On subspaces of classical polar spaces

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Finite Geometries 2022 — Sixth Irsee Conference

Polar spaces

Notation

- $\Gamma = (\mathscr{P}, \mathscr{L})$ point-line geometry;
- $\forall p, q \in \mathscr{P} : p \perp q \equiv \exists \ell \in \mathscr{L} : p, q \in \ell$ (collinearity);
- $X \subseteq \mathscr{P}$, $X^\perp := \{p \in \mathscr{P} : \forall x \in X, p \perp x\}$ (perp).

Definition (One/all axiom)

Γ is a *polar space* if and only if

$$\forall p \in \mathscr{P}, \ell \in \mathscr{L}, \text{ either } \ell \subseteq p^\perp \text{ or } |\ell \cap p^\perp| = 1.$$

Polar spaces: subspaces

Definition

- Γ *non-degenerate* if $\mathrm{Rad}(\Gamma) := \mathscr{P}^\perp = \emptyset$.
- $X \subseteq \mathscr{P}$ *subspace* of Γ if

$$\forall \ell \in \mathscr{L} : |\ell \cap X| \geq 2 \Rightarrow \ell \subseteq X.$$

- $X \leq \Gamma$ *singular* if $X \subseteq X^\perp$.

Remark

- A subspace $X \leq \mathscr{P}$ can be endowed with the structure of a polar space $(\mathscr{P}_{|X}, \mathscr{L}_{|X})$ where

$$\mathscr{P}_X = X, \qquad \mathscr{L}_X := \{\ell \in \mathscr{L} : \ell \subseteq X\}.$$

Hyperplanes

Definition

- A *hyperplane* $\mathscr{S}$ of a polar space $\Gamma := (\mathscr{P}, \mathscr{L})$ is a proper subspace $\mathscr{S} < \Gamma$ such that $\forall \ell \in \mathscr{L}, \ell \cap \mathscr{S} \neq \emptyset$.

Theorem (Shult)

Hyperplanes are maximal subspaces of Γ.

Rank

Definition

- $\mathscr{S}(\Gamma) := \{X \leq \Gamma : X \leq X^{\perp}\}$;
- $\mathfrak{S}(\Gamma) :=$ {well ordered chains of elements of $\mathscr{S}(\Gamma)$};
- *Rank*: $\mathrm{Rk}(\Gamma) := \max\{|\mathfrak{z}| : \mathfrak{z} \in \mathfrak{S}(\Gamma)\}$.

Definition

- Γ of rank n *thick* if every line of Γ contains at least 3 points and every singular subspace of rank $n-2$ is contained in at least 3 maximal singular subspaces.

Remarks

Remarks

- If Γ is thick, then all of its maximal singular subspaces are self-dual projective spaces.
- All singular subspaces of Γ are projective spaces.
- The planes of a thick polar space of rank 3 are Moufang.
- If $r := \max\{\dim(X) + 1 : X \leq \mathscr{P}, X \text{ singular}\} < \infty$, then all maximal singular subspaces (*generators*) of Γ have the same dimension $\mathrm{Rk}(\Gamma) - 1 = r - 1$.
- If it is finite, the rank of Γ is the common (projective) dimension of its generators plus 1.

Stars

Stars

- $X \leq \mathscr{P}$: singular subspace of Γ;
- $\mathrm{Res}_X(\Gamma) := (\mathscr{P}_X, \mathscr{L}_X)$ where
  - $\mathscr{P}_X := \{X \oplus \langle t \rangle : t \in X^\perp\}$;
  - $\mathscr{L}_X := \{X \oplus \ell : \ell \in \mathscr{L}, \ell \subseteq X^\perp\}$.
- $\mathrm{Res}_X(\Gamma)$ with incidence given by $\subseteq$ is a polar space.

Remarks

- By construction, the elements of $\mathscr{P}_X$ and $\mathscr{L}_X$ are singular subspaces.
- If $X = \mathscr{P}^\perp$, then $\Gamma_{nd} := \mathrm{Res}_{\mathscr{P}^\perp}(\Gamma)$ is non–degenerate.

Non-degenerate rank

Definition

- $\mathscr{S} \leq \Gamma$ subspace;
- *Non-degenerate rank of* $\mathscr{S}$:

$$\mathrm{Rk}_{nd}(\mathscr{S}) := \mathrm{Rk}(\mathrm{Res}_{\mathrm{Rad}(\mathscr{S})}(\mathscr{S})).$$

Remark

- $\mathrm{Rad}(\mathscr{S}) = \mathscr{S} \cap \mathscr{S}^{\perp}$;
- $\mathrm{Rk}_{nd}(\mathscr{S}) = \mathrm{Rk}(\mathscr{S}) - \mathrm{Rk}(\mathrm{Rad}(\mathscr{S}))$;
- *if* $\mathscr{S}$ *singular subspace, then* $\mathrm{Rk}_{nd}(\mathscr{S}) = 0$.

