_1$, $A_2 \subseteq E_2$.

Not true for vector spaces!

Example

Let $E_1 = \langle 100, 010 \rangle$ and $E_2 = \langle 001 \rangle$. Then $A = \langle 111 \rangle$ has trivial intersection with both E_1 and E_2.

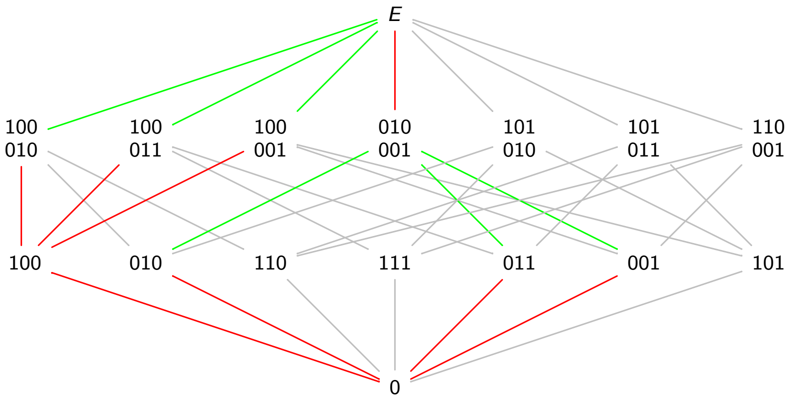
Definition (Naive attempt)

The direct sum of the q-matroids $M_1 = (E_1, r_1)$ and $M_2 = (E_2, r_2)$ is a q-matroid M on ground space $E = E_1 \oplus E_2$ such that

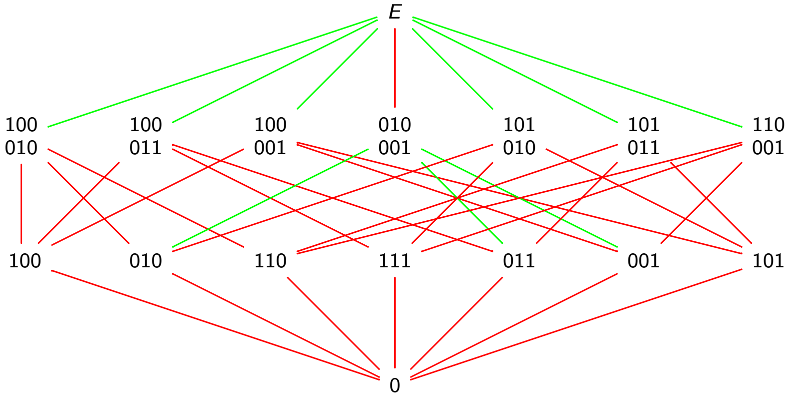
$$M|_{E_1} = M/E_2 = M_1 \text{ and } M|_{E_2} = M/E_1 = M_2.$$

Let's hope the rank axioms take care of the rest of the subspaces!

Example ($U_{1,1} \oplus U_{1,2}$ over $\mathbb{F}_2$)



Example ($U_{1,1} \oplus U_{1,2}$ over $\mathbb{F}_2$)



Unfortunately, this construction becomes not unique already in dimension 4. . .

Goal: find some equivalent description of the direct sum that *does* allow for a q-analogue.

Definition

The matroid union $M_1 \vee M_2$ of two matroids $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$ is a matroid on ground set $E_1 \cup E_2$ with independent sets

$$\mathcal{I} = \{I_1 \cup I_2 : I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}.$$

Its rank function is, for all $A \subseteq E_1 \cup E_2$:

$$r(A) = \min_{X \subseteq E} \{r_{M_1}(X) + r_{M_2}(X) + |A \backslash X|\}.$$

How to make the direct sum of the matroids $M_1 = (E_1, r_1)$ and $M_2 = (E_2, r_2)$, using matroid union?

- Let $E = E_1 \sqcup E_2$.
- Let M_1' be the matroid on E such that $M_1'|_{E_1} = M_1$ and $M_1'|_{E_2}$ consists of only loops.
- Let M_2' be the matroid on E such that $M_2'|_{E_2} = M_2$ and $M_2'|_{E_1}$ consists of only loops.
- Now $M_1 \oplus M_2 = M_1' \vee M_2'$.

Definition (Ceria & J., 2021)

The direct sum of the q-matroids $M_1 = (E_1, r_1)$ and $M_2 = (E_2, r_2)$ is constructed as follows:

- Let $E = E_1 \oplus E_2$ (such that $E_1^{\perp} = E_2$).
- Let M_1' be the q-matroid on E such that $M_1'|_{E_1} = M_1$ and $M_1'|_{E_2}$ consists of only loops.
- Let M_2' be the q-matroid on E such that $M_2'|_{E_2} = M_2$ and $M_2'|_{E_1}$ consists of only loops.
- Now $M_1 \oplus M_2 = M_1' \vee M_2'$.

Theorem (Ceria & J., 2021)

The direct sum has M_1 and M_2 both twice as minors:

$$M|_{E_1} = M/E_2 = M_1 \text{ and } M|_{E_2} = M/E_1 = M_2.$$

Theorem (Ceria & J., 2021)

The dual of the direct sum is the direct sum of the duals:
$(M_1 \oplus M_2)^* = M_1^* \oplus M_2^*$.

Thank you for your attention!

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The direct sum of q-matroids
`arXiv:2109.13637`