Conditions on Large Caps

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Caps

$\mathrm{PG}(n,q)$: n-dimensional projective space over $\mathbb{F}_q$.

Points: 1-spaces of $\mathbb{F}_q^{n+1}$. **Lines**: 2-spaces of $\mathbb{F}_q^{n+1}$.

Definition

A **cap** is a set of points in $\mathrm{PG}(n,q)$, no 3 collinear.

Easy examples:

- Ovals, e.g. $x_0^2 + x_1^2 + x_2^2 = 0$ for $n = 2$.
- Ovoids, e.g. $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$ for $q \equiv 1 \pmod 4$.

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Motivation:

- Linear Codes,
- Extremal Graphs,
- Partial Geometries,
- Strongly Regular Graphs.

Bounds on Caps

Consider a cap $\mathcal{C}$ of $\mathrm{PG}(n, q)$.

Lemma

We have $|\mathcal{C}| \leq (1 + o(1))q^{n-1}$ (as $q \to \infty$, n fix).

Proof.

Look at lines through $p \in \mathcal{C}$. $\square$

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What about regime $n \to \infty$, q fix?

Trivial: A cap has at most size $O(q^n)$ (as $n \to \infty$).

Ellenberg-Gijswijt (2017): For $q = 3$, a cap has at most size $o(2.76^n)$.

More Examples

Bound $(1+o(1))q^{n-1}$ is **tight** for $n = 2, 3$.

Bierbrauer, Edel (2004): Construction of size $\sim 3q^2$ for $n = 4$ for q even.

Segre (1959): Construction of size $(1+o(1))q^{\lfloor \frac{2}{3}n \rfloor}$.

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Hence, if $\mathcal{C}$ has **maximum size**, then

$$q^{\frac{2}{3}n-\frac{2}{3}} \lesssim |\mathcal{C}| \lesssim q^{n-1}.$$

What is the **truth**?

My Hope: $O(q^{\frac{3}{4}n-\frac{1}{4}})$.

That is $O(q^{2.75})$ for $n=4$ and $O(q^5)$ for $n=7$.

Ovoids

Take for $\mathcal{C}$ an **elliptic quadric** of $\mathrm{PG}(3, q)$.

Size: $q^2 + 1$.
Exterior points: Each on precisely $\frac{q^2-q}{2}$ secants.

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Define a **graph**:

Vertices: vectors x of $\mathbb{F}_q^4$ with $\mathrm{PG}(3,q)$ at infinity.
Adjacency: x, y adjacent iff $\langle x, y \rangle$ meets $\mathcal{C}$ at infinity.

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Strongly regular with parameters $(q^4, (q^2+1)(q-1), q-2, q^2-q)$:

- order q^4,
- degree $(q^2+1)(q-1)$,
- two **adjacent** vertices share $q-2$ neighbors,
- two **nonadjacent** vertices share q^2-q neighbors.

Cap of size 11 in $\mathrm{PG}(4,3)$: $(243, 22, 1, 2)$ (Berlekamp-Van Lint-Seidel).
Cap of size 729 in $\mathrm{PG}(5,3)$: $(729, 112, 1, 20)$ (Games graph).

A Fictional Cap

Suppose that $\mathcal{C}$ is a cap in $\mathrm{PG}(6, 3)$.

Size: 91.
Exterior Points: Each on precisely 30 secants.

We obtain a **strongly regular graph**.
Parameters: $(729, 182, 1, 60)$.

(**Partial GQ** with parameters $(s, t, \mu) = (2, 90, 60)$.)

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Cannot exist!

Reason 1: Krein condition.
Reason 2: Absolute bound.
Reason 3: Coclique of size 91, but only 26 possible (**inertia bound**).

Stability? Say, $\leq \frac{1}{12}$ ext. points on 29 secants, $\geq \frac{10}{12}$ on 30, $\leq \frac{1}{12}$ on 31.

Inertia bound: Bound is ≤ 39! Still **impossible**!

Approximately Strongly Regular Graphs

Consider a **k-regular graph of order v**.

Define λ_{ab} (μ_{ab}) as the **size of common neighborhood** of vertices a, b adjacent (nonadjacent).

Strongly regular with parameters (v, k, λ, μ): k-regular of order v with $\lambda = \lambda_{ab}$ and $\mu = \mu_{ab}$.

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Approximately strongly regular with parameters $(v, k, \lambda, \mu; \sigma)$:

$$\mathbb{E}(\lambda_{ab}) = \lambda, \qquad \mathbb{E}(\mu_{ab}) = \mu,$$
$$\mathrm{Var}(\lambda_{ab}) \leq \sigma^2, \qquad \mathrm{Var}(\mu_{ab}) \leq \sigma^2.$$

Some fun facts:

1. SRGs: precisely ASRGs with $\sigma = 0$.
2. Equation $(v - k - 1)\mu = k(k - \lambda - 1)$ holds!
3. **Complement** of ASRG is ASRG with $(v, v-k-1, v-2k+\mu, v-2k+\lambda; \sigma)$.
4. **All regular graphs** are approximately ASRG with $\sigma = k$.