Our main problem

Problem

- Describe the subspaces of a given polar space.

Projective embeddings

Notation

- $\mathbb{K}$: division ring;
- $V := V(\mathbb{K})$ vector space over $\mathbb{K}$;
- $\Gamma = (\mathscr{P}, \mathscr{L})$: polar space.

Definition

- $\varepsilon : \mathscr{P} \to \mathrm{PG}(V)$ *(full) projective embedding* of Γ if
  1. ε is injective;
  2. $\forall \ell \in \mathscr{L} : \{\varepsilon(p) : p \in \ell\}$ is a line of $\mathrm{PG}(V)$;
  3. $\langle \varepsilon(\mathscr{P}) \rangle = \mathrm{PG}(V)$.
- $\dim(\varepsilon) := \dim(V)$.

Natural questions

- When a polar space Γ is embeddable?
- How to describe a projective embedding? How many of them there are? Are there any preferred embeddings?
- What properties of Γ are "easy" to read in the embedding?
- How to characterize the possible embeddings in terms of Γ?

Embeddable polar spaces

Theorem (Tits)

- *All polar spaces of rank $n \geq 4$ are embeddable.*
- *The non-embeddable polar spaces of rank $n = 3$ are:*
    1. *Line Grassmannians of projective spaces of rank 3 over a non-commutative division ring $\mathbb{K}$ (not thick);*
    2. *A class of polar spaces related to Cayley-Dickson division algebras whose totally singular planes are Moufang but not Desarguesian (thick).*

Embeddable polar spaces

- $\Gamma := (\mathscr{P}, \mathscr{L})$ embeddable, non-degenerate, thick polar space of rank n with $2 \leq n < \infty$;
- $\varepsilon : \Gamma \to \mathrm{PG}(V)$ projective embedding.

Remark

- $\dim(\varepsilon) \geq 2n$.

Theorem (I. Cardinali, LG, A. Pasini)

An embeddable non-degenerate thick polar space Γ admits an embedding ε of dimension $2n$ if and only if

- *for every generator M of Γ, $a, b \in \mathscr{P}$ with $a \not\perp b$,*

$$\dim(M/(M \cap \{a,b\}^{\perp})) = 1 \Rightarrow M \cap \{a,b\}^{\perp\perp} \neq \emptyset.$$

Covering embeddings

Definition

- $\varepsilon_1 : \mathscr{P} \to \mathrm{PG}(V_1)$, $\varepsilon_2 : \mathscr{P} \to \mathrm{PG}(V_2)$ p. embeddings;
- $\varepsilon_2 \leq \varepsilon_1$ (ε_1 *covers* ε_2) if $\exists f : V_1 \to V_2$ such that

$$\begin{array}{ccc} \mathscr{P} & \xrightarrow{\varepsilon_1} & \mathrm{PG}(V_1) \\ & \searrow^{\varepsilon_2} & \downarrow [f] \\ & & \mathrm{PG}(V_2) \end{array}$$

- ε *dominant* (or *relatively universal*) if

$$\forall \varepsilon' : \mathscr{P} \to \mathrm{PG}(V) \text{ embedding}, \varepsilon \leq \varepsilon' \Rightarrow \varepsilon \cong \varepsilon';$$

- $\varepsilon_{\text{univ}}$ *(absolutely) universal* if

$$\forall \varepsilon : \mathscr{P} \to \mathrm{PG}(V) \text{ embedding}, \varepsilon \leq \varepsilon_{\text{univ}}.$$

Embeddable polar spaces

Definition

- $\Gamma := (\mathscr{P}, \mathscr{L})$ *classical polar space* when it admits the universal embedding $\varepsilon_{\text{univ}} : \mathscr{P} \to \mathrm{PG}(V)$ for suitable $V := V(\mathbb{K})$.

Theorem (Tits)

If Γ is a classical polar space and $\mathrm{char}(\mathbb{K}) \neq 2$ then $\varepsilon_{\text{univ}}$ is the unique embedding of Γ.

Describing the embeddings

Given

- $\Gamma := (\mathscr{P}, \mathscr{L})$: polar space;
- $\varepsilon : \mathscr{P} \to \mathrm{PG}(V)$: projective embedding.