The Inertia Bound for ASRGs

Theorem

Let Γ be an ASRG with $k = o(v)$ and $k = o(|\lambda - \mu|^2)$.[a]
Then a ***coclique*** *in Γ has at most size*

$$(1 + o(1)) \left(\frac{vk}{(\mu - \lambda)^2} + \frac{v^2 \sigma^2}{k^2} \right).$$

[a] Family $(\Gamma_i)_i$ of ASRGs with $(v_i, k_i, \lambda_i, \mu_i; \sigma_i)$, $k_i = o(v_i)$ and $k_i = o(|\lambda_i - \mu_i|^2)$.

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Let $\mathcal{C}$ be a cap of $\mathrm{PG}(n, q)$.

Corollary

If $\sigma^2 = o(q^{\frac{1}{4}n})$, then $|\mathcal{C}| = o(q^{\frac{3}{4}n})$.

If $|\mathcal{C}| = \Omega(q^{n-1})$, then $\sigma = \Omega(q^{\frac{1}{2}n - \frac{3}{2}})$.

Ellenberg-Gijswijt (2016): $o(2.76^n)$. (for $q = 3$.)
Edel (2003): $\omega(2.21^n)$.
Bound here for small σ: $o(2.28^n)$.

Krein Bound for ASRGs

Consider an **approximately SRG** Γ with parameters $(v, k, \lambda, \mu; \sigma)$.

Theorem (Krein Bound for ASRGs)

If $\mu > \lambda$, $k = o(v)$, $k = o(|\mu - \lambda|^{\frac{3}{2}})$, *then* $\sigma \geq (1+o(1))(\mu - \lambda)^{\frac{3}{2}} v^{-1}$.

Theorem (Krein Bound for Special 1-Walk-Regular ASRGs)

Same plus ***regularity conditions****. Then* $\sigma \geq (1 + o(1))(\mu - \lambda)^{\frac{5}{4}} v^{-\frac{3}{4}} k^{\frac{1}{2}}$.

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Let $\mathcal{C}$ be a cap of $\mathrm{PG}(n, q)$.

Corollary

If $\sigma^2 = o(q^{\frac{1}{2}n})$ *and* ***regularity conditions****, then* $|\mathcal{C}| = O(q^{\frac{3}{4}n - \frac{1}{4}})$.

If $|\mathcal{C}| = \Omega(q^{n-1})$ *and* ***regularity conditions****, then* $\sigma = \Omega(q^{n-2})$.

Now σ **large enough** for most reasonable construction!

Why do this?

(1) **Ihringer-Verstraëte (2022*):**
Random constructions for **cap variants**.

- **Failure** to improve $\Omega(q^{\frac{2}{3}n})$ bound for caps.
- Constructions should satisfy results, so $O(q^{\frac{3}{4}n-\frac{1}{4}})$ **best possible**.

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(2) Different application of ASRGs:

Mubayi, Verstraëte (2018): Lower bounds on **off-diagonal Ramsey numbers** from **clique-free pseudorandom graphs**.

ASRG Krein bounds imply

Corollary (informal)

*If subconstituents **close to SRGs**, then graphs are (relatively) **sparse**.*

Details in: FI, Approximately Strongly Regular Graphs, arXiv:2205.05792 [math.CO].

Very Small ASRGs

One can also study very small parameters:

v	k	λ	μ	σ	nr	remarks
8	3	0	1.5	0.5	1	D_8
10	3	0	1	0	1	Petersen graph, $NO^{-\perp}_{3,5}$
12	3	0	0.75	~ 0.43	2	D_8, D_9
14	3	0	0.8	~ 0.49	9	
16	3	0.625	0.34375	~ 0.48	2	D_6, D_9
18	3	$0.\overline{6}$	$0.3\overline{571428}$	~ 0.47	2	D_6, $S_3^2 \rtimes C_2$
20	3	0.3	0.31875	~ 0.47	5993	
22	3	$0.\overline{27}$	$0.2\overline{87}$	~ 0.45	86977	
9	4	1	2	0	1	Paley(9)
10	4	0.75	1.8	~ 0.43	1	D_5
11	4	$1.\overline{09}$	$1.\overline{27}$	~ 0.44	1	$C_2^2 \times S_3$
12	4	1	$1.\overline{142857}$	0.41	1	$C_2 \times D_4$
13	4	$0.\overline{692307}$	$1.\overline{153846}$	~ 0.46	1	D_8
14	4	$0.32\overline{142857}$	$1.\overline{190476}$	~ 0.47	2	id, C_2^2
15	4	0.1	1.16	~ 0.37	1	D_6
16	4	0	$1.\overline{09}$	0.36	1	$C_2^4 \rtimes C_2$
12	5	0.7	2.75	~ 0.46	1	S_3^2
14	5	$1.0\overline{285714}$	$1.\overline{857142}$	~ 0.45	1	$C_2 \times D_4$
13	6	2	3	0	1	Paley(13)

Thank you for your attention!