Describe

- $\varepsilon(\mathscr{P}) \subseteq \mathrm{PG}(V)$;
- $\varepsilon(\mathscr{L}) \subseteq \mathrm{Gr}_2(\mathrm{PG}(V))$.

Constructing polar spaces from forms

Theorem

Let f be a reflexive (σ, ϵ)-sesquilinear form on a vector space V. Then

1. $\Gamma(f) := (\mathscr{P}(f), \mathscr{L}(f))$ *is a polar space where*
   - $\mathscr{P}(f)$*: set of the f-isotropic points of* $\mathrm{PG}(V)$*;*
   - $\mathscr{L}(f)$*: set of the totally f-isotropic lines of* $\mathrm{PG}(V)$*.*
2. *The identity mapping* $\iota : \mathscr{P}(f) \to \mathrm{PG}(V)$ *is an embedding for* $\Gamma(f)$*.*
3. *The polar space* $\Gamma(f)$ *is non-degenerate if and only if* $[\mathrm{Rad}(f)] = \emptyset$*.*

Theorem

Let Q be a generalized (σ, ϵ)-pseudoquadratic form on a vector space V over $\mathbb{K}$ with $(\sigma, \epsilon) \neq (\mathrm{Id}_{\mathbb{K}}, -1)$ if $\mathrm{char}(\mathbb{K}) \neq 2$. Then

1. $\Gamma(Q) := (\mathscr{P}(Q), \mathscr{L}(Q))$ *is a polar space where*
   - $\mathscr{P}(Q)$*: set of the Q-singular points of* $\mathrm{PG}(V)$*;*
   - $\mathscr{L}(Q)$*: set of the totally Q-singular lines of* $\mathrm{PG}(V)$*.*
2. *The identity mapping* $\iota : \mathscr{P}(f) \to \mathrm{PG}(V)$ *is an embedding for* $\Gamma(Q)$*.*
3. *The polar space* $\Gamma(Q)$ *is non-degenerate if and only if* $[\mathrm{Rad}(Q)] = \emptyset$*.*

Description of the embeddings

Theorem (Tits)

Let Γ be an embeddable non-degenerate polar space of rank $n \geq 2$ and $\varepsilon : \Gamma \to \mathrm{PG}(V)$ be an embedding.

- *If ε is dominant, then $\varepsilon(\Gamma) = \Gamma(Q)$ for Q a non-degenerate pseudoquadratic form Q defined over V or $\mathrm{char}(\mathbb{K}) \neq 2$ and $\varepsilon(\Gamma) = \Gamma(f)$ for $f : V \times V \to \mathbb{K}$ a non-degenerate alternating form.*
- *Furthermore, ε is also* absolutely universal *except in the following two cases:*
  1. *Γ is a* bi-embeddable quaternion quadrangle*; i.e. $\mathrm{Rk}(\Gamma) = 2$, $\dim(V) = 4$, $\mathbb{K}$ is a quaternion division ring, Q is (σ, ϵ)-quadratic with σ the standard involution of $\mathbb{K}$ and $\mathbb{K}_{\sigma,\epsilon}$ is a 1-dimensional vector space over $Z(\mathbb{K})$.*
  2. *Γ is a* grid of order at least 5*; i.e. $\mathrm{Rk}(\Gamma) = 2$, $\dim(V) = 4$ and $\varepsilon(\Gamma)$ is a hyperbolic quadric of the projective 3-space $\mathrm{PG}(V)$.*

Subspaces from an embedding

Remark

- $\Gamma := (\mathscr{P}, \mathscr{L})$ *embeddable polar space;*
- $\varepsilon : \mathscr{P} \to \mathrm{PG}(V)$ *embedding of* Γ*;*
- $[X] \leq \mathrm{PG}(V)$*;*
- $\mathscr{S} := \varepsilon^{-1}([X])$ *is a subspace of* Γ*.*

Definition

We say that a subspace $\mathscr{S}$ of Γ *arises* from the embedding $\varepsilon : \mathscr{P} \to \mathrm{PG}(V)$ if there is $[X] \leq \mathrm{PG}(V)$ such that $\mathscr{S} = \varepsilon^{-1}([X])$.

Remark

In general, not all subspaces of Γ *arise from (any) embeddings.*

Example 1: wrong embedding

- $\mathbb{K} := \mathbb{F}_{2^n}$, $V := \mathbb{K}^6$, $f(\mathbf{x}) = x_1x_2 + x_3x_4 + x_5x_6$, $(\mathscr{P}, \mathscr{L}) := \Gamma(f)$;
- $\iota : \mathscr{P} \to \mathrm{PG}(V)$ given by the identity is an embedding;
- $\varepsilon_{\text{univ}} : \mathscr{P} \to \mathrm{PG}(\mathbb{K}^7)$ universal embedding;
- the image of $\varepsilon_{\text{univ}}$ is the quadric

$$Q(\mathbf{x}') = x_1x_2 + x_3x_4 + x_5x_6 + x_7^2;$$

- $\pi : x_1 + x_7 = 0$;
- $\mathscr{S} := \varepsilon_{\text{univ}}^{-1}(\pi)$: subspace of $\Gamma(f)$ arising from $\varepsilon_{\text{univ}}$ but not ι;
- $\iota(\mathscr{S})$ is a quadric.

Example 2: Subspaces of rank 1

- $\mathbb{K} := \mathbb{F}_{q^2}$, $V := \mathbb{K}^4$, $f(\mathbf{x}) = x_1 x_2^q + x_3 x_4^q$.
- $(\mathscr{P}, \mathscr{L}) := \Gamma(f)$ polar space;
- $\iota : \mathscr{P} \to \mathrm{PG}(V)$: identity mapping;
- ι is the universal embedding of $\Gamma(f)$;
- The pointset of Γ is identified by ι with an Hermitian surface $\mathscr{H}(3, q^2)$;
- It is well known that $\mathscr{H}(3, q^2)$ has non-classical ovoids (e.g. obtained by derivation) $\mathscr{O}$;
- The preimage $\iota^{-1}(\mathscr{O})$ of a non-classical ovoid $\mathscr{O}$ of $\mathscr{H}(3, q^2)$ is a hyperplane of $\Gamma(f)$ which does not arise from ι.

Main theorem

Theorem (A. Cohen, E.E. Shult)

- *Let* $\Gamma := (\mathscr{P}, \mathscr{L})$ *to be a polar space of rank* $n > 2$. *Then all hyperplanes of* Γ *arise from* $\varepsilon_{univ} : \mathscr{P} \to \mathrm{PG}(V_{univ})$.

Theorem (I. Cardinali, LG, A. Pasini)

Suppose

- Γ: *classical polar space of finite rank* $n \geq 2$ *with universal embedding* $\varepsilon_{univ} : \Gamma \to \mathrm{PG}(V_{univ})$;
- $\mathscr{S}$: *proper non-singular subspace of* Γ *with* $\mathrm{Rk}_{nd}(\mathscr{S}) \geq 2$.

Then

- $\mathscr{S}$ *arises from* ε_{univ}.

Remarks

Remarks

- Γ has finite rank but we do not assume it is finitely generated.
- The hypothesis $\mathrm{Rk}_{nd}(\mathscr{S}) \geq 2$ cannot be removed from the theorem.
- If $\mathrm{Rk}_{nd}(\mathscr{S}) = 1$, then $\mathscr{S}$ is a "cone over a partial ovoid", in the sense that $\mathscr{S}$ is a collection of singular subspaces of rank $k+1$ containing a fixed subspace of rank k, no two of them contained in a common singular subspace.
- The two embeddable polar spaces which do not admit the universal embedding do not admit proper non-singular subspaces of nondegenerate rank at least 2 (so we do not need to exclude them explicitly).

Theorem (I. Cardinali, LG, A. Pasini)

- *Every maximal proper subspace of rank at least* 2 *of a classical polar space* Γ *is a hyperplane.*

Corollary (I. Cardinali, LG, A. Pasini)

Suppose

- Γ*: polar space with* $\mathrm{Rk}(\Gamma) = n > 2$.

Then

- *The hyperplanes of* Γ *are precisely the maximal subspaces of* Γ *of rank at least* 2 *(actually either* $n-1$ *or n).*

Remark (Anonymous referee)

- *When* $\mathrm{Rk}(\Gamma) = 2$ *there are maximal subspaces of rank* 1 *which are not hyperplanes.*
- *(actually also when* $\mathrm{Rk}(\Gamma) > 2$ *there might be such subspaces)*

Theorem (I. Cardinali, LG, A. Pasini)

Suppose

- $\Gamma := (\mathscr{P}, \mathscr{L})$: *polar space of rank* 2.

Then

- Γ *does not admit a* 2*-dimensional embedding;*
- *if* Γ *admits a relatively universal* 3*-dimensional embedding, then all of its proper subspaces have non-degenerate rank at most* 1.

Further developments

Theorem (A. Pasini)

Let Γ be an embeddable polar space of rank n > 2. Then any subspace $\mathscr{S}$ of Γ of non-degenerate rank at least 2 arises from an embedding, except possibly when $\mathscr{S}$ is a rosette.

References

